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A PRODUCT CONSTRUCTION FOR HYPERBOLIC METRIC SPACES

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ABSTRACT. For hyperbolic metric spaces X_1, X_2 we define and study a one parameter family of "hyperbolic products" Y_Δ, $\Delta \geq 0$, of X_1 and X_2. In particular, we investigate the relation between the boundaries at infinity of the factor spaces and the boundary at infinity of their hyperbolic products.

1. Introduction

A triple $(a_1, a_2, a_3) \in \mathbb{R}^3$ of three real numbers is called a *δ-triple* for $\delta \geq 0$ if $a_\mu \geq \min\{a_{\mu+1}, a_{\mu+2}\} - \delta$ for $\mu = 1, 2, 3$, where the indices are taken modulo 3. Thus (a_1, a_2, a_3) is a δ-triple, if the two smallest of the three numbers differ by at most δ.

Let X be a metric space, and let $|xy|$ denote the distance between points. For $x, y, z \in X$ let

$$(x|y)_z := \frac{1}{2}(|zx| + |zy| - |xy|).$$

The space X is called *δ-hyperbolic* (compare [G]) if for all $o, x, y, z \in X$

$$((x|y)_o, (y|z)_o, (x|z)_o) \text{ is a } \delta\text{-triple}.$$

X is called *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

Given two hyperbolic metric spaces, their metric product will typically fail to be hyperbolic itself. In [FS2] we introduced a hyperbolic product construction for proper, geodesic, hyperbolic metric spaces. Given two such spaces, their hyperbolic product was shown to be a proper, geodesic, hyperbolic metric space itself.

The purpose of this paper is to generalize this hyperbolic product construction to arbitrary hyperbolic metric spaces.

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Let X_1, X_2 be metric spaces and $Y := X_1 \times X_2$. On Y we will always consider the maximum metric, i.e.,

$$|(x_1, x_2)(y_1, y_2)| := \max\{|x_1 y_1|, |x_2 y_2|\} \quad \text{for all } x_\nu, y_\nu \in X_\nu,\ \nu = 1, 2.$$

For $a, b, c \in \mathbb{R}$ and $c \geq 0$ we define

$$a \doteq_c b \quad :\Longleftrightarrow \quad |a - b| \leq c.$$

Given two pointed hyperbolic metric spaces (X_1, o_1) and (X_2, o_2) and a number $\Delta \geq 0$, we write $o := (o_1, o_2) \in Y$ and define

$$Y_{\Delta,o} := \Big\{ (x_1, x_2) \in Y \;\Big|\; |o_1 x_1| \doteq_\Delta |o_2 x_2| \Big\}.$$

The space $Y_{\Delta,o} \subset Y$ is endowed with the restriction of the maximum metric on Y.

THEOREM 1.1. *If X_1, X_2 are δ-hyperbolic, then $Y_{\Delta,o}$ is δ'-hyperbolic for some $\delta' = \delta'(\delta, \Delta)$.*

We also discuss a version of this result where the base point lies at infinity. For a hyperbolic metric space X one can define its boundary at infinity $\partial_\infty X$ (for details see Section 3). Let (X_ν, o_ν), $\nu = 1, 2$, be two pointed hyperbolic spaces with non-empty boundaries at infinity and fix $\xi_\nu \in \partial_\infty X_\nu$, $\nu = 1, 2$. Let b_ν be the Busemann function associated to o_ν and ξ_ν, $\nu = 1, 2$ (for the definition of the Busemann function see Section 3). Let $\Delta \geq 0$. We write $\xi := (\xi_1, \xi_2)$ and define

$$Y_{\Delta,\xi,o} := \Big\{ (x_1, x_2) \in Y \;\Big|\; b_1(x_1) \doteq_\Delta b_2(x_2) \Big\}.$$

THEOREM 1.2. *If X_1, X_2 are δ-hyperbolic metric spaces with non-empty boundaries at infinity, then $Y_{\Delta,\xi,o}$ is δ'-hyperbolic for some $\delta' = \delta'(\delta, \Delta)$.*

In order to investigate the boundaries of $Y_{\Delta,o}$ and $Y_{\Delta,\xi,o}$ we need more structure:

Let $k \geq 0$. A *k-rough geodesic* is a map $\gamma : I \to X$ from an interval $I \subset \mathbb{R}$ to a metric space X with

$$|\gamma(s)\gamma(t)| \doteq_k |s - t| \qquad \text{for all } s, t \in I.$$

The space X is called *k-roughly geodesic*, if for every pair $x, y \in X$ there exists a k-rough geodesic $\gamma : [0, |xy|] \to X$ with $\gamma(0) = x$ and $\gamma(|xy|) = y$. X is called *roughly geodesic* if X is k-roughly geodesic for some $k \geq 0$.

THEOREM 1.3. *If X_1, X_2 are δ-hyperbolic and k-roughly geodesic, then there exists $\Delta_0 = \Delta_0(\delta, k) \geq 0$ such that for all $\Delta \geq \Delta_0$ the space $Y_{\Delta,o}$ is k'-roughly geodesic for some $k'(\delta, k, \Delta)$.*

THEOREM 1.4. *Let X_1, X_2 be δ-hyperbolic and k-roughly geodesic metric spaces with non-empty boundaries at infinity. Then there exists some $\Delta_0 = \Delta_0(\delta, k) \geq 0$ such that $Y_{\Delta,\xi,o}$ is roughly geodesic for all $\Delta \geq \Delta_0$.*

Finally, we relate the topology of the boundary at infinity of our hyperbolic products to those of the boundary at infinity of its factors, by proving the following two theorems:

THEOREM 1.5. *Let X_ν, $\nu = 1, 2$, be δ-hyperbolic and k-roughly geodesic metric spaces. Then there exists $\Delta_0 = \Delta_0(\delta, k) \geq 0$ such that for all $\Delta \geq \Delta_0$ there is a natural homeomorphism $\partial_\infty Y_{\Delta,o} \approx \partial_\infty X_1 \times \partial_\infty X_2$.*

THEOREM 1.6. *Let X_ν, $\nu = 1, 2$, be δ-hyperbolic and k-roughly geodesic metric spaces. Then there exists $\Delta_0 = \Delta_0(\delta, k) \geq 0$ such that for all $\Delta \geq \Delta_0$ there is a natural homeomorphism $\partial_\infty Y_{\Delta,\xi,o} \approx (\partial_\infty X_1, \xi_1) \wedge (\partial_\infty X_2, \xi_2)$. Here $(\partial_\infty X_1, \xi_1) \wedge (\partial_\infty X_2, \xi_2)$ is the coarse smashed product of the pointed topological spaces $(\partial_\infty X_1, \xi_1)$ and $(\partial_\infty X_2, \xi_2)$.*

For the precise definition of the coarse smashed product of two pointed topological spaces, we refer the reader to Section 7.2.

Outline of the paper. In Sections 2 and 3 we start with some preliminaries and the notion of general hyperbolic metric spaces. In Section 4 we discuss hyperbolic products and prove Theorems 1.1 and 1.2. In Section 5 we introduce the notion of roughly geodesic metric spaces and in Section 6 we prove Theorems 1.3 and 1.4. In Section 7 we investigate the boundary structure and prove Theorems 1.5 and 1.6.

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2. Preliminaries

For $a, b, c \in \mathbb{R}$ and $c \geq 0$ we define

$$a \doteq_c b \quad :\Longleftrightarrow \quad |a - b| \leq c.$$

If $\{a_i\}_i, \{b_i\}_i$ are sequences, where $i \in \mathbb{N}$, then we define

$$\{a_i\}_i \doteq_c a \quad :\Longleftrightarrow \quad \limsup |a_i - a| \leq c$$

and

$$\{a_i\}_i \doteq_c \{b_i\}_i \quad :\Longleftrightarrow \quad \limsup |a_i - b_i| \leq c.$$

Let $\delta \geq 0$. A triple $(a_1, a_2, a_3) \in \mathbb{R}^3$ is called a *δ-triple*, if $a_\mu \geq \min\{a_{\mu+1}, a_{\mu+2}\} - \delta$ for $\mu = 1, 2, 3$, where the indices are taken modulo 3.

The following is easily proved:

LEMMA 2.1.

(1) *If* (a_1, a_2, a_3) *and* (b_1, b_2, b_3) *are* δ*-triples, then*

$$(\min\{a_1, b_1\}, \min\{a_2, b_2\}, \min\{a_3, b_3\})$$

is a δ*-triple.*

(2) *If* $\{(a_{1i}, a_{2i}, a_{3i})\}_i$ *are* δ*-triples for* $i \in \mathbb{N}$*, then*

$$(\inf a_{1i}, \inf a_{2i}, \inf a_{3i})$$

and

$$(\liminf a_{1i}, \liminf a_{2i}, \liminf a_{3i})$$

are δ*-triples.*

We call the following result the Tetrahedron Lemma.

LEMMA 2.2 (Tetrahedron Lemma). *Let* $d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}$ *be six numbers, such that the four triples* $A_1 = (d_{23}, d_{24}, d_{34})$*,* $A_2 = (d_{13}, d_{14}, d_{34})$*,* $A_3 = (d_{12}, d_{14}, d_{24})$ *and* $A_4 = (d_{12}, d_{13}, d_{23})$ *are* δ*-triples. Then*

$$B = (d_{12} + d_{34}, d_{13} + d_{24}, d_{14} + d_{23})$$

is a 2δ*-triple.*

Proof. Without loss of generality we can assume that d_{34} is maximal among the listed numbers. Then $d_{13} \doteq_\delta d_{14}$ since A_2 is a δ-triple, and $d_{23} \doteq_\delta d_{24}$ since A_1 is a δ-triple. Adding these approximate equalities we obtain that $d_{13} + d_{24} \doteq_{2\delta} d_{23} + d_{14}$. Since d_{34} is maximal, this means, if we assume that B is not a 2δ-triple, that $d_{12} < \min\{d_{13}, d_{14}, d_{23}, d_{24}\} - 2\delta$. But this contradicts the assumption that A_3 and A_4 are δ-triples. Thus B is a 2δ-triple. □

3. Hyperbolic spaces

3.1. δ-hyperbolic spaces. Let X be a metric space. For $x, y, z \in X$ let

$$(x|y)_z := \frac{1}{2}(|zx| + |zy| - |xy|).$$

The space X is called δ*-hyperbolic* if for $o, x, y, z \in X$

$$((x|y)_o, (y|z)_o, (x|z)_o) \text{ is a } \delta\text{-triple.} \tag{3.1}$$

X is called *hyperbolic*, if it is δ-hyperbolic for some $\delta \geq 0$. The relation (3.1) is called the δ*-inequality* with respect to the point $o \in X$. This condition is equivalent to the inequality

$$|ox| + |yz| \leq \max\{|oy| + |xz|, |oz| + |xy|\} + 2\delta. \tag{3.2}$$

The inequality (3.2) is called the *4-point inequality* for the points $o, x, y, z \in X$. If X satisfies the δ-inequality for one individual base point $o \in X$, then it satisfies the 2δ-inequality for any other base point $o' \in X$ (see, for example,

[G]). Thus, to check hyperbolicity, one only has to check this inequality at a single point.

Let X be a hyperbolic space and $o \in X$ be a base point. A sequence $\{x_i\}$ of points $x_i \in X$ *converges to infinity,* if

$$\lim_{i,j\to\infty}(x_i|x_j)_o = \infty.$$

Two sequences $\{x_i\}$, $\{x_i'\}$ that converge to infinity are *equivalent* if

$$\lim_{i\to\infty}(x_i|x_i')_o = \infty.$$

Using the δ-inequality, one easily sees that this defines an equivalence relation for sequences in X converging to infinity. The *boundary at infinity* $\partial_\infty X$ of X is defined as the set of equivalence classes of sequences converging to infinity.

For points ξ, $\xi' \in \partial_\infty X$ we define their Gromov product by

$$(\xi|\xi')_o = \inf \liminf_{i\to\infty}(x_i|x_i')_o,$$

where the infimum is taken over all sequences $\{x_i\} \in \xi$, $\{x_i'\} \in \xi'$. Note that $(\xi|\xi')_o$ takes values in $[0, \infty]$ and that $(\xi|\xi')_o = \infty$ if and only if $\xi = \xi'$. In a similar way we define for $\xi \in \partial_\infty X$, $x \in X$

$$(\xi|x)_o = \inf \liminf_{i\to\infty}(x_i|x)_o.$$

From Lemma 2.1(2) we obtain:

LEMMA 3.1. *Let X be δ-hyperbolic.*

(1) *If $\overline{x}, \overline{y}, \overline{z} \in \overline{X} := X \cup \partial_\infty X$, then $((\overline{x}|\overline{y})_o, (\overline{y}|\overline{z})_o, (\overline{x}|\overline{z})_o)$ is a δ-triple.*

(2) *If $\{x_i\} \in \xi$ and $\{y_i\} \in \eta$, then*

$$(x|\xi)_o \doteq_\delta \{(x|x_i)_o\}_i \quad \text{and} \quad (\xi|\eta)_o \doteq_{2\delta} \{(x_i|y_i)_o\}_i.$$

We define for points $\overline{x}, \overline{y} \in \overline{X}$

$$\sigma_{\xi,o}(\overline{x}, \overline{y}) := (\overline{x}|\xi)_o + (\overline{y}|\xi)_o.$$

The following result is obvious.

LEMMA 3.2.

(1) $\{x_i\} \in \xi$ *iff* $\sigma_{\xi,o}(x_i, x_j) \to \infty$.

(2) $\{x_i\} \in \eta \in \partial_\infty X \setminus \{\xi\}$ *iff* $(x_i|x_j)_o \to \infty$ *and* $\sigma_{\xi,o}(x_i, x_j)$ *is bounded.*

We define the Busemann function of $\xi \in \partial_\infty X$ by

$$b_\xi(x, y) = \inf \liminf_{i\to\infty}(|xz_i| - |yz_i|),$$

where the infimum is taken over all sequences $\{z_i\} \in \xi$. We state some properties of this function.

Lemma 3.3.

(1) *If* $\{z_i\} \in \xi$, *then* $b_\xi(x,y) \doteq_{2\delta} \{(|xz_i| - |yz_i|)\}_i$.
(2) *Let* $\{x_i\} \in \eta \in \partial_\infty X$, $o \in X$. *If* $\eta \neq \xi$, *then* $b_\xi(x_i, o) \to \infty$.

Proof. (1) Note that for sequences $\{z_i\}, \{z_i'\} \in \xi$

$$\{(|xz_i| - |yz_i|) - (|xz_i'| - |yz_i'|)\}_i = 2\{((y|z_i)_x - (y|z_i')_x)\}_i \doteq_{2\delta} 0,$$

since $(y|z_i)_x, (y|z_i')_x, (z_i|z_i')_x$ is a δ-triple and $(z_i|z_i')_x \to \infty$. This implies that

$$b_\xi(x,y) \doteq_{2\delta} \{(|xz_i| - |yz_i|)\}_i$$

for any sequence $\{z_i\} \in \xi$.

(2) Let $\{z_i\} \in \xi$ and $\{x_i\} \in \eta$. If $\eta \neq \xi$, then the numbers $2(x_i|z_j)_o$ are bounded by some number D, which implies $|z_j x_i| - |oz_j| \geq |ox_i| - D$. Since $b_\xi(x_i, o) \doteq_{2\delta} \{|x_i z_j| - |oz_j|\}_j$ and $|x_i o| \to \infty$, this yields the result. □

For $o \in X$, $\xi \in \partial_\infty X$ and $x, y \in X$ we define

$$(x|y)_{\xi,o} := \frac{1}{2}(b_\xi(x,o) + b_\xi(y,o) - |xy|).$$

We extend $(x|y)_{\xi,o}$ to points $\overline{x}, \overline{y} \in \overline{X} \setminus \{\xi\}$ by setting

$$(\overline{x}|\overline{y})_{\xi,o} := \inf \liminf_{i\to\infty} (x_i|y_i)_{\xi,o},$$

where the infimum is taken over all sequences $\{x_i\} \in \overline{x}$ and $\{y_i\} \in \overline{y}$. In the case that $\overline{x} \in X$, $\{x_i\} \in \overline{x}$ means any sequence $\{x_i\}$ converging to $\overline{x}$.

Lemma 3.4.

(1) *If* $x, y, z \in X$, *then* $((x|y)_{\xi,o}, (y|z)_{\xi,o}, (z|x)_{\xi,o})$ *is a* 3δ*-triple.*
(2) *If* $x, y \in X$, *then*

$$(x|y)_{\xi,o} + \sigma_{\xi,o}(x,y) \doteq_{4\delta} (x|y)_o.$$

(3) *If* $\overline{x}, \overline{y} \in \overline{X} \setminus \{\xi\}$, *then*

$$(\overline{x}|\overline{y})_{\xi,o} + \sigma_{\xi,o}(\overline{x}, \overline{y}) \doteq_{8\delta} (\overline{x}|\overline{y})_o.$$

Proof. We only prove (2) and leave (1) and (3) to the reader. Let $\{z_i\} \in \xi$ be given. Then

$$\begin{aligned}
(x|y)_{\xi,o} &= \frac{1}{2}(b_\xi(x,o) + b_\xi(y,o) - |xy|) \\
&\doteq_{2\delta} \frac{1}{2}\{(|xz_i| - |oz_i| + |yz_i| - |oz_i| - |xy|)\}_i \\
&= \{(x|y)_o - (x|z_i)_o - (y|z_i)_o\}_i \\
&\doteq_{2\delta} (x|y)_o - (x|\xi)_o - (y|\xi)_o.
\end{aligned}$$

□

3.2. A criterion for hyperbolicity. At the end of this section we give a criterion for hyperbolicity. Let therefore X be an arbitrary metric space. We define a map $A : X^4 \to \mathbb{R}$, where $A = A(x, y, z, t)$ is given by

$$A = \max\{(x|y)_u + (z|t)_u - (x|z)_u - (y|t)_u,\\ (x|y)_u + (z|t)_u - (x|t)_u - (y|z)_u\},$$

where $u \in X$ is arbitrary. An easy calculation shows that A is independent of u. By specializing $u = t$ we see that $A = (x|y)_t - \min\{(x|z)_t, (y|z)_t\}$. Thus it follows that X is δ-hyperbolic iff $A \geq -\delta$ for all $x, y, z, t \in X$.

Remark 3.5. One can write $A(x, y, z, t)$ in an even more complicated manner as the maximum of the two numbers

$$[(x|y)_u - |uv|] + [(z|t)_u - |uv|] - [(x|z)_u - |uv|] - [(y|t)_u - |uv|]$$

and

$$[(x|y)_u - |uv|] + [(z|t)_u - |uv|] - [(x|t)_u - |uv|] - [(y|z)_u\} - |uv|],$$

where $u, v \in X$ are arbitrary. This follows from a trivial computation and will be useful later on.

4. Products

Let X_1, X_2 be metric spaces. Let $Y = X_1 \times X_2$. On Y we will always consider the maximum metric, i.e., for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ let

$$|xy| = \max\{|x_1y_1|, |x_2y_2|\}.$$

For a point $o = (o_1, o_2) \in Y$ one easily checks that

$$(x|y)_o \geq \min\{(x_1|y_1)_{o_1}, (x_2|y_2)_{o_2}\}. \tag{4.1}$$

We define

$$Y_{\Delta,o} := \Big\{(x_1, x_2) \in Y \;\Big|\; |o_1x_1| \doteq_\Delta |o_2x_2|\Big\}.$$

It is easy to check that for points $x, y \in Y_{\Delta,o}$ we have

$$(x|y)_o \doteq_\Delta \min\{(x_1|y_1)_{o_1}, (x_2|y_2)_{o_2}\}. \tag{4.2}$$

For later reference we restate equations (4.1) and (4.2) in the following lemma.

Lemma 4.1. *If $x, y \in Y_{\Delta,o}$ then*

$$0 \leq (x|y)_o - \min\{(x_1|y_1)_{o_1}, (x_2|y_2)_{o_2}\} \leq \Delta.$$

Theorem 1.1. *If X_1, X_2 are δ-hyperbolic, then $Y_{\Delta,o}$ is $(\Delta+\delta)$-hyperbolic.*

Proof. Let $\delta \geq 0$ and $o_\nu \in X_\nu$ be such that X_ν satisfies the δ-inequality with respect to o_ν. Then Lemma 2.1(1) and Lemma 4.1 give (omitting base points)

$$\begin{aligned}(x|z) &\geq \min\{(x_1|z_1),(x_2|z_2)\} \geq \min\{(x_1|y_1),(y_1|z_1),(x_2,y_2),(y_2|z_2)\} - \delta \\ &\geq \min\{(x|y),(y|z)\} - \Delta - \delta. \qquad \square\end{aligned}$$

Consider $\xi_\nu \in \partial_\infty X_\nu$ and let $b_\nu(x) := b_{\xi_\nu}(x, o_\nu)$, $\nu = 1, 2$. We define

$$Y_{\Delta,\xi,o} := \Big\{(x_1,x_2) \in Y \;\Big|\; b_1(x_1) \doteq_\Delta b_2(x_2)\Big\}.$$

We will show that $Y_{\Delta,\xi,o}$ is hyperbolic. To prove this we need the following lemma.

Lemma 4.2. *Let X_ν be δ-hyperbolic spaces for $\nu = 1, 2$. For $i \in \mathbb{N}$ let $\{u_{1i}\} \in \xi_1$, $\{u_{2i}\} \in \xi_2$ and $u_i = (u_{1i}, u_{2i}) \in X_1 \times X_2$. Then, for $x, y \in Y_{\Delta,\xi,o}$, we have*

$$\{|u_i x| - |u_i o|\}_i \doteq_{\Delta+2\delta} b_\nu(x_\nu), \qquad \nu = 1, 2,$$

and

$$\{(x|y)_{u_i} - |u_i o|\}_i \doteq_{\Delta+2\delta} \min\{(x_1|y_1)_{\xi_1,o_1}, (x_2|y_2)_{\xi_2,o_2}\}.$$

Proof. We have by Lemma 3.3 $\{|u_{\nu i}x_\nu| - |u_{\nu i}o_\nu|\}_i \doteq_{2\delta} b_\nu(x_\nu)$ for $\nu = 1, 2$ and $b_1(x_1) \doteq_\Delta b_2(x_2)$.

Now the first inequality follows from the general fact that if $r_\nu - s_\nu \doteq_\delta b_\nu$ and $b_1 \doteq_\Delta b_2$ for some real numbers r_ν, s_ν, b_ν, then $\max\{r_1, r_2\} \doteq_{\delta+\Delta} \max\{s_1, s_2\} + b_\nu$. To prove this we may assume $s_1 \leq s_2$. Then $\max\{r_1, r_2\} \geq r_2 \geq s_2 + b_2 - \delta \geq \max\{s_1, s_2\} + b_\nu - \Delta - \delta$. Moreover, $r_1 \leq s_1 + b_1 + \delta$, $r_2 \leq s_2 + b_2 + \delta$, and hence $\max\{r_1, r_2\} \leq \max\{s_1, s_2\} + b_\nu + \Delta + \delta$.

To obtain the second inequality we compute

$$\begin{aligned}\{(x|y)_{u_i} - |u_i o|\}_i &= \frac{1}{2}\{|u_i x| - |u_i o| + |u_i y| - |u_i o| - |xy|\}_i \\ &\doteq_{\Delta+2\delta} \min_{\nu\in\{1,2\}} \frac{1}{2}\{b_\nu(x_\nu) + b_\nu(y_\nu) - |x_\nu y_\nu|\} \\ &= \min\{(x_1|y_1)_{\xi_1,o_1}, (x_2|y_2)_{\xi_2,o_2}\}. \qquad \square\end{aligned}$$

Theorem 1.2. *If X_1, X_2 are δ-hyperbolic, then $Y_{\Delta,\xi,o}$ is $(4\Delta + 14\delta)$-hyperbolic.*

Proof. Consider on $X_1 \times X_2$ the function A from Section 3.2. We have to show that $A_{|Y^4_{\Delta,\xi,o}} \geq -(4\Delta + 14\delta)$.

Choose $\{u_{1i}\}_i \in \xi_1$ and $\{u_{2i}\}_i \in \xi_2$ and let $u_i = (u_{1i}, u_{2i}) \in X_1 \times X_2$. (Note that u_i is not necessarily in $Y_{\Delta,\xi,o}$.) We use for A the complicated expression from Remark 3.5 with $u = u_i$ and $v = (o_1, o_2)$. The typical terms in this expression are then of the form $[(x|y)_{u_i} - |u_i o|]$.

By Lemma 4.2 we have that $x, y \in Y_{\Delta,\xi,o}$ implies

$$\{[(x|y)_{u_i} - |u_i o|]\}_i \doteq_{\Delta+2\delta} \min\{(x_1|y_1)_{\xi_1,o_1}, (x_2|y_2)_{\xi_2,o_2}\}.$$

Let now $y^1, y^2, y^3, y^4 \in Y_{\Delta,\xi,o}$, where $y^j = (y_1^j, y_2^j)$, $j = 1,2,3,4$, and consider the expression $A = A(y^1, y^2, y^3, y^4)$. Then

$$A = \max\{d_{12} + d_{34} - d_{13} - d_{24}, d_{12} + d_{34} - d_{14} - d_{23}\},$$

where, by Lemma 4.2, $d_{jk} \doteq_{\Delta+2\delta} \min\{d_{jk}^1, d_{jk}^2\}$ with $d_{jk}^\nu = (y_\nu^j|y_\nu^k)_{\xi_\nu,o_\nu}$. By Lemma 3.4(1), for every $\nu \in \{1,2\}$ the six numbers d_{jk}^ν satisfy the conditions of the Tetrahedron Lemma 2.2 with constant 3δ. Thus, by Lemma 2.1(1), the six numbers $\min\{d_{jk}^1, d_{jk}^2\}$ also satisfy the assumptions of the Tetrahedron Lemma with constant 3δ. Thus the six numbers d_{jk} satisfy the assumptions of the Tetrahedron Lemma with constant $3\delta + 2(\Delta + 2\delta) = 2\Delta + 7\delta$. The Tetrahedron Lemma then shows that $(d_{12} + d_{34}, d_{13} + d_{24}, d_{14} + d_{23})$ is a $4\Delta + 14\delta$ triple which implies $A \geq -(4\Delta + 14\delta)$. □

5. Roughly geodesic spaces

Let $k \geq 0$. A *k-rough geodesic* is a map $\gamma : I \to X$ from an interval $I \subset \mathbb{R}$ to a metric space X with

$$|\gamma(s)\gamma(t)| \doteq_k |s - t| \qquad \text{for all } s, t \in I.$$

The space X is called *k-roughly geodesic* if for every pair $x, y \in X$ there exists a k-rough geodesic $\gamma : [0, |xy|] \to X$ with $\gamma(0) = x$ and $\gamma(|xy|) = y$. X is called *roughly geodesic* if X is k-roughly geodesic for some $k \geq 0$. Parts of the results of this section are contained in [BoS]; compare also [V].

In this section we consider a fixed δ-hyperbolic and k-roughly geodesic space X with a base point $o \in X$.

To avoid notational complications, we will use in this section the following convention: We write $a \doteq b$ if $a \doteq_c b$ and the constant c depends only on δ and k. We will also say that $\gamma : I \to X$ from an interval $I \subset \mathbb{R}$ is a rough geodesic when $|\gamma(s)\gamma(t)| \doteq |s - t|$ (where we already used the first part of the convention).

LEMMA 5.1. *Let X be a δ-hyperbolic, k-roughly geodesic metric space, $\xi \in \partial_\infty X$ and $b : X \to \mathbb{R}$ be the Busemann function $b(x) = b_\xi(x, o)$. Then there exists a $k' = k'(\delta, k)$ such that*

1. *for every $x \in X$ there exists a k'-rough geodesic $\gamma_{\xi,x} : (-\infty, b(x)] \to X$ with $\{\gamma_{\xi,x}(-i)\}_i \in \xi$, $\gamma_{\xi,x}(b(x)) = x$ and $b(\gamma_{\xi,x}(t)) \doteq t$, and*
2. *for every $\eta \in \partial_\infty X \setminus \{\xi\}$ there exists a k'-rough geodesic $\gamma_{\xi,\eta} : \mathbb{R} \to X$ with $\{\gamma_{\xi,\eta}(-i)\}_i \in \xi$, $\{\gamma_{\xi,\eta}(i)\}_i \in \eta$ such that $b(\gamma_{\xi,\eta}(t)) \doteq t$.*

Proof. (1) By [BoS, Proposition 5.2(2)] we find a k'-rough geodesic $\alpha : [0,\infty) \to X$ from x to ξ. By Lemma 3.3(1) we have

$$b_\xi(\alpha(t),x) \doteq_{2\delta} \{|\alpha(t)\alpha(i)| - |x\alpha(i)|\}_i \doteq_{2k'} \{i-t-i\}_i = -t.$$

Setting $\gamma(t) = \alpha(b(x)-t)$ we obtain a k'-rough geodesic $\gamma : (-\infty, b(x)] \to X$, and then $b_\xi(\gamma(t),x) = b_\xi(\alpha(b(x)-t),x) \doteq t-b(x)$. Using again Lemma 3.3(1), we get $b_\xi(\gamma(t),x) + b_\xi(x,o) \doteq b_\xi(\gamma(t),o) = b(\gamma(t))$, and hence $b(\gamma(t)) \doteq t$.

(2) By [BoS, Proposition 5.2(3)] we find a k'-rough geodesic $\alpha : \mathbb{R} \to X$ from ξ to η, that is, $\{\alpha(-i)\}_i \in \xi$, $\{\alpha(i)\}_i \in \eta$. By Lemma 3.3(1) we get $b_\xi(\alpha(t),\alpha(0)) \doteq \{|\alpha(t)\alpha(-i)| - |\alpha(0)\alpha(-i)|\}_i \doteq \{t+i-i\}_i = t$, and hence $b(\alpha(t)) \doteq b_\xi(\alpha(t),\alpha(0)) + b_\xi(\alpha(0),o) \doteq t + b(\alpha(0))$. The desired rough geodesic γ is now given by $\gamma(t) = \alpha(t - b(\alpha(0)))$. □

Lemma 5.2.

(1) *Let $y, x_1, x_2 \in X$, let $\gamma_i : [0,r_i] \to X$, $i = 1,2$, be k-rough geodesics from y to x_i, $r_i = |yx_i|$. Then $|\gamma_1(t)\gamma_2(t)| \doteq 0$ for $t \le (x_1|x_2)_y$.*

(2) *Let $x_1, x_2 \in X$, $\xi \in \partial_\infty X$, let $\gamma_i = \gamma_{\xi,x_i} : (-\infty, b(x_i)] \to X$ be k'-rough geodesics given by Lemma 5.1(1). Then $|\gamma_1(t)\gamma_2(t)| \doteq 0$ for $t \le (x_1|x_2)_{\xi,o}$.*

Proof. (1) Let $0 \le t \le (x_1|x_2)_y$ and set $x_i' = \gamma_i(t)$. Then

$$2(x_i|x_i')_y = |x_iy| + |x_i'y| - |x_ix_i'| \ge r_i + (t-k) - (r_i - t + k) = 2t - 2k,$$

which implies

$$(x_1'|x_2')_y \ge \min\{(x_1'|x_1)_y, (x_1|x_2)_y, (x_2x_2')_y\} - 2\delta \ge t - k - 2\delta.$$

Since

$$2(x_1'|x_2')_y = |x_1'y| + |x_2'y| - |x_1'x_2'| \le 2t + 2k - |x_1'x_2'|,$$

we get $|x_1'x_2'| \le 4k + 4\delta$.

(2) Let $t \le (x_1|x_2)_{\xi,o}$ and set $x_i' = \gamma_i(t)$. By Lemma 5.1(1) we have $b(x_i') \doteq t$. Hence

$$2(x_i|x_i')_{\xi,o} = b(x_i) + b(x_i') - |x_ix_i'| \doteq b(x_i) + t - (b(x_i) - t) = 2t.$$

By Lemma 3.4(1) we obtain

$$(x_1|x_2')_{\xi,o} \ge \min\{(x_1'|x_1)_{\xi,o}, (x_1|x_2)_{\xi,o}, (x_2|x_2')_{\xi,o}\} - 6\delta \ge t - c(\delta,k).$$

Since

$$2(x_1'|x_2')_{\xi,o} = b(x_1') + b(x_2') - |x_1'x_2'| \doteq 2t - |x_1'x_2'|,$$

the lemma follows. □

Lemma 5.3. *Let X and ξ be as in Lemma 5.1 and $o, x, y \in X$. Then:*

$$(1) \qquad |\gamma_x(t)\gamma_y(s)| \doteq \begin{cases} s + t - 2(x|y)_o & \text{if } s,t \ge (x|y)_o, \\ |s-t| & \text{otherwise.} \end{cases}$$

$$
(2) \qquad |\gamma_{\xi,x}(t)\gamma_{\xi,y}(s)| \doteq \begin{cases} s+t-2(x|y)_{\xi,o} & \text{if } s,t \geq (x|y)_{\xi,o}, \\ |s-t| & \text{otherwise.} \end{cases}
$$

Proof. (1) For every $x \in X$ let $\gamma_x : [0, |ox|] \to X$ be a k-rough geodesic from o to x. Set $x' = \gamma_x((x|y)_o)$ and $y' = \gamma_y((x|y)_o)$.

We assume first $s, t \geq (x|y)_o$.

Since by Lemma 5.2(1) $|x'y'| \doteq 0$, we have

$$
\begin{aligned}
|\gamma_x(t)\gamma_y(s)| &\leq |\gamma_x(t)x'| + |x'y'| + |y'\gamma_y(s)| \\
&\doteq (t-(x|y)_o) + 0 + (s-(x|y)_o) \\
&= s+t-2(x|y)_o
\end{aligned}
$$

and

$$
\begin{aligned}
|\gamma_x(t)\gamma_y(s)| &\geq |xy| - |x\gamma_x(t)| - |y\gamma_y(s)| \\
&\doteq |xy| - (|ox|-t) - (|oy|-s) \\
&= s+t-2(x|y)_o.
\end{aligned}
$$

To consider the second case, let without loss of generality $t \leq (x|y)_o$, $t \leq s$. Then by Lemma 5.2(1)

$$
\begin{aligned}
|\gamma_x(t)\gamma_y(s)| &\leq |\gamma_x(t)\gamma_y(t)| + |\gamma_y(t)\gamma_y(s)| \\
&\doteq 0 + |t-s| \\
&= |t-s|
\end{aligned}
$$

and

$$
\begin{aligned}
|\gamma_x(t)\gamma_y(s)| &\geq |o\gamma_y(s)| - |o\gamma_x(t)| \\
&\doteq |t-s|.
\end{aligned}
$$

(2) We may assume that $t \leq s$. Set $t_0 = (x|y)_{\xi,o}$.

Case 1: $t \geq t_0$. Set $x' = \gamma_{\xi,x}(t_0), y' = \gamma_{\xi,y}(t_0)$. By Lemma 5.2(2) we have $|x'y'| \doteq 0$. Hence

$$
\begin{aligned}
|\gamma_{\xi,x}(t)\gamma_{\xi,y}(s)| &\leq |\gamma_{\xi,x}(t)x'| + |x'y'| + |y'\gamma_{\xi,y}(s)| \\
&\doteq t-t_0+0+s-t_0 = s+t-2t_0.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
|\gamma_{\xi,x}(t)\gamma_{\xi,y}(s)| &\geq |xy| - |x\gamma_{\xi,x}(t)| - |y\gamma_{\xi,y}(s)| \\
&\geq |xy| - (b(x)-t+k) - (b(y)-s+k) \\
&= s+t-2t_0-2k.
\end{aligned}
$$

Case 2: $t \leq t_0$. As $|\gamma_{\xi,x}(t)\gamma_{\xi,y}(t)| \doteq 0$ by Lemma 5.2(2), we obtain

$$
|\gamma_{\xi,x}(t)\gamma_{\xi,y}(s)| \doteq |\gamma_{\xi,y}(t)\gamma_{\xi,y}(s)| \doteq |t-s|. \qquad \square
$$

Let $x, y \in X$ and assume that a is a number with $a \geq |xy|$. Then we define $\gamma^a_{x,y} : [0, a] \to X$ by

$$\gamma^a_{x,y}(t) = \begin{cases} \gamma_x(|ox| - t) & \text{for } 0 \leq t \leq \min\{|ox|, \frac{1}{2}(|ox| - |oy| + a)\}, \\ o & \text{for } |ox| \leq t \leq a - |oy|, \\ \gamma_y(|oy| - a + t) & \text{for } \max\{a - |oy|, \frac{1}{2}(|ox| - |oy| + a)\} \leq t \leq a. \end{cases}$$

Thus, if $a = |ox| + |oy|$, then $\gamma^a_{x,y}$ is just the concatenation of γ_x^{-1} and γ_y. If $a > |ox| + |oy|$, then $\gamma^a_{x,y}$ is the concatenation of γ_x^{-1}, a constant curve at o and γ_y. If $|xy| \leq a < |ox| + |oy|$, then $\gamma^a_{x,y}$ is the concatenation of the inverse of $\gamma_x|_{[\tau,|ox|]}$ and $\gamma_y|_{[\tau,|oy|]}$, where $\tau = \frac{1}{2}(|ox| - |oy| + a)$. Note that for $a \leq |ox| + |oy|$ the curve $\gamma^a_{x,y}$ has two definitions at the parameter value τ. However, we have

$$\sigma := |ox| - \tau = \frac{1}{2}(|ox| + |oy| - a) = |oy| - a + \tau,$$

and in the case $|xy| \leq a \leq |xo| + |yo|$ also $0 \leq \sigma \leq (x|y)_o$. Hence, by Lemma 5.1(1), we have $|\gamma_x(\sigma)\gamma_y(\sigma)| \doteq 0$, which says that $\gamma^a_{x,y}$ is well defined up to a uniformly bounded error. This is enough for our considerations.

Lemma 5.4. *Let $x, y \in X$ and $|xy| \leq a$. Then:*

(1) *There exists a constant c depending only on δ and k such that*

$$|\gamma^a_{x,y}(t)\gamma^a_{x,y}(s)| \leq |s - t| + c \text{ for all } 0 \leq s, t \leq a.$$

(2) *If $a \doteq |xy|$, then $|\gamma^a_{x,y}(t)\gamma^a_{x,y}(s)| \doteq |s - t|$ for all $0 \leq s, t \leq a$.*

(3) *$|\gamma^a_{x,y}(t)o| \doteq \max\{|ox| - t, 0, |oy| - a + t\}$ for all $0 \leq t \leq a$.*

Proof. (1) follows from the fact that $\gamma^a_{x,y}$ is, up to a uniformly bounded error, the concatenation of rough geodesics and constant curves. (2) follows from (1) and $|\gamma^a_{x,y}(a)\gamma^a_{x,y}(0)| \doteq a$. (3) follows from the definition of $\gamma^a_{x,y}$, $|\gamma_x(t)o| \doteq t$ and $|\gamma_y(t)o| \doteq t$. □

The above results have straightforward generalizations to the case where we fix a "base point" at infinity. We only replace the distance to o by the Busemann function b.

For $x, y \in X$ and $a \geq |xy|$ we define $\gamma^a_{\xi,x,y} : [0, a] \to X$ by

$$\gamma^a_{\xi,x,y}(t) = \begin{cases} \gamma_{\xi,x}(b(x) - t) & \text{for } 0 \leq t \leq \frac{1}{2}(b(x) - b(y) + a), \\ \gamma_{\xi,y}(b(y) - a + t) & \text{for } \frac{1}{2}(b(x) - b(y) + a) \leq t \leq a. \end{cases}$$

Lemma 5.5. *Let X and ξ be as in Lemma 5.1, $x, y \in X$ and $a \geq |xy|$. Then:*

(1) *There exists a constant c depending only on δ and k such that*

$$|\gamma^a_{\xi,x,y}(t)\gamma^a_{\xi,x,y}(s)| \leq |s - t| + c \text{ for all } 0 \leq s, t \leq a.$$

(2) *If* $a \doteq |xy|$, *then* $|\gamma^a_{\xi,x,y}(t)\gamma^a_{\xi,x,y}(s)| \doteq |s-t|$ *for all* $0 \le s,t \le a$.
(3) $b(\gamma^a_{\xi,x,y}(t)) \doteq \max\{b(x)-t, b(y)-a+t\}$ *for all* $0 \le t \le a$.

6. Hyperbolic products of roughly geodesic spaces

In this section we show that hyperbolic products of roughly geodesic spaces are roughly geodesic. We assume that X_1, X_2 are metric spaces which are δ-hyperbolic and k-roughly geodesic. Let $o_\nu \in X_\nu$, $\nu = 1,2$, be base points.

LEMMA 6.1. *If* $x \in Y_{\Delta,o}$, *then there exists* $x' \in Y_{k,o}$ *with* $|xx'| \le \Delta + k$.

Proof. We assume without loss of generality that $a_1 := |x_1o_1| \ge |x_2o_2| =: a_2$. By assumption $a_1 - a_2 \le \Delta$. Let $\gamma_1 : [0,a_1] \to X_1$ be a k-rough geodesic with $\gamma_1(0) = o$ and $\gamma_1(a_1) = x_1$, and define $x_1' := \gamma_1(a_2)$. By construction $x' = (x_1', x_2)$ satisfies the required properties. □

LEMMA 6.2. *There exists* $k' = k'(\delta,k) \ge 0$ *with the following property: If* $x, y \in Y_{k,o}$, *then there exists a* k'*-rough geodesic* $\gamma : I \to X_1 \times X_2$ *from* x *to* y *such that* $\gamma(t) \in Y_{k',o}$ *for all* $t \in I$.

Proof. Let $a := \max\{|x_1y_1|, |x_2y_2|\}$ and consider

$$\begin{array}{rclcrcl} \gamma_1 & := & \gamma^a_{x_1,y_1}: & [0,a] & \longrightarrow & X_1, \\ \gamma_2 & := & \gamma^a_{x_2,y_2}: & [0,a] & \longrightarrow & X_2, \\ \gamma & := & (\gamma_1,\gamma_2): & [0,a] & \longrightarrow & X_1 \times X_2. \end{array}$$

It follows from Lemma 5.4(1),(2) that γ is a rough geodesic with a constant that depends only on δ and k. From Lemma 5.4(3) we obtain that $|\gamma_1(t)o_1| \doteq_{k'} |\gamma_2(t)o_2|$ for a constant k' depending only on δ and k. □

THEOREM 1.3. *If* X_1, X_2 *are* δ*-hyperbolic and* k*-roughly geodesic, then there exists* $\Delta_0 \ge 0$ *such that for all* $\Delta \ge \Delta_0$ *the space* $Y_{\Delta,o}$ *is roughly geodesic.*

Proof. Let k' be the constant from Lemma 6.2. We claim that $\Delta_0 := \max\{k,k'\}$ satisfies the required properties. Let $\Delta \ge \Delta_0$ and let $x, y \in Y_{\Delta,o}$. Let $x', y' \in Y_{k,o}$ be points according to Lemma 6.1 with $|xx'| \le \Delta + k$ and $|yy'| \le \Delta + k$. Let $a' = |x'y'|$ and $a = |xy|$. Then $a \doteq_{2\Delta+2k} a'$. Let $\overline{a} = \min\{a,a'\}$. By Lemma 6.2 there exists a k'-rough geodesic $\gamma' : [0,a'] \to Y_{k',o}$ from x' to y'. Let $\gamma : [0,a] \to Y_{\Delta,o}$ be defined by

$$\gamma(t) = \begin{cases} x & \text{for } \ t = 0, \\ \gamma'(t) & \text{for } 0 < t < \overline{a}, \\ y & \text{for } \overline{a} \le t \le a. \end{cases}$$

Since $|xx'| \le \Delta + k$, $|yy'| \le \Delta + k$ and $a \doteq a'$, the curve γ is a $\overline{k}$-rough geodesic, where $\overline{k}$ depends only on δ, k and Δ. □

In essentially the same way one shows:

Theorem 1.4. *Let X_1, X_2 be δ-hyperbolic and k-roughly geodesic. Let $\xi_\nu \in \partial_\infty X_\nu$. Then there exists $\Delta_0 \geq 0$ such that $Y_{\Delta,\xi,o}$ is roughly geodesic for all $\Delta \geq \Delta_0$.*

7. The boundary of hyperbolic products

In this section we study the boundary of hyperbolic products. We start from spaces X_ν, $\nu = 1, 2$, which are hyperbolic and roughly geodesic.

7.1. The boundary of $Y_{\Delta,o}$. We consider the product $Y_{\Delta,o}$.

Theorem 1.5. *Let X_ν, $\nu = 1, 2$, be δ-hyperbolic and k-roughly geodesic metric spaces. Then there exists $\Delta_0 = \Delta_0(\delta, k) \geq 0$ such that for all $\Delta \geq \Delta_0$ the space $\partial_\infty Y_{\Delta,o}$ is naturally homeomorphic to $\partial_\infty X_1 \times \partial_\infty X_2$.*

Proof. Let $\Delta_0 = 2k'(\delta, k)$, where k' is the constant from Lemma 5.1(1). Then for $\Delta \geq \Delta_0$ the space $Y_{\Delta,o}$ is hyperbolic by Theorem 1.1.

We first show that by setting

$$\begin{aligned} \psi : \partial_\infty Y_{\Delta,o} &\to \partial_\infty X_1 \times \partial_\infty X_2 \\ [\{z_i\}] &\mapsto ([\{z_{1i}\}], [\{z_{2i}\}]) \end{aligned}$$

we obtain a well defined map. Let $\{z_i\}$ be a sequence converging to infinity. Then $(z_i|z_j)_o \to \infty$. Since by Lemma 4.1 $(z_i|z_j)_o \doteq \min\{(z_{1i}|z_{1j})_{o_1}, (z_{2i}|z_{2j})_{o_2}\}$, where $\doteq$ means $\doteq_{c(\delta,k,\Delta)}$, we see that $\{z_{\nu i}\}$ is also converging to infinity for $\nu = 1, 2$. If $\{z'_i\}$ is equivalent to $\{z_i\}$, then $(z_i|z'_i)_o \to \infty$, which implies $(z_{\nu i}|z_{\nu i'})_o \to \infty$ for $\nu = 1, 2$. Thus ψ is well defined.

It follows easily from Lemma 4.1 that for $\eta, \eta' \in \partial_\infty Y_{\Delta,o}$ with $\psi(\eta) = (\eta_1, \eta_2)$ and $\psi(\eta') = (\eta'_1, \eta'_2)$ we have

$$(\eta|\eta')_o \doteq \min\{(\eta_1|\eta'_1)_{o_1}, (\eta_2|\eta'_2)_{o_2}\}.$$

This implies the continuity and injectivity of ψ, and it will also show the continuity of ψ^{-1} once we have proved the bijectivity of the map. That the map ψ is also surjective can be seen as follows: Let $\eta_\nu \in \partial_\infty X_\nu$ and let $\gamma_\nu : [0, \infty) \to X_\nu$ be rough geodesics with $\gamma_\nu(0) = o_\nu$ and $\gamma_\nu(i) \to \eta_\nu$. Then $\gamma(t) = (\gamma_1(t), \gamma_2(t)) \in Y_{\Delta,o}$ for Δ large enough, since $|o_\nu \gamma_\nu(t)| \doteq t$. It follows that $\psi(\gamma(\infty)) = (\eta_1, \eta_2)$. □

7.2. Coarse smashed product. Let (Z_ν, ξ_ν) be pointed topological spaces, $\nu = 1, 2$. We call the subset $(Z_1 \times \{\xi_2\}) \cup (\{\xi_1\} \times Z_2)$ the *cross at* $(\xi_1, \xi_2) \in Z_1 \times Z_2$. The smashed product $(Z_1, \xi_1) \wedge (Z_2, \xi_2)$ is the space $Z_1 \times Z_2$, where we identify (smash) the cross at (ξ_1, ξ_2) to one point. Formally we define an equivalence relation $\sim$ on $Z_1 \times Z_2$ by letting

$$(\eta_1, \eta_2) \ \sim \ (\eta'_1, \eta'_2)$$

if and only if

$$\Big\{\eta_1 = \eta_1' \ \wedge \ \eta_2 = \eta_2'\Big\} \ \vee \ \Big[\Big\{\eta_1 = \xi_1 \vee \eta_2 = \xi_2\Big\} \ \wedge \ \Big\{\eta_1' = \xi_1 \vee \eta_2' = \xi_2\Big\}\Big].$$

The coarse smashed product topology is defined as follows: A basis of the open sets are the sets $U_1 \times U_2$, where $U_\nu \subset X_\nu \setminus \{\xi_\nu\}$, $\nu = 1, 2$, are open, and the sets $(W_1 \times Z_2) \cup (Z_1 \times W_2)$, where $W_\nu \subset Z_\nu$ are open neighborhoods of ξ_ν.

Thus a sequence $[(\eta_{1i}, \eta_{2i})] \in (Z_1, \xi_1) \wedge (Z_2, \xi_2)$ converges to $[(\xi_1, \xi_2)]$, iff for all open neighborhoods $W_\nu \subset Z_\nu$ of ξ_ν there exists $i_0 = i_0(W_1, W_2) \in \mathbb{N}$ such that for all $i \geq i_0$ one has $\eta_{1i} \in W_1$ or $\eta_{2i} \in W_2$.

If the spaces Z_ν are second countable for $\nu = 1, 2$, then so is $(Z_1, \xi_1) \wedge (Z_2, \xi_2)$.

Remark 7.1. Note that in the literature the smashed product of two pointed topological spaces (Z_1, ξ_1) and (Z_2, ξ_2) is defined as the set $Z_1 \times Z_2/\sim$ endowed with the quotient topology. In general, the coarse smashed product is coarser than the smashed product. However, in the case when Z_1 and Z_2 are compact, the two topologies are equivalent. Since in [FS2] we considered proper geodesic spaces and the boundaries at infinity of such spaces are compact, the smashed product topology we considered in Theorem 2 of [FS2] agrees with the coarse smashed product topology as introduced above.

7.3. Boundary of $Y_{\Delta,\xi,o}$. We assume that the spaces X_ν are hyperbolic, roughly geodesic spaces and that $\xi_\nu \in \partial_\infty Y_{\Delta,\xi,o}$. By Theorem 1.2 $Y_{\Delta,\xi,o}$ is δ'-hyperbolic for some $\delta'(\delta, \Delta)$. Let $k'(\delta, k)$ and $c'(\delta, k)$ be the numbers given by Lemma 5.1 such that $b(\gamma_{\xi,x}(t)) \doteq_{c'} t$ and $b(\gamma_{\xi,\eta}(t)) \doteq_{c'} t$. Let $\Delta_0(\delta, k) = 2c'$ and let $\Delta \geq \Delta_0$.

We use in this section the convention that $\doteq$ means $\doteq_c$, where c depends only on δ and k and Δ. Let $\gamma_\nu : [0, \infty) \to X_\nu$ be rough geodesics from o_ν with $\{\gamma_\nu(i)\}_i \in \xi_\nu$. Then $b_\nu(\gamma_\nu(t)) \doteq -t$ and hence $\gamma(t) = (\gamma_1(t), \gamma_2(t)) \in Y_{\Delta,\xi,o}$ for Δ large enough. Clearly, $\gamma(i)$ converges to infinity and we define $\xi := [\{\gamma(i)\}]$.

Lemma 7.2. *If $x, y \in Y_{\Delta,\xi,o}$, then:*

(1) $(x|y)_{\xi,o} \doteq \min\{(x_1|y_1)_{\xi_1,o_1}, (x_2|y_2)_{\xi_2,o_2}\}$.

(2) $(x|\xi)_o \doteq \max\{(x_1|\xi_1)_{o_1}, (x_2|\xi_2)_{o_2}\}$.

Proof. (1) Let $\{u_i\} \in \xi$. Then we have

$$\begin{aligned}(x|y)_{\xi,o} &\doteq \frac{1}{2}\{|xu_i| - |ou_i| + |yu_i| - |ou_i| - |xy|\}_i \\ &= \{(x|y)_{u_i} - |u_i o|\}_i \\ &\doteq \min\{(x_1|y_1)_{\xi_1,o_1}, (x_2|y_2)_{\xi_2,o_2}\},\end{aligned}$$

where the last step follows from Lemma 4.2.

(2) Set $u_i = \gamma(i)$, $u_{\nu i} = \gamma_\nu(i)$. We first show that

$$\{|ou_i|\}_i \doteq \{|o_\nu u_{\nu i}|\}_i, \tag{7.1}$$

$$\{|xu_i|\}_i \doteq \{|x_\nu u_{\nu i}|\}_i. \tag{7.2}$$

As γ_ν is a k'-geodesic, we have $|o_\nu u_{\nu i}| \doteq i$, which implies (7.1). By Lemma 3.3(1) we have

$$\{|o_1 u_{1i}| - |x_1 u_{1i}|\}_i \doteq_{2\delta} -b_1(x_1) \doteq_\Delta -b_2(x_2) \doteq_{2\delta} \{|o_2 u_{2i}| - |x_2 u_{2i}|\}_i.$$

This and (7.1) imply (7.2).

By Lemma 3.1(2) we have

$$2(x|\xi) \doteq_{2\delta'} \{2(x|u_i)_o\}_i = \{\max\{|o_1 x_1|, |o_2 x_2|\} + |ou_i| - |xu_i|\}_i.$$

Now (7.1) and (7.2) imply the assertion. □

THEOREM 1.6. *The boundary $\partial_\infty Y_{\Delta,\xi,o}$ is naturally homeomorphic to the coarse smashed product $(\partial_\infty X_1, \xi_1) \wedge (\partial_\infty X_2, \xi_2)$.*

Proof. The proof uses the following formulae from Lemmata 7.2 and 3.4:

$$(x|y)_o \doteq (x|y)_{\xi,o} + (x|\xi)_o + (y|\xi)_o \qquad \text{for all } x, y \in Y_{\Delta,\xi,o}, \tag{7.3}$$

$$(x|y)_{\xi,o} \doteq \min\{(x_1|y_1)_{\xi_1,o_1}, (x_2|y_2)_{\xi_2,o_2}\} \quad \text{for all } x, y \in Y_{\Delta,\xi,o}, \tag{7.4}$$

$$(x|\xi)_o \doteq \max\{(x_1|\xi_1)_{o_1}, (x_2|\xi_2)_{o_2}\} \qquad \text{for all } x \in Y_{\Delta,\xi,o}. \tag{7.5}$$

We have

x_i converges to a point in $\partial_\infty Y_{\Delta,\xi,o} \setminus \{\xi\}$

$\iff (x_i|x_j)_o \to \infty$ and $(x_i|\xi)_o$ bounded

$\iff_{(7.3)} (x_i|x_j)_{\xi,o} \to \infty$ and $(x_i|\xi)_o$ bounded

$\iff_{(7.4),(7.5)} (x_{\nu i}|x_{\nu j})_{\xi_\nu,o_\nu} \to \infty$ and $(x_{\nu i}|\xi_\nu)_{o_\nu}$ bounded for $\nu = 1, 2$

$\iff x_{\nu i}$ converges to a point in $\partial_\infty X_\nu \setminus \{\xi_\nu\}$ for $\nu = 1, 2$.

This calculation shows that the map

$$\psi : \partial_\infty Y_{\Delta,\xi,o} \setminus \{\xi\} \to (\partial_\infty X_1 \setminus \{\xi_1\}) \times (\partial_\infty X_2 \setminus \{\xi_2\})$$

given by $\psi(\eta) = ([\{x_{1i}\}], [\{x_{2i}\}])$, where $\{x_i\}$ is a sequence in $Y_{\Delta,\xi,o}$ with $[\{x_i\}] = \eta$, is well defined.

The formulae (7.3)–(7.5) have extensions to the ideal boundary: If $\eta, \eta' \in \partial_\infty Y_{\Delta,\xi,o} \setminus \{\xi\}$, $\psi(\eta) = (\eta_1, \eta_2)$ and $\psi(\eta') = (\eta_1', \eta_2')$, then

$$(\eta|\eta')_o \doteq (\eta|\eta')_{\xi,o} + (\eta|\xi)_o + (\eta'|\xi)_o, \tag{7.6}$$

$$(\eta|\eta')_{\xi,o} \doteq \min\{(\eta_1|\eta_1')_{\xi_1,o_1}, (\eta_2|\eta_2')_{\xi_2,o_2}\}, \tag{7.7}$$

$$(\eta|\xi)_o \doteq \max\{(\eta_1|\xi_1)_{o_1}, (\eta_2|\xi_2)_{o_2}\}. \tag{7.8}$$

Let $\eta_i, \eta \in \partial_\infty Y_{\Delta,\xi,o} \setminus \{\xi\}$, $i \in \mathbb{N}$. Then

$$\begin{aligned}
\eta_i \to \eta &\iff (\eta_i|\eta)_o \to \infty \text{ and } (\eta_i|\xi)_o \text{ bounded} \\
&\iff_{(7.6)} (\eta_i|\eta)_{\xi,o} \to \infty \text{ and } (\eta_i|\xi)_o \text{ bounded} \\
&\iff_{(7.7),(7.8)} (\eta_{\nu i}|\eta_\nu)_{\xi_\nu,o_\nu} \to \infty \text{ and } (\eta_{\nu i}|\xi_\nu)_{o_\nu} \text{ bounded for } \nu = 1,2 \\
&\iff_{(7.6\nu)} (\eta_{\nu i}|\eta_\nu)_{o_\nu} \to \infty \text{ and } (\eta_{\nu i}|\xi_\nu)_{o_\nu} \text{ bounded for } \nu = 1,2 \\
&\iff \eta_{\nu i} \to \eta_\nu \text{ for } \nu = 1,2,
\end{aligned}$$

where (7.6ν) is the formula (7.6) applied to the factors. This computation shows in particular the continuity of ψ. It will also show the continuity of ψ^{-1} after we have proved the bijectivity. If $\eta, \eta' \in \partial_\infty Y_{\Delta,\xi,o} \setminus \{\xi\}$, then $\psi(\eta) = \psi(\eta')$ implies by (7.7) that $(\eta|\eta')_{\xi,o} = \infty$, and hence $\eta = \eta'$. Thus ψ is injective.

We next show that the map is also surjective. Let $\eta_\nu \in \partial_\infty X_\nu \setminus \xi_\nu$ be given. Due to Lemma 5.1 there are rough geodesics $\gamma_{\xi_\nu,\eta_\nu} : \mathbb{R} \to X_\nu$ with $\{\gamma_{\xi_\nu,\eta_\nu}(-i)\}_i \in \xi_\nu$, $\{\gamma_{\xi_\nu,\eta_\nu}(i)\}_i \in \eta_\nu$ and $b(\gamma_{\xi_\nu,\eta_\nu}(t)) \doteq_{c'} t$, $\nu = 1,2$. By our choice of Δ_0 we obtain $(\gamma_{\xi_1,\eta_1}(t), \gamma_{\xi_2,\eta_2}(t)) \in Y_{\Delta,\xi,o}$, from which the surjectivity of ψ immediately follows.

Finally we show that ψ can be extended continuously to a homeomorphism

$$\overline{\psi} : \partial_\infty Y_{\Delta,\xi,o} \to (\partial_\infty X_1, \xi_1) \wedge (\partial_\infty X_2, \xi_2)$$

by defining $\overline{\psi}(\xi) = [(\xi_1, \xi_2)]$. Observe

$$\begin{aligned}
\eta_i \to \xi &\iff (\eta_i|\xi)_o \to \infty \\
&\iff_{(7.8)} \max\{(\eta_{1i}|\xi_1)_{o_1}, (\eta_{2i}|\xi_2)_{o_2}\} \to \infty \\
&\iff \forall D \geq 0 \;\; \exists i_0 \in \mathbb{N} \;\text{ such that }\; \forall i \geq i_0 \\
&\qquad (\eta_{1i}|\xi_1)_{o_1} \geq D \;\text{ or }\; (\eta_{2i}|\xi_2)_{o_2} \geq D \\
&\iff [(\eta_{1i}, \eta_{2i})] \to [(\xi_1, \xi_2)],
\end{aligned}$$

where in the last line the convergence is in $(\partial_\infty X_1, \xi_1) \wedge (\partial_\infty X_2, \xi_2)$. ☐

8. Maximum metric versus Euclidean metric

We finally point out that when starting off with two proper geodesic metric spaces one has to consider the length metric d induced by the maximum metric d_m on $Y_{0,o}$ or $Y_{0,\xi,o}$, in order to obtain a proper geodesic space again. In this case, we might as well endow Y_0 with the length metric induced by the Euclidean product metric d_e instead of the maximum metric d_m. Since both are geodesic spaces which are bilipschitz related, one of them is Gromov hyperbolic if and only if the other one is (see, e.g., Theorem 1.9 in Chapter III.1 of [BrH]).

In fact, when starting off with two Riemannian manifolds and fixing points at infinity, the construction using the Euclidean product metric has the advantage that it once again yields a Riemannian manifold (compare [FS1]).

However, we emphasize that in neither of the Theorems 1.1–1.6 we can replace the maximum metric by the Euclidean metric, as the following example shows.

Example 8.1. Consider two copies of the real hyperbolic space $\mathbb{H}^2$. Fix points $o_1 = o_2 \in \mathbb{H}^2$ and $\xi_1 = \xi_2 \in \partial_\infty \mathbb{H}^2$. Now consider sequences of points $\{x^i = (x_1^i, x_2^i)\}$, $\{y^i = (y_1^i, y_2^i)\}$, $\{z^i = (z_1^i, z_2^i)\}$ and $\{w^i = (w_1^i, w_2^i)\}$ such that $x_1^i = x_2^i$, $y_1^i = y_2^i$, $z_1^i = z_2^i$, $b_\nu(z_\nu^i) = b_\nu(y_\nu^i)$, $|x_\nu^i y_\nu^i| = |x_\nu^i z_\nu^i| = \frac{1}{2}|y_\nu^i z_\nu^i|$, $w_1^i = y_1^i$ and $w_2^i = z_2^i$ for all $i \in \mathbb{N}$, $\nu = 1, 2$, as well as $|y_\nu^i z_\nu^i| \overset{i\to\infty}{\longrightarrow} \infty$, $\nu = 1, 2$.

We claim that $(Y_{0,\xi,o}, d_e)$ is not hyperbolic. Suppose to the contrary that $(Y_{0,\xi,o}, d_e)$ is hyperbolic. Then there exists a $\delta \geq 0$ such that for all $i \in \mathbb{N}$

$$\begin{aligned} & d_e(y^i, z^i) \;+\; d_e(x^i, w^i) \\ & \qquad\qquad \leq \max\{d_e(x^i, y^i) + d_e(z^i, w^i), d_e(y^i, w^i) + d_e(x^i, z^i)\} \;+\; 2\delta \\ \Longleftrightarrow\; & d_e(y^i, z^i) \;\leq\; \max\{d_e(z^i, w^i), d_e(y^i, w^i)\} \;+\; 2\delta \\ \Longleftrightarrow\; & \sqrt{2}\,|y_1^i z_1^i| \;\leq\; |y_1^i z_1^i| \;+\; 2\delta, \end{aligned}$$

which contradicts our choices of sequences.

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