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REDUCIBILITY OF DUPIN SUBMANIFOLDS

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ABSTRACT. We introduce the notion of weak reduciblity for Dupin submanifolds with arbitrary codimension. We give a complete characterization of all weakly reducible Dupin submanifolds, as a consequence of a general result on a broader class of Euclidean submanifolds. As a main application, we derive an explicit recursive procedure to generate all holonomic Dupin submanifolds in terms of solutions of completely integrable systems of linear partial differential equations of first order. We obtain several additional results on Dupin submanifolds.

1. Introduction

A hypersurface $f\colon M^n \to \mathbb{Q}_c^{n+1}$ of a simply connected space form of sectional curvature c is called *proper Dupin* if the number of principal curvatures is constant and each one of them is constant along the corresponding eigenbundle. These conditions are invariant under conformal transformations of the ambient space, which makes the theory of Dupin hypersurfaces essentially the same whether it is considered in Euclidean space $\mathbb{R}^{n+1}$, in the sphere $\mathbb{S}^{n+1}$ or in hyperbolic space $\mathbb{H}^{n+1}$. More generally, the class of proper Dupin hypersurfaces in Euclidean space $\mathbb{R}^{n+1}$ is invariant under the Lie sphere group generated by the subgroup of conformal (Moebius) transformations of $\mathbb{R}^{n+1}$ together with the 1-parameter subgroup of parallel translations that transform a hypersurface to its parallel ones at a fixed distance in the normal direction. Two hypersurfaces that differ by a Lie transformation are said to be Lie equivalent.

An important class of proper Dupin hypersurfaces in $\mathbb{R}^{n+1}$ is that of stereographic projections of isoparametric hypersurfaces in $\mathbb{S}^{n+1}$. The latter are abundant and have not yet been completely classified, although several interesting results are known including strong restrictions on the numbers of distinct principal curvatures and their multiplicities; see [Th1] for a nice recent survey that also discusses Dupin hypersurfaces.

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It was observed by Pinkall [Pi] that further local examples of proper Dupin hypersurfaces in $\mathbb{R}^{n+1}$ having any given number of principal curvatures with arbitrarily prescribed multiplicities can be constructed by means of one of the following procedures, the last two of which yielding submanifolds that are Lie equivalent. Start with a proper Dupin hypersurface L^{n-s} in $\mathbb{R}^{n-s+1}$, the latter regarded as a linear subspace $\mathbb{R}^{n-s+1} \times \{0\}$ of $\mathbb{R}^{n+1}$, and let M^n be defined as

(i) M^n is the cylinder $L^{n-s} \times \mathbb{R}^s$;
(ii) M^n is obtained by rotating L^{n-s} around an axis $\mathbb{R}^{n-s} \subset \mathbb{R}^{n-s+1}$;
(iii) M^n is the cylinder $CV^{n-s} \times \mathbb{R}^{s-1}$, where CV^{n-s} is the cone over the inverse image $V^{n-s} \subset \mathbb{S}^{n-s+1} \subset \mathbb{R}^{n-s+2}$ of L^{n-s} by the stereographic projection;
(iii′) M^n is a tube around L^{n-s}.

These constructions introduce a new principal curvature λ of multiplicity s that is easily seen to be constant along its eigenbundle E_λ. The other principal curvatures of M^n are determined from those of L^{n-s}, and they are constant along the corresponding eigenbundles because L^{n-s} is Dupin. Moreover, the *conullity* distribution $E_\lambda^\perp$ of λ, that is, the orthogonal distribution to E_λ in the tangent bundle TM, is always integrable. In fact, in the first three constructions $E_\lambda^\perp$ is *spherical* in M^n, that is, the leaves of $E_\lambda^\perp$ are umbilical submanifolds of M^n with parallel mean curvature vector.

It was pointed out in recent work due to Cecil and Jensen that there are two natural settings for attempting to obtain classification results for proper Dupin hypersurfaces. One can either assume compactness and look for global results or work locally and search for hypersurfaces that are locally irreducible. A Dupin hypersurface is *reducible* if it is Lie equivalent to a hypersurface obtained by one of Pinkall's constructions. Cecil and Jensen [CJ] showed that a locally irreducible proper Dupin hypersurface with three distinct principal curvatures must be Lie equivalent to an isoparametric hypersurface. On the other hand, Pinkall and Thorbergsson [PT], and independently Miyaoka and Ozawa [MO], produced compact proper Dupin hypersurfaces with 4 distinct principal curvatures that are neither locally Lie equivalent to isoparametric hypersurfaces nor locally reducible. Thus, classifying locally or globally proper Dupin hypersurfaces with at least 4 distinct principal curvatures remains wide open and seems to be a rather difficult problem.

The results in this article give strong support to our belief that a weaker notion of reducibility is more appropriate for the local study of Dupin hypersurfaces with an arbitrary number of principal curvatures. We say that a proper Dupin hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$ is *weakly reducible* if it has a principal curvature λ with integrable conullity $E_\lambda^\perp$, a property that is also invariant under Lie transformations. As observed before, every reducible Dupin hypersurface is also weakly reducible, but we will show that the converse does not hold. Another important subclass of weakly reducible Dupin hypersurfaces is that of the *holonomic* ones, that is, hypersurfaces that can be endowed with principal coordinates. In fact, holonomicity can be characterized by the fact that the submanifold is weakly reducible with respect to every principal curvature. On the other hand, no isoparametric hypersurface of the sphere with at least three principal curvatures nor any of the examples in [PT] and [MO] is weakly reducible.

In this paper we give a complete local characterization of weakly reducible Dupin hypersurfaces. In fact, we solve a more general problem with an interest of its own in the theory of Euclidean submanifolds of arbitrary codimension. Namely, we characterize the submanifolds that carry a Dupin principal normal with integrable conullity.

Recall that a smooth normal vector field η of an isometric immersion $f\colon M^n \to \mathbb{R}^N$ is called a *principal normal* with multiplicity $s \geq 1$ if the tangent subspaces

$$\mathcal{E}_\eta = \ker(\alpha_f - \langle\ ,\ \rangle\eta)$$

have constant dimension s, where $\alpha_f\colon TM \times TM \to T_f^\perp M$ stands for the second fundamental form of f with values in the normal bundle. This is a natural generalization for submanifolds of higher codimension of the notion of principal curvature of a hypersurface. We say that a principal normal η of multiplicity s is *Dupin* if it is parallel in the normal connection of f along the (conformal) *nullity* distribution $\mathcal{E}_\eta$ associated to η. This condition is automatic for multiplicity $s \geq 2$ (cf. [Re1] or [DFT1]). If η is nonvanishing, it is well-known that $\mathcal{E}_\eta$ is an involutive distribution whose leaves are round s-dimensional spheres in $\mathbb{R}^N$; see [Re2] or [DFT1] for details. When η vanishes identically, the distribution $\mathcal{E}_\eta = \mathcal{E}_0$ is known as the relative nullity distribution, in which case the leaves are open subsets of affine subspaces of $\mathbb{R}^N$.

If one of the first three constructions due to Pinkall is applied to an arbitrary submanifold L^{n-s} in $\mathbb{R}^{N-s}$ with any codimension $N-n$, then the process introduces a Dupin principal normal η with multiplicity s, which has the additional property that the conullity $\mathcal{E}_\eta^\perp$ is a spherical distribution on M^n. It was proved in [DFT1] (see also Theorem 4.7 below) that this last property characterizes these examples up to conformal transformations of the ambient space.

A simple way to construct submanifolds carrying a relative nullity distribution with integrable conullity is as follows. Let $g\colon L^{n-s} \to \mathbb{Q}_\epsilon^N$, $\epsilon = 0, 1, -1$, be an isometric immersion with a parallel flat normal subbundle $\mathcal{V}$ of rank s. Then the n-dimensional *generalized cylinder* in $\mathbb{Q}_\epsilon^N$ over g determined by $\mathcal{V}$ is the submanifold parametrized by means of the exponential map of $\mathbb{Q}_\epsilon^N$ as

$$\gamma \in \mathcal{V} \mapsto \exp^\epsilon_{g(\pi(\gamma))}(\gamma).$$

Any such submanifold carries a relative nullity distribution of dimension s, whose leaves are the fibers of $\mathcal{V}$. Moreover, the conullity distribution is integrable, its leaves being given by the parallel sections of $\mathcal{V}$. Our first result concerning generalized cylinders is that these are the only submanifolds having a relative nullity distribution with integrable conullity.

The property of having a Dupin principal normal with integrable conullity is invariant under $\mathcal{L}$-transformations. By an *$\mathcal{L}$-transformation* of an Euclidean submanifold we mean a diffeomorphism that is a composition of conformal transformations of the ambient space and parallel translations, the latter being translations of the submanifold by parallel normal vector fields. Of course, in the case of hypersurfaces $\mathcal{L}$-transformations are the usual transformations of Lie sphere geometry. We call two submanifolds *$\mathcal{L}$-equivalent* if they differ by an $\mathcal{L}$-transformation. Therefore, a class of submanifolds carrying a Dupin principal normal with integrable conullity is obtained by applying $\mathcal{L}$-transformations to the family of (stereographic projections of) generalized cylinders. In the hypersurface case, this class properly contains those submanifolds obtained by Pinkall's constructions. However, they are far from exhausting the whole family of submanifolds carrying a Dupin principal normal with integrable conullity, as will be made clear below.

The key observation in the characterization of submanifolds carrying a Dupin principal normal with integrable conullity is that the leaves of the conullity distribution of such a submanifold are always Ribaucour transforms of each other. This is in the sense of the extended notion of Ribaucour transformation for submanifolds of arbitrary dimension and codimension developed in [DT1] and [DT2] from the classical notion for surfaces in three dimensional Euclidean space. This observation can be seen as a generalization of the classical fact (see [Bi]) that the orthogonal surfaces of a cyclic system are Ribaucour transforms of each other. It has also been made recently by Corro [Co] in the particular case of holonomic Dupin hypersurfaces with a principal curvature of constant multiplicity one.

In order to turn the above observation into an explicit description of all such submanifolds, it was convenient to introduce the notion of $\mathcal{N}$–Ribaucour transform of a submanifold $h\colon\ L^{n-s} \to \mathbb{R}^N$ carrying a parallel flat normal subbundle $\mathcal{N}$ of rank s. This is an explicitly parametrized n-dimensional submanifold foliated by Ribaucour transforms of h, each corresponding to a parallel section of $\mathcal{N}$. The distribution orthogonal to this foliation is precisely the nullity distribution of a Dupin principal normal. One of the main results of this paper is that any submanifold that carries a Dupin principal normal with integrable conullity arises locally this way.

Each $\mathcal{N}$-Ribaucour transform of $h\colon\ L^{n-s} \to \mathbb{R}^N$ is essentially determined by a Codazzi tensor on L^{n-s} that commutes with the second fundamental form of h. We show that submanifolds that are $\mathcal{L}$-equivalent to (stereographic projections of) generalized cylinders are precisely those $\mathcal{N}$–Ribaucour transforms of h that are determined by commuting Codazzi tensors on L^{n-s} that can be expressed as linear combinations of the identity tensor and shape operators with respect to parallel normal vector fields.

The results discussed in the preceding paragraphs are then applied to the class of k–Dupin submanifolds, that is, Euclidean submanifolds with flat normal bundle that have exactly k principal normals all of which are Dupin. Our main result is that any k–Dupin submanifold that is *weakly reducible* carrying a principal normal with integrable conullity, is the $\mathcal{N}$-Ribaucour transform of a $(k-1)$–Dupin submanifold determined by a commuting Codazzi tensor of Dupin type. As an important application of this result, we obtain an explicit recursive procedure to generate all such submanifolds in terms of solutions of completely integrable systems of linear partial differential equations of first order.

Thorbergsson [Th1] raised the question whether the number of principal curvatures of an irreducible proper Dupin hypersurface must be $1, 2, 3, 4$ or 6, as shown by himself in the case of compact Dupin hypersurfaces [Th2]. We produce counterexamples (see (5.5) and Proposition 5.12), which are, nonetheless, weakly reducible. Therefore, the problem remains open under the assumption of weak irreducibility. This is another indication that the appropriate assumption in questions of such a local nature should be weak irreducibility.

We obtain several additional results on k–Dupin submanifolds. We show that the maximal possible value for the conformal codimension is $k-1$. Moreover, the submanifold is necessarily holonomic if its conformal codimension is $k-1$, and it is necessarily weakly reducible if its conformal codimension is at least $(2/3)k-1$, the latter estimate being sharp. Finally, we give a complete description of the weakly reducible 4–Dupin submanifolds. We show that the submanifold is either holonomic or is $\mathcal{L}$-equivalent to (the stereographic projection of) a generalized cylinder over a submanifold that is Lie equivalent to an isoparametric hypersurface.

To conclude this introduction, we point out that our description of submanifolds that carry a Dupin principal normal with integrable conullity as $\mathcal{N}$–Ribaucour transforms of submanifolds carrying a parallel flat normal subbundle has been recently used in [Fl] in order to characterize doubly (conformally) ruled submanifolds in space forms. On the other hand, the problem of locally describing submanifolds that carry a parallel flat normal subbundle, the starting point for the $\mathcal{N}$–Ribaucour transformation to be applied, is addressed in a forthcoming paper [DFT2].

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2. Preliminaries

In this section, we first recall the notions of Combescure and Ribaucour transformations of an Euclidean submanifold. Then, we discuss several basic facts about them that are used throughout the paper. We refer to [DT1] and [DT2] for further details and results on the subject.

A smooth map $\mathcal{F}\colon M^n \to \mathbb{R}^{n+p}$ is said to be a *Combescure transform* of a given isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ when there exists a symmetric endomorphism Φ of TM such that

$$\mathcal{F}_* = f_* \circ \Phi.$$

This condition implies that Φ belongs to the real vector space of (symmetric) Codazzi tensors on M^n that are *commuting* in the sense that

$$\alpha_f(X, \Phi Y) = \alpha_f(\Phi X, Y) \quad \text{for all } \; X, Y \in TM.$$

Conversely, any commuting Codazzi tensor Φ on a simply connected M^n gives rise to a Combescure transform $\mathcal{F}$ of f. Moreover, Φ and $\mathcal{F}$ can be given as

$$\Phi = \operatorname{Hess}\varphi - A^f_\beta \quad \text{and} \quad \mathcal{F} = f_*\nabla\varphi + \beta,$$

where $\varphi \in C^\infty(M)$ and $\beta \in T_f^\perp M$ satisfy

$$\alpha_f(\nabla\varphi, X) + \nabla^\perp_X \beta = 0 \quad \text{for all } \; X \in TM, \tag{2.1}$$

A^f_β denotes the shape operator of f with respect to β and $\nabla^\perp$ stands for the induced connection on the normal bundle.

Definition. An immersion $\tilde{f}\colon M^n \to \mathbb{R}^{n+p}$ is called a *Ribaucour transform* of a given immersion $f\colon M^n \to \mathbb{R}^{n+p}$ if $\tilde{f} \neq f$ everywhere, and there are a vector bundle isometry $\mathcal{P}\colon f^*T\mathbb{R}^{n+p} \to \tilde{f}^*T\mathbb{R}^{n+p}$ with $\mathcal{P}\, T_f M = T_{\tilde{f}} M$, and a nowhere vanishing smooth map $\delta\colon M^n \to \mathbb{R}^{n+p}$ such that:

(a) $\mathcal{P}Z - Z = \langle \delta, Z\rangle (f - \tilde{f})$ for all $Z \in f^*T\mathbb{R}^{n+p}$;
(b) $D = f_*^{-1}\mathcal{P}^{-1}\tilde{f}_*\colon TM \to TM$ is self adjoint in the metric induced by f.

Condition (a) says that for any $Z \in T_{f(x)}\mathbb{R}^{n+p}$ the straight lines in $\mathbb{R}^{n+p}$ through $f(x)$ and $\tilde{f}(x)$ tangent to Z and $\mathcal{P}Z$, respectively, are either parallel or intersect at a point equidistant to $f(x)$ and $\tilde{f}(x)$.

The following statement contains the basic facts on the Ribaucour transformation that will be used throughout this paper without further reference. We denote by $\mathcal{S}(f)$ the set of pairs (φ, β) satisfying (2.1) such that $\varphi\mathcal{F} \neq 0$ everywhere. Then $\mathcal{S}_0 = \mathcal{S}_0(f)$ stands for the corresponding real projective set, that is, $(\varphi, \beta) \sim (\varphi', \beta')$ if and only if $\varphi' = \lambda\varphi$ and $\beta' = \lambda\beta$ for some $0 \neq \lambda \in \mathbb{R}$.

Theorem 2.1 ([DT2]). *Let $f\colon M^n \to \mathbb{R}^{n+p}$ be an isometric immersion of a simply connected Riemannian manifold and let $\tilde f$ be a Ribaucour transform of f. Then there exists a unique $[(\varphi,\beta)] \in \mathcal{S}_0$ such that*

$$\tilde f = f - 2\varphi\nu\mathcal{F}, \tag{2.2}$$

where $\mathcal{F} = f_*\nabla\varphi + \beta$ and $\nu = \|\mathcal{F}\|^{-2}$. Moreover, we have that

$$\mathcal{P} = I - 2\nu\mathcal{F}\mathcal{F}^*, \quad \delta = -\varphi^{-1}\mathcal{F} \quad \text{and} \quad D = I - 2\varphi\nu\Phi,$$

where $\mathcal{F}^*(Z) = \langle \mathcal{F}, Z\rangle$ for any $Z \in f^*T\mathbb{R}^{n+p}$. Conversely, given $[(\varphi,\beta)] \in \mathcal{S}_0$, let $\mathcal{P}$, δ and D be defined by the preceding expressions on an open subset $U \subset M^n$ where D is invertible. Then $\tilde f|_U$ given by (2.2) is a Ribaucour transform of $f|_U$ for $\mathcal{P}$, δ and D. Moreover:

(i) *The second fundamental forms of f and $\tilde f$ are related by*

$$A^{\tilde f}_{\mathcal{P}\xi} = D^{-1}(A^f_\xi + 2\nu\langle\beta,\xi\rangle\Phi).$$

(ii) *The restriction $\mathcal{P}|_{T_f^\perp M}\colon T_f^\perp M \to T_{\tilde f}^\perp M$ is parallel.*

We denote by $\mathcal{R}_w(f)$ the Ribaucour transform $\tilde f$ of f determined by $w \in \mathcal{S}_0$. Since w determines $\mathcal{P}, \delta$ and D completely and $\mathcal{F}$ and Φ up to constants, when convenient we will use it as a subscript for these maps.

We see next that inversions and parallel translations are special cases of Ribaucour transformations. In the following and elsewhere writing a vector subspace as a subscript of a vector indicates taking the orthogonal projection of the vector onto that subspace.

Examples 2.2.

(i) Given a point $P_0 \in \mathbb{R}^{n+p}$ and $r > 0$, set $w = [(\varphi_1,\beta_1)]$ where $2\varphi_1 = \|f - P_0\|^2 - r^2$ and $\beta_1 = (f - P_0)_{T_f^\perp M}$. Then $\mathcal{F} = f - P_0$, $\Phi = I$, and

$$\tilde f = \mathcal{R}_w(f) = P_0 + r^2\|f - P_0\|^{-2}(f - P_0)$$

is obtained from f by an inversion with respect to the sphere of radius r centered at P_0. Moreover, $\mathcal{P} = I - 2\|f - P_0\|^{-2}(f - P_0)^*(f - P_0)$ and

$$r^2 A^{\tilde f}_{\mathcal{P}\mu} = \|f - P_0\|^2 A^f_\mu + 2\langle f - P_0, \mu\rangle I \quad \text{for all } \mu \in T_f^\perp M. \tag{2.3}$$

(ii) Given a parallel normal vector field ξ, set $w = [(\varphi_2,\beta_2)]$ where $2\varphi_2 = \|\xi\|^2$ and $\beta_2 = -\xi$. Then $\mathcal{F} = -\xi$, $\Phi = A^f_\xi$, and

$$\tilde f = \mathcal{R}_w(f) = f + \xi$$

is the parallel translation L_ξ of f. Moreover, $\mathcal{P} = I - 2\|\xi\|^{-2}\xi^*\xi$ and

$$A^{\tilde f}_\mu = (I - A^f_\xi)^{-1}A^f_\mu \quad \text{for all } \mu \in T_f^\perp M. \tag{2.4}$$

The Ribaucour transformation has the following invariance property under $\mathcal{L}$- transformations (see Proposition 33 in [DT2]). By an *$\mathcal{L}$-transformation* of an Euclidean submanifold we mean a diffeomorphism that is a composition of conformal transformations of the ambient space and parallel translations, the latter being translations of the submanifold by parallel normal vector fields. To each $\mathcal{L}$-transformation T of f we associate an $\mathcal{L}$-transformation $\tilde{T}$ of a given Ribaucour transform $\mathcal{R}_w(f)$ as follows:

(i) $\tilde{T} = T$ when T is a conformal map of $\mathbb{R}^{n+p}$.
(ii) $\tilde{T} = L_{\mathcal{P}\xi}$ when $T = L_\xi$.

Then, there is a correspondence $T \longmapsto w^T \in \mathcal{S}_0(T(f))$ such that

$$\tilde{T}(\mathcal{R}_w(f)) = \mathcal{R}_{w^T}(T(f)). \tag{2.5}$$

For later use we describe this correspondence explicitly for each of the following types of transformations. We omit the computations, which are straightforward with the exception of case (iv). For the latter we refer to Proposition 31 in [DT2].

(i) *Euclidean translation:* $T_u(f) = f + u$, where $u \in \mathbb{R}^{n+p}$. Then,

$$\mathcal{F}^{T_u} = \mathcal{F} \text{ and } \varphi^{T_u} = \varphi.$$

(ii) *Orthogonal transformation:* $O(f) = O \circ f$, where $O \in O(n+p)$. Then,

$$\mathcal{F}^O = O(\mathcal{F}) \text{ and } \varphi^O = \varphi.$$

(iii) *Homothety:* $H_k(f) = kf$, where $k \in \mathbb{R}$. Then,

$$\mathcal{F}^{H_k} = \mathcal{F} \text{ and } \varphi^{H_k} = k\varphi.$$

(iv) *Inversion:* $i(f) = f/\|f\|^2$. Then,

$$\mathcal{F}^i = \mathcal{P}^i(\mathcal{F} - 2\varphi\|f\|^{-2}f) = \mathcal{F} - 2(\langle \mathcal{F}, f\rangle - \varphi)\|f\|^{-2}f \text{ and } \varphi^i = \varphi\|f\|^{-2}.$$

(v) *Parallel translation:* $L_\xi(f) = f + \xi$, where $\xi \in T_f^\perp M$ is parallel. Then,

$$\mathcal{F}^{L_\xi} = \mathcal{F} \text{ and } \varphi^{L_\xi} = \varphi + \langle \mathcal{F}, \xi\rangle.$$

The *conformal codimension* of a submanifold $g\colon M^n \to \mathbb{R}^{n+p}$ is the number $c(g)$ such that $n + c(g)$ is the least dimension of a sphere or an affine subspace in $\mathbb{R}^{n+p}$ that contains the submanifold. If $c(g) = p$, then g is said to be *conformally substantial*.

PROPOSITION 2.3. *The conformal codimension is invariant under $\mathcal{L}$-transformations.*

Proof. The invariance under conformal transformations is clear. Thus it suffices to check that a submanifold $g\colon M^n \to \mathbb{R}^{n+p}$ and a parallel translate $L_\xi(g)$ have the same conformal codimension. Since the immersions have the same normal spaces at every point, it suffices to argue that $L_\xi(g)(M)$ is contained in a sphere whenever $g(M)$ is contained in a sphere centered, say, at the origin. We can write $\xi = ag + b\eta$, where $a, b \in \mathbb{R}$ and η is a parallel normal vector field tangent to the sphere. Hence $g + \xi = (1 + a)g + b\eta$ has also constant norm. □

On a submanifold $f\colon M^n \to \mathbb{R}^{n+p}$ consider a commuting Codazzi tensor

$$\Phi = aI - A^f_\delta, \quad a \in \mathbb{R}, \tag{2.6}$$

where $\delta \in T^\perp_f M$ is a parallel vector field in the normal connection. Then, a Combescure transform $\mathcal{F}$ of f determined by Φ, that is, $\mathcal{F}_* = f_* \circ \Phi$, can be written as

$$\mathcal{F} = af + v + \delta, \quad v \in \mathbb{R}^{n+p}. \tag{2.7}$$

Proposition 2.4. *The decomposition* (2.7) *is unique if f is conformally substantial.*

Proof. If $\mathcal{F} = \tilde{a}h + \tilde{v} + \tilde{\delta}$, set $a' = \tilde{a} - a$, $v' = \tilde{v} - v$ and $\delta' = \tilde{\delta} - \delta$. We obtain by differentiating $a'h + v' + \delta' = 0$ that $A^h_{\delta'} = a'I$. Since $\|\delta'\|$ is constant and h is conformally substantial, we conclude that $\delta' = 0 = a' = v'$. □

If $w = [(\varphi, \beta)] \in \mathcal{S}_0(f)$ is such that $\mathcal{F} = \mathcal{F}_w = f_* \nabla\varphi + \beta$ is as in (2.7), then

$$2\varphi = a\|f\|^2 + 2\langle f, v\rangle + c, \quad c \in \mathbb{R}. \tag{2.8}$$

Definitions. We say that w, Φ or $\mathcal{F}$ are *$\mathcal{L}$-trivial* when they are given by (2.6), (2.7) and (2.8). They are *conformally trivial* if they can be given by those expressions with $\delta = 0$.

If $w \in \mathcal{S}_0(f)$ is $\mathcal{L}$-trivial, then the corresponding Ribaucour transform is

$$\tilde{f} = f - \big(a\|f\|^2 + 2\langle f, v\rangle + c\big)\,\frac{af + v + \delta}{\|af + v + \delta\|^2}. \tag{2.9}$$

Notice that special cases of (2.9) are inversions and parallel translations.

Proposition 2.5. *If $w \in \mathcal{S}_0(f)$ is $\mathcal{L}$-trivial (respectively, conformally trivial), then the same holds for $w^T \in \mathcal{S}_0(T(f))$ for any $\mathcal{L}$-transformation T of f. More precisely, if $\mathcal{F}$ and φ are given by* (2.7) *and* (2.8)*, then $\mathcal{F}^T$ and φ^T are as follows:*

(i) $$\begin{cases} \mathcal{F}^{T_u} = aT_u f + v - au + \delta, \\ 2\varphi^{T_u} = a\|T_u f\|^2 + 2\langle T_u f, v - au\rangle + c - 2\langle u, v\rangle + a\|u\|^2, \end{cases}$$

(ii) $\mathcal{F}^O = aOf + Ov + O\delta, \quad 2\varphi^O = a\|Of\|^2 + 2\langle Of, Ov\rangle + c,$

(iii) $\mathcal{F}^{H_k} = (a/k)kf + v + \delta, \quad 2\varphi^{H_k} = (a/k)\|kf\|^2 + 2\langle kf, v\rangle + ck,$

(iv) $\mathcal{F}^i = ci(f) + v + \delta - 2\langle\delta, f\rangle i(f), \quad 2\varphi^i = c\|i(f)\|^2 + 2\langle i(f), v\rangle + a,$

(v) $\begin{cases} \mathcal{F}^{L_\xi} = aL_\xi f + v + \delta - a\xi, \\ 2\varphi^{L_\xi} = a\|L_\xi f\|^2 + 2\langle L_\xi f, v\rangle + c + 2\langle \delta, \xi\rangle - a\|\xi\|^2. \end{cases}$

Proof. It is straightforward using the expressions after (2.5). □

3. Dupin principal normals

Our goal in this section is to describe a procedure to generate all Euclidean submanifolds carrying a Dupin principal normal with integrable conullity.

We start by observing that the property of carrying a Dupin principal normal with integrable conullity is invariant under $\mathcal{L}$-transformations. In fact, let $f\colon M^n \to \mathbb{R}^{n+p}$ be an isometric immersion and let $\tilde f$ be an $\mathcal{L}$-transform of f. Then, to each principal normal η of f there corresponds a principal normal $\tilde\eta$ of $\tilde f$ such that $\mathcal{E}_{\tilde\eta} = \mathcal{E}_\eta$, and thus $\tilde\eta$ has integrable conullity if and only if η does. Moreover, $\tilde\eta$ is Dupin if and only if the same holds for η. Namely,

$$\tilde\eta = r^{-2}\mathcal{P}(\|f - P_0\|^2\eta + 2(f - P_0)_{T_f^\perp M}),$$

for an inversion as in Examples 2.2(i), whereas

$$\tilde\eta = (1 - \langle\xi,\eta\rangle)^{-1}\eta = (1 - \langle\xi,\eta\rangle)^{-1}\mathcal{P}(\eta - 2\|\xi\|^{-2}\langle\eta,\xi\rangle)$$

for a parallel translation L_ξ as in Examples 2.2(ii), as one can easily check using (2.3) and (2.4). That $\tilde\eta$ is Dupin if and only if the same is true for η follows from (ii) in Theorem 2.1.

We now introduce the main tool of the paper, which is an extension of the notion of Ribaucour transformation of an Euclidean submanifold. Fix a simply connected submanifold $h\colon L^{n-s} \to \mathbb{R}^{n+p}$ with a normal subbundle $\mathcal{N}$ of rank s that is parallel and flat with respect to the normal connection. Flatness of $\mathcal{N}$ means that the normal curvature tensor satisfies $R_h^\perp|_{\mathcal{N}} = 0$. We denote by $\mathcal{S}_{\mathcal{N}} = \mathcal{S}_{\mathcal{N}}(h)$ the set of equivalence classes of pairs $(\varphi,\beta) \in \mathcal{S}(h)$ under the equivalence relation that identifies two pairs (φ,β) and (φ',β') whenever

$$\varphi' = \lambda\varphi \quad\text{and}\quad \beta' - \lambda\beta \in \mathcal{N}^{\|}$$

for some $0 \neq \lambda \in \mathbb{R}$. Here and elsewhere $\mathcal{N}^{\|}$ stands for the s-dimensional real vector space of parallel sections of $\mathcal{N}$.

Definition. The $\mathcal{N}$-*Ribaucour transform* $\mathcal{R}_w^{\mathcal{N}}(h)$ of $h\colon L^{n-s} \to \mathbb{R}^{n+p}$ determined by $w = [(\varphi,\beta)] \in \mathcal{S}_{\mathcal{N}}$ is the n–dimensional immersed submanifold parametrized, at regular points, by the map $f\colon L^{n-s} \times \mathcal{N}^{\|} \to \mathbb{R}^{n+p}$ given by

$$f(u,t) = h_t(u), \tag{3.1}$$

where $h_t = \mathcal{R}_{[(\varphi,\beta+t)]}(h) = h - 2\varphi\nu(h_*\nabla\varphi + \beta + t)$ and $\nu = \|h_*\nabla\varphi + \beta + t\|^{-2}$.

Observe that h itself is the leaf "at infinity" of the foliation parametrized by t in the sense that $h = \lim_{\|t\|\to\infty} h_t$. After choosing a parallel orthonormal frame $\xi_1, \ldots, \xi_s$ of $\mathcal{N}$, the map f can be rewritten as $f\colon L^{n-s} \times \mathbb{R}^s \to \mathbb{R}^{n+p}$ given by

$$f = h - 2\varphi\nu \left(h_* \nabla\varphi + \beta + \sum_{j=1}^{s} y_j \xi_j(u) \right).$$

Given $(\varphi, \beta) \in \mathcal{S}(h)$ and $t \in \mathcal{N}^{\parallel}$, we denote $\mathcal{F}_t = h_*\nabla\varphi + \beta + t$, $\nu_t = \|\mathcal{F}_t\|^{-2}$, $\Phi_t = \operatorname{Hess}\varphi - A^f_{\beta+t}$, $\mathcal{P}_t = \mathcal{P}_{[(\varphi,\beta+t)]}$ and $D_t = D_{[(\varphi,\beta+t)]}$.

Proposition 3.1. *Let $\mathcal{R}^{\mathcal{N}}_w(h)\colon M^n = L^{n-s} \times \mathcal{N}^{\parallel} \to \mathbb{R}^{n+p}$ be an $\mathcal{N}$-Ribaucour transform of $h\colon L^{n-s} \to \mathbb{R}^{n+p}$. Then the following facts hold:*

(i) *$f = \mathcal{R}^{\mathcal{N}}_w(h) = \mathcal{R}^{\mathcal{N}}_{\bar w}(h)$ if and only if $w = \bar w$.*

(ii) *The normal space of f at $(u,t) \in M^n$ is $T^{\perp}_{(u,t)}M = \mathcal{P}_t(\mathcal{N}^{\perp}(u))$.*

(iii) *The normal connection of f is given by*

$$\nabla^{\perp}_X \hat\delta = \mathcal{P}_t \nabla^{\perp}_X \delta \quad \text{and} \quad \nabla^{\perp}_S \hat\delta = 0$$

for all $\delta \in \mathcal{N}^{\perp}, X \in TL$ and $S \in \mathcal{N}^{\parallel}$, where $\hat\delta(u,t) = \delta_t(u) = \mathcal{P}_t\delta(u)$. In particular, δ is a parallel normal vector field of h if and only if $\hat\delta$ is a parallel normal vector field of f, and hence, f has flat normal bundle if and only if h does.

(iv) *The second fundamental form of f at $(u,t) \in M^n$ for all $S \in \mathcal{N}^{\parallel}$ and $Z \in T_{(u,t)}M$ is given by*

$$\alpha_f(S,Z) = \langle S, Z\rangle \mathcal{P}_t \bar\beta(u), \tag{3.2}$$

where $\bar\beta = -\varphi^{-1}\beta_{\mathcal{N}^{\perp}}$, and by

$$\alpha_f(X,Y) = \mathcal{P}_t((\alpha_h(D_tX, Y) + 2\nu_t(u)\langle D_tX, \Phi_tY\rangle\beta(u))_{\mathcal{N}^{\perp}}) \tag{3.3}$$

for all $X, Y \in T_uL$.

(v) *For each point $u_0 \in L^{n-s}$ the map $f(u_0, \cdot)\colon \mathcal{N}^{\parallel} \to \mathbb{R}^{n+p}$ is a conformal parametrization of a sphere or an affine subspace, the latter occurring if and only if $\mathcal{F}_{\varphi,\beta}(u_0) \in \mathcal{N}$.*

Proof. The proof of (i) is straightforward. An easy computation at $(\cdot, t)$ yields

$$f_*X = \mathcal{P}_t D_t X \quad \text{and} \quad f_*S = -2\varphi\nu_t\mathcal{P}_t S \tag{3.4}$$

for all $X \in TL$ and $S \in \mathcal{N}^{\parallel}$, and (ii) follows. The assertions in (iii) are consequences of (ii) in Theorem 2.1 and the fact that $\tilde\nabla_S\hat\delta = -2\nu_t\langle\beta,\delta\rangle\mathcal{P}_tS \in TM$, where $\tilde\nabla$ denotes the Euclidean connection. The proof of (iv) is similar to that of Corollary 21 of [DT2]. Finally, (v) follows from (3.2) and (3.4). □

We show next that the $\mathcal{N}$-Ribaucour transform $\mathcal{R}^{\mathcal{N}}_w(h)$ of an isometric immersion $h\colon L^{n-s} \to \mathbb{R}^{n+p}$ with a parallel flat normal subbundle $\mathcal{N}$ of rank s always carries a Dupin principal normal of multiplicity s if a certain regularity assumption is satisfied. For the precise statement we need to introduce some terminology. For an isometric immersion $h\colon\ L^{n-s}\to\mathbb{R}^N$, a vector subbundle $\mathcal{V}\subset T_h^\perp L$ and a vector field $\delta\in\mathcal{V}$, we define at $x\in L^{n-s}$ the subspace

$$\mathcal{E}_\delta^{\mathcal{V}}(x)=\{Z\in T_xL:\ \alpha_h(Z,X)_{\mathcal{V}}=\langle Z,X\rangle\delta\quad\text{for all}\ \ X\in T_xL\}.$$

Given $w=[(\varphi,\beta)]\in\mathcal{S}_{\mathcal{N}}(h)$, we denote $\bar\beta=-\varphi^{-1}\beta_{\mathcal{N}^\perp}$ and $\mathcal{E}(w)=\mathcal{E}_{\bar\beta}^{\mathcal{N}^\perp}$.

The only purpose of the regularity condition $\mathcal{E}(w)=0$ in the following result is to assure that the Dupin principal normal generated by the $\mathcal{N}$-Ribaucour transformation has the lowest possible multiplicity everywhere.

Proposition 3.2. *Let $h\colon L^{n-s}\to\mathbb{R}^{n+p}$ be a simply connected submanifold carrying a parallel flat normal subbundle $\mathcal{N}$ of rank s, and let*

$$f=\mathcal{R}_w^{\mathcal{N}}(h)\colon\ M^n=L^{n-s}\times\mathcal{N}^{\parallel}\to\mathbb{R}^{n+p}$$

be an $\mathcal{N}$-Ribaucour transform of h determined by $w\in\mathcal{S}_{\mathcal{N}}$ with $\mathcal{E}(w)=0$. Then the vector field $\eta(u,t)=\mathcal{P}_t\bar\beta(u)$ is a Dupin principal normal of f with integrable conullity, the leaves being parametrized by $(\,\cdot\,,t_0)$ with $t_0\in\mathcal{N}^{\parallel}$.

Proof. It follows from (3.2) that $\mathcal{N}^{\parallel}\subset\mathcal{E}_\eta$. To see that equality holds, take $X\in TL\cap\mathcal{E}_\eta$ and $Y\in TL$. Then

$$\alpha_f(X,Y)=\langle D_tX,D_tY\rangle\mathcal{P}_t\bar\beta.$$

Using (3.3), we obtain that

$$\alpha_h(D_tX,Y)_{\mathcal{N}^\perp}=\langle D_tX,Y\rangle\bar\beta,$$

and hence $X=0$ by the assumption that $\mathcal{E}(w)=0$. That η is Dupin follows from Proposition 3.1(iii). Finally, by (3.4) we have that $\langle f_*S,f_*X\rangle=0$, and we conclude that the conullity is integrable, the leaves being parametrized by $(\,\cdot\,,t_0)$ with $t_0\in\mathcal{N}^{\parallel}$. ☐

One of our main results is that, conversely, any submanifold carrying a Dupin principal normal with integrable conullity arises locally this way.

Theorem 3.3. *Let $f\colon M^n\to\mathbb{R}^{n+p}$ be an isometric immersion carrying a Dupin principal normal η of multiplicity s with integrable conullity. Then $\mathcal{N}=\mathcal{E}_\eta|_L$ is a parallel flat normal subbundle of $h=f|_L$ for any given leaf L^{n-s} of conullity, and f is an $\mathcal{N}$-Ribaucour transform of h determined by a unique $w\in\mathcal{S}_{\mathcal{N}}(h)$ with $\mathcal{E}(w)=0$.*

Theorem 3.3 will be derived from Proposition 3.4, where the conullity is only assumed to admit one *maximal integral submanifold*, that is, a submanifold L satisfying that $T_xL=\mathcal{E}_\eta^\perp(x)$ for any $x\in L$, and that each leaf of Δ_f intersects L exactly once. First we need some further terminology and notations. We refer to a vector bundle (E,π,M) with total space E and projection $\pi\colon\ E\to M$ simply by E, and denote by $\Gamma(E)$ the space of its smooth sections.

The kernel of π_{*t} at $t \in E(x)$ is the vertical subspace of T_tE. Clearly, T_tE can be identified with $E(x)$ itself, thus we can write $E(x) \subset T_tE$ without risk of confusion. We call the *vertical subbundle* of TE the subbundle $E \subset TE$ formed by its vertical subspaces.

PROPOSITION 3.4. *Let $h\colon L^{n-s} \to \mathbb{R}^{n+p}$ be a submanifold with a parallel normal subbundle $\mathcal{N}$ of rank s and let $\mu\colon L^{n-s} \to \mathbb{R}^{n+p}$ be a smooth map such that $\eta_h = \mu_{\mathcal{N}^\perp}$ satisfies $\mathcal{E}_{\eta_h}^{\mathcal{N}^\perp} = 0$ and*

$$(\mu_* X)_{\mathcal{N}^\perp} = \langle h_* X, \mu\rangle \eta_h \quad \text{for all } X \in TL. \tag{3.5}$$

Define $f\colon \mathcal{N} \to \mathbb{R}^{n+p}$ by

$$f \circ t = h + 2\|\mu + t\|^{-2}(\mu + t) \quad \text{for all } t \in \Gamma(\mathcal{N}), \tag{3.6}$$

and $\eta\colon \mathcal{N} \to \mathbb{R}^{n+p}$ by

$$\eta \circ t = \eta_h - 2\|\eta_h\|^2\|\mu + t\|^{-2}(\mu + t). \tag{3.7}$$

Then f parametrizes, at regular points, a submanifold M^n with η as a Dupin principal normal such that the nullity $\mathcal{E}_\eta$ is the vertical subbundle $\mathcal{N} \subset T\mathcal{N}$, and L^{n-s} is a maximal integral submanifold of the conullity.

Conversely, let $f\colon M^n \to \mathbb{R}^{n+p}$ be a submanifold carrying a Dupin principal normal η of multiplicity $s \geq 1$ such that the conullity has a maximal integral submanifold L^{n-s}. Then $\mathcal{N} = \mathcal{E}_\eta|_L$ is normal and parallel along $h = f|_L$, and there exists a smooth map $\mu\colon L^{n-s} \to \mathbb{R}^{n+p}$ orthogonal to $\mathcal{N}$ such that f and η can be locally parametrized by (3.6) *and* (3.7) *on an open neighborhood of L^{n-s}.*

Before proving Proposition 3.4 we make the following useful observation.

LEMMA 3.5. *Let $h\colon L^{n-s} \to \mathbb{R}^{n+p}$ be a submanifold with a parallel normal subbundle $\mathcal{N}$ of rank $s \geq 0$ and let $\eta \in \mathcal{N}^\perp$ be a nowhere vanishing vector field. Then the subspace $\{T \in \mathcal{E}_\eta^{\mathcal{N}^\perp}(x) : \nabla_T^\perp \eta = 0\}$ coincides with*

$$\{T \in T_xM : (h + \|\eta\|^{-2}\eta)_* T = 0;\ \tilde{\nabla}_T \xi \in \{\eta\}^\perp \cap \mathcal{N}^\perp \ \text{for all } \xi \in \{\eta\}^\perp \cap \mathcal{N}^\perp\}.$$

Proof. We have that $T \in \mathcal{E}_\eta^{\mathcal{N}^\perp}(x)$ and $\nabla_T^\perp \eta = 0$ if and only if the right hand sides of the following equations vanish for any $\xi \in \{\eta\}^\perp \cap \mathcal{N}^\perp$:

$$\begin{aligned}(h + \|\eta\|^{-2}\eta)_* T &= h_*(T - \|\eta\|^{-2}A_\eta T) + \nabla_T^\perp \|\eta\|^{-2}\eta;\\ \langle \tilde{\nabla}_T \xi, \eta\rangle &= -\langle \nabla_T^\perp \eta, \xi\rangle;\\ \langle \tilde{\nabla}_T \xi, h_* X\rangle &= -\langle \alpha_h(T, X), \xi\rangle.\end{aligned}$$

□

Proof of Proposition 3.4. We start the proof of the direct statement by determining the normal bundle of f. Differentiating (3.6) along $X \in TL$, we obtain that

$$(f \circ t)_* X = h_* X + 2X(\|\mu + t\|^{-2})(\mu + t) + 2\|\mu + t\|^{-2}(\mu_* X + \tilde{\nabla}_X t). \tag{3.8}$$

It follows from (3.5), (3.8) and the parallelism of $\mathcal{N}$ that $\mathcal{N}^\perp \cap \{\eta_h\}^\perp$ is normal to f. Moreover, (3.5) and (3.8) also imply that

$$\langle (f\circ t)_* X, \eta\circ t\rangle = 2\|\mu + t\|^{-2}\left(\langle \mu_* X, \eta_h\rangle - \|\eta_h\|^2\langle h_* X, \mu\rangle\right) = 0.$$

Using that $\|\eta\circ t\| = \|\eta_h\|$, we obtain the orthogonal splitting

$$T_f^\perp M(t(x)) = \left(\mathcal{N}^\perp \cap \{\eta_h\}^\perp\right)(x) \oplus \mathrm{span}\{\eta(t(x)\}, \tag{3.9}$$

which shows that $\eta^\perp$ is constant along $\mathcal{N} \subset T\mathcal{N}$. For a point $x \in L^{n-s}$ where $\eta_h(x) = 0$, we conclude from (3.9) that

$$T_f^\perp M(t(x)) = \mathcal{N}^\perp(x), \tag{3.10}$$

and thus $\mathcal{N}(x) \subset \mathcal{E}_0(t(x))$. On the open subset where $\eta_h \neq 0$, we have that

$$f\circ t + \|\eta\circ t\|^{-2}\eta\circ t = h + \|\eta_h\|^{-2}\eta_h, \tag{3.11}$$

and hence the left hand side is constant along the leaves of $\mathcal{N} \subset T\mathcal{N}$. We conclude from Lemma 3.5 that $\mathcal{N} \subset \mathcal{E}_\eta$ and that η is parallel along $\mathcal{N}$.

It remains to show that $\mathcal{N} = \mathcal{E}_\eta$ under our regularity assumption. This holds by (3.10) at the points of L^{n-s} where $\eta_h = 0$. Hence, we may assume that $\eta_h(x) \neq 0$, and then the same holds for η on $\mathcal{N}(x)$. Any transversal tangent vector to the leaves of $\mathcal{N} \subset T\mathcal{N}$ can be written as $t_* X$ for some $X \in T_x L$ and $t \in \Gamma(\mathcal{N})$. Assume that $t_* X \in \mathcal{E}_\eta(t(x))$. Then, we have from Lemma 3.5 that

$$\left((f + \|\eta\|^{-2}\eta)_*(t_* X)\right)_{T_f M} = 0 = (\tilde\nabla_{t_* X}\xi)_{T_f M}$$

for any normal vector field $\xi \in \{\eta\}^\perp \subset T_f^\perp M$. By (3.9), any normal vector field $\hat\xi \in \mathcal{N}^\perp \cap \{\eta_h\}^\perp$ to h gives rise to a normal vector field $\xi \in \{\eta\}^\perp$ to f along t by setting $\xi\circ t = \hat\xi$. Therefore, $(\tilde\nabla_X \hat\xi)_{T_h L} = 0$ for any $\hat\xi \in \mathcal{N}^\perp \cap \{\eta_h\}^\perp$. Moreover, by (3.11) we obtain that $\left((h + \|\eta_h\|^{-2}\eta_h)_*(X)\right)_{T_h L} = 0$. From our assumption on η_h and Lemma 3.5, we conclude that $X = 0$, and thus $t_* X = 0$ as we wanted.

We first prove the converse under the assumption that η is nowhere vanishing. Let $\sigma(x)$ be the leaf of $\mathcal{E}_\eta$ through $x \in L^{n-s}$. Then $f(\sigma(x))$ is an open subset of a round sphere $\mathbb{S}^s(x)$ through $f(x)$. Now let $\mu\colon L^{n-s} \to \mathbb{R}^{n+p}$ be the vector field defined so that $h(x) + \|\mu(x)\|^{-2}\mu(x)$ is the center of $\mathbb{S}^s(x)$. Then $\mu(x)$ is orthogonal to the tangent space $\mathcal{N}(x) = \mathcal{E}_\eta(x)$ of $\mathbb{S}^s(x)$ at $h(x)$, and the inversion with respect to the sphere of radius $\sqrt{2}$ centered at the origin followed by translation by $h(x)$ maps the affine hyperplane $\mu(x) + \mathcal{N}(x)$ onto $\mathbb{S}^s(x)$ minus the point $h(x)$. Since L is a maximal integral submanifold of the conullity, f is parametrized by (3.6) on an open neighborhood of the zero section of $\mathcal{N}$ along h. Moreover, since η is a Dupin principal normal, it follows from Lemma 3.5 that $f + \|\eta\|^{-2}\eta$ is constant along $\mathcal{E}_\eta$. Therefore (3.11) holds and η is given by (3.7). Finally, the proof of the direct statement shows that η being normal to f and the normal subspace $\{\eta\}^\perp$ being constant along $\mathcal{E}_\eta$ imply that μ satisfies (3.5) and that $\mathcal{N}$ is parallel.

For the general case, we may compose f with a translation, if necessary, and an inversion i so that the corresponding principal normal η^i of $i(f)$ is nowhere vanishing. Thus, we have a submanifold h, a subbundle $\mathcal{N}$ and a map μ as before, and we can describe $i(f)$ by (3.6) as

$$i(f) = h + 2\|\mu + t\|^{-2}(\mu + t),$$

with μ satisfying (3.5). Applying the inversion i to $i(f)$, it is easy to see that

$$f = i(h) + 2\|\bar{\mu} + \bar{t}\|^{-2}(\bar{\mu} + \bar{t}),$$

where $\bar{\mu} = \mathcal{P}_i(2h + \|h\|^2\mu)$, $\bar{t} = \mathcal{P}_i\,\|h\|^2 t$, and $\mathcal{P}_i = I - 2\langle h, \cdot\rangle i(h)$ is the vector bundle isometry associated to i seen as a Ribaucour transformation of $i(h)$. In particular, we have that $\bar{\mathcal{N}}^\perp = \mathcal{P}_i\mathcal{N}^\perp$. It is also easy to check that $(\bar{\mu}_* X)_{\bar{\mathcal{N}}^\perp} = \langle i(h)_* X, \bar{\mu}\rangle \bar{\mu}_{\bar{\mathcal{N}}^\perp}$, and this completes the proof. □

Proof of Theorem 3.3. By Proposition 3.4, we have that $\mathcal{N} = \mathcal{E}_\eta|_L$ is parallel along $h = f|_L$ for any given leaf L^{n-s} of the conullity, and that f and η can be parametrized by (3.6) and (3.7), respectively, for some smooth map $\mu\colon L^{n-s} \to \mathbb{R}^{n+p}$ everywhere orthogonal to $\mathcal{N}$ satisfying (3.5). Since the conullity is integrable, for any $T \in \mathcal{N}(x_0)$ there is a section $t \in \Gamma(\mathcal{N})$ with $t(x_0) = T$ everywhere orthogonal to the vertical subbundle of $T\mathcal{N}$ with respect to the metric induced by f. We have that

$$f_* S = 2\|\mu + t\|^{-2}(S - 2\|\mu + t\|^{-2}\langle t, S\rangle(\mu + t))$$

for any section S of $\mathcal{N} \subset T\mathcal{N}$ along t. A straightforward computation using (3.8) shows that $\langle (f \circ t)_* X, f_* S\rangle = 0$ if and only if

$$\langle \mu_* X + \tilde{\nabla}_X t, S\rangle - \langle t, S\rangle\langle h_* X, \mu\rangle = 0.$$

Since $\mathcal{N}$ is parallel, this is equivalent to

$$\nabla^\perp_X t = -(\mu_* X)_{\mathcal{N}} + \langle h_* X, \mu\rangle t. \tag{3.12}$$

Given $t_0 \in \mathcal{N}(x_0)$, take $t_1, t_2 \in \Gamma(\mathcal{N})$ orthogonal to the vertical subbundle of $T\mathcal{N}$ such that $(t_1 - t_2)(x_0) = t_0$. It follows from (3.12) that $t = t_1 - t_2$ satisfies $t(x_0) = t_0$ and

$$\nabla^\perp_X t = \omega(X) t, \quad \text{where } \omega(X) = \langle h_* X, \mu\rangle. \tag{3.13}$$

Set $\tau = \log(\|t\|/\|t_0\|)$. Then $\tau(x_0) = 0$ and $d\tau = \omega$. It follows from (3.13) and the closeness of ω that $R^\perp(X, Y)t = 0$ for all $X, Y \in TL$. In particular, we have that $\mathcal{N}$ is flat. Moreover, (3.5) and (3.12) yield

$$(\mu_* X)_{T_h^\perp L} = X(\tau)(t + \mu_{\mathcal{N}^\perp}) - (\tilde{\nabla}_X t)_{\mathcal{N}}. \tag{3.14}$$

Set $t = e^\tau t_T + \bar{t}$, where t_T is the parallel section of $\mathcal{N}$ with $t_T(x_0) = T = t_0$. Then (3.14) can be written as

$$((\mu + \bar{t})_* X)_{T_h^\perp L} = X(\tau)(\mu + \bar{t})_{T_h^\perp L},$$

or equivalently,

$$(e^{-\tau}(\mu+\bar{t}))_*X \in T_hL \quad \text{for all } \ X \in TL.$$

Setting $\varphi = -e^{-\tau}$ and $\beta = e^{-\tau}(\eta_h+\bar{t})$, we obtain that $e^{-\tau}(\mu+\bar{t}) = h_*\nabla\varphi+\beta = \mathcal{F}$ is a Combescure transform of h. Moreover, $\mu + t = -\varphi^{-1}(\mathcal{F} + t_T)$. Thus

$$f \circ t = h - 2\varphi\|\mathcal{F} + t_T\|^{-2}(\mathcal{F} + t_T).$$

Finally, the uniqueness of $[(\varphi, \beta)]$ was observed in Proposition 3.1(i). □

Remark. In view of Theorem 3.3, the invariance under $\mathcal{L}$-transformations of the property of admitting a Dupin principal normal with integrable conullity can also be derived from the invariance of the $\mathcal{N}$-Ribaucour transformation under $\mathcal{L}$-transformations. The latter is as follows. To each $\mathcal{L}$-transformation T of h we associate an $\mathcal{L}$-transformation $\tilde{T}$ for $f = \mathcal{R}_w^{\mathcal{N}}(h)$ given by

(i) $\tilde{T} = T$ when T is a conformal map of $\mathbb{R}^{n+p}$.
(ii) $\tilde{T} = L_{\hat{\xi}}$ when $T = L_\xi$ for $\xi \in \mathcal{N}^\perp$, where $\hat{\xi} = \mathcal{P}_t\xi$.

Then, the correspondence $T \mapsto w^T \in \mathcal{S}_{\mathcal{N}}(T(h))$ given in (2.5) is such that

$$\tilde{T}f = \mathcal{R}_{w^T}^{\mathcal{N}^T}(Th), \tag{3.15}$$

where $\mathcal{N}^T = T\mathcal{N}$ if $T \in O(n+p)$, $\mathcal{N}^T = \mathcal{P}^i\mathcal{N}$ in the case of an inversion, and $\mathcal{N}^T = \mathcal{N}$ if T is either an Euclidean translation, an homothety or a parallel translation.

We conclude this section by showing that the $\mathcal{N}$-Ribaucour transformation for holonomic submanifolds admits a simple coordinate description in terms of solutions of completely integrable first order systems of partial differential equations.

Let $g\colon L^{n-s} \to \mathbb{R}^{n+p}$ be a holonomic submanifold endowed with principal coordinates $(u_1, \ldots, u_{n-s})$, let $\mathcal{N}$ be a parallel flat normal subbundle, and let $\xi_1, \ldots, \xi_{p+s}$ be a parallel orthonormal normal frame such that $\mathcal{N} = \operatorname{span}\{\xi_{p+1}, \ldots \xi_{p+s}\}$. Set $ds^2 = \sum_{j=1}^{n-s} v_j^2 du_j^2$, define $h_{ij} \in C^\infty(M)$ by $v_ih_{ij} = \partial v_j/\partial u_i$, $1 \le i, j \le n-s$, and $V_j^r \in C^\infty(M)$ by $v_jA^g_{\xi_r}X_j = V_j^rX_j$, $1 \le r \le s+p$, where $X_j = v_j^{-1}\partial/\partial u_j$. It follows from the Gauss, Codazzi and Ricci equations for g that the triple (v, h, V) satisfies the completely integrable system of partial differential equations

$$\begin{cases} \text{(i)}\ \dfrac{\partial v_i}{\partial u_j} = h_{ji}v_j, \ \ \text{(ii)}\ \dfrac{\partial h_{ij}}{\partial u_i} + \dfrac{\partial h_{ji}}{\partial u_j} + \displaystyle\sum_k h_{ki}h_{kj} + \sum_r V_i^rV_j^r = 0, \\ \text{(iii)}\ \dfrac{\partial h_{ik}}{\partial u_j} = h_{ij}h_{jk}, \ \ \text{(iv)}\ \dfrac{\partial V_i^r}{\partial u_j} = h_{ji}V_j^r, \end{cases} \tag{3.16}$$

where always $i \neq j \neq k \neq i$. Now consider the linear system of partial differential equations of first order

$$\begin{cases} \partial\varphi/\partial u_i \ = v_i\gamma_i, \\ \partial\gamma_j/\partial u_i = h_{ji}\gamma_i, \quad i \neq j, \\ \partial\beta_r/\partial u_i = -V_i^r\gamma_i. \end{cases} \tag{3.17}$$

System (3.17) is also completely integrable, the compatibility conditions being satisfied by virtue of (3.16). Moreover, $(\varphi, \gamma, \beta) = (\varphi, \gamma_1, \ldots, \gamma_{n-s}, \beta_1, \ldots, \beta_{p+s})$ is a solution of (3.17) if and only if the pair $(\varphi, \beta = \sum_{r=1}^{p+s} \beta_r \xi_r)$, satisfies (2.1). Therefore, $\mathcal{R}^{\mathcal{N}}_{\varphi,\beta}(g)$ can be parametrized as $f\colon L^{n-s} \times \mathbb{R}^s \to \mathbb{R}^{n+p}$ given by

$$f(u,y) = g - 2\varphi\nu \left(\sum_{j=1}^{n-s} \gamma_j g_* X_j + \sum_{r=1}^{p} \beta_r \xi_r + \sum_{\ell=p+1}^{s+p} (\beta_\ell + y_{\ell-p})\xi_\ell \right), \tag{3.18}$$

where $\nu^{-1} = \sum_j \gamma_j^2 + \sum_r \beta_r^2 + \sum_\ell (\beta_\ell + y_{\ell-p})^2$.

4. Generalized cylinders

This section is devoted to a characterization of the class of submanifolds that are $\mathcal{L}$-equivalent to the generalized cylinders within the class of submanifolds that carry a Dupin principal normal with integrable conullity. Let us first recall their precise definition.

Definition. Let $g\colon L^{n-s} \to \mathbb{Q}_\epsilon^N$, $\epsilon = 0, 1, -1$, be an isometric immersion with a parallel flat normal subbundle $\mathcal{V}$ of rank s. The *generalized cylinder* over g determined by $\mathcal{V}$ is the n-dimensional submanifold parametrized by means of the exponential map of $\mathbb{Q}_\epsilon^N$ as

$$\gamma \in \mathcal{V} \mapsto \exp^\epsilon_{g(\pi(\gamma))}(\gamma).$$

We start by showing that the generalized cylinders are the only submanifolds that carry a relative nullity distribution with integrable conullity.

Proposition 4.1. *Let $h\colon L^{n-s} \to \mathbb{Q}_\epsilon^{n+p}$ be a simply connected submanifold with a parallel flat normal subbundle $\mathcal{N}$ of rank s such that $\mathcal{E}_0^{\mathcal{N}^\perp} = 0$. Then the generalized cylinder over h determined by $\mathcal{N}$ has relative nullity of constant dimension s and integrable conullity.*

Conversely, any submanifold $f\colon M^n \to \mathbb{Q}_\epsilon^{n+p}$ with relative nullity of constant dimension s and integrable conullity arises this way locally. That is, $\mathcal{N} = \mathcal{E}_0|_L$ is a parallel flat normal subbundle of $h = f|_L$ for any leaf L^{n-s} of the conullity and f is an open neighborhood of h of the generalized cylinder over h determined by $\mathcal{N}$.

Proof. We give the proof for the converse in the case where $\epsilon = 0$, the proof of the direct statement being straightforward. By Theorem 3.3, we have that $\mathcal{N} = \mathcal{E}_0|_L$ is a parallel flat subbundle of the normal bundle of $h = f|_L$ for any given leaf L^{n-s} of the conullity, and that f is an $\mathcal{N}$-Ribaucour transform of h determined by a unique $w \in \mathcal{S}_{\mathcal{N}}(h)$. Since the leaves of relative nullity are affine subspaces, then we must have that $\mathcal{F}_w \in \mathcal{N}$ by Proposition 3.1(v), and hence f is a generalized cylinder in $\mathbb{R}^{n+p}$. The case $\epsilon = 1$ can be easily reduced to the Euclidean one by taking the cone in $\mathbb{R}^{N+1}$ over the submanifold; details are left to the reader. The proof of the case $\epsilon = -1$ can be done similarly. □

Now let $f\colon M^n \to \mathbb{R}^{n+p}$ be an isometric immersion that carries a Dupin principal normal η of multiplicity s and integrable conullity. By Theorem 3.3, there exist a submanifold $h\colon L^{n-s} \to \mathbb{R}^{n+p}$, a parallel flat normal subbundle $\mathcal{N}$ of rank s and an element $w \in \mathcal{S}_{\mathcal{N}}$ such that $f = \mathcal{R}_w^{\mathcal{N}}(h)$. In the following result, we characterize those $w \in \mathcal{S}_{\mathcal{N}}$ for which f is $\mathcal{L}$-equivalent to (the stereographic projection of) a generalized cylinder in $\mathbb{Q}_\epsilon^{n+p}$.

THEOREM 4.2. *The following assertions are equivalent:*

(a) *$f = \mathcal{R}_w^{\mathcal{N}}(h)$ for some $\mathcal{L}$-trivial (respectively, conformally trivial) $w \in \mathcal{S}_{\mathcal{N}}$.*

(b) *f is $\mathcal{L}$-equivalent (resp., conformally equivalent) to (the stereographic projection if $\epsilon \neq 0$ of) a generalized cylinder in $\mathbb{Q}_\epsilon^{n+p}$.*

Moreover, if h is conformally substantial, then $\epsilon = \epsilon(w)$ is uniquely determined.

Proof. First we give a parametric description of the generalized cylinders in $\mathbb{Q}_\epsilon^N$ as $\mathcal{N}$-Ribaucour transforms. Consider $\mathbb{R}^N = \{0\} \times \mathbb{R}^N$ inside $\mathbb{R}^{N+1}$ if $\epsilon = 1$, or inside the Lorentzian space $\mathbb{L}^{N+1} = \mathbb{R}^{1,N}$ if $\epsilon = -1$, and then take $\mathbb{S}^N \subset \mathbb{R}^{N+1}$ and $\mathbb{H}^N \subset \mathbb{L}^{N+1}$. Then, the generalized cylinder in $\mathbb{Q}_\epsilon^N$ over h determined by $\mathcal{V}$ can be parametrized as

$$\gamma \in \mathcal{V} \mapsto h(x) - (1+\epsilon^2)\frac{\epsilon h(x) + \gamma}{\epsilon + \|\gamma\|^2}, \quad x = \pi(\gamma). \tag{4.1}$$

On the other hand, if $w \in \mathcal{S}_{\mathcal{N}}$ is $\mathcal{L}$-trivial, it follows from (2.8) and (2.9) that $f = \mathcal{R}_w^{\mathcal{N}}(h)$ can be parametrized as $f(u,t) = h_t(u)$, where

$$h_t = h - \left(a\|h\|^2 + 2\langle h, v\rangle + c\right)\frac{ah + v + \delta + t}{\|ah + v + \delta + t\|^2}. \tag{4.2}$$

Moreover, we can assume that $\delta \in \mathcal{N}^\perp$ for $\delta_{\mathcal{N}}$ can always be canceled by a reparametrization in (4.2). If h is conformally substantial, then a, v, δ and c are now completely determined up to a common multiplicative constant by Proposition 2.4. In particular,

$$\epsilon(w) = \operatorname{sign}\left(ac - \|v\|^2 + \|\delta\|^2\right)$$

is well defined.

LEMMA 4.3. *If h is conformally substantial, then $\epsilon(w)$ is invariant by conformal transformations and parallel translations L_ξ for any $\xi \in \mathcal{N}^\perp$.*

Proof. It follows easily from Proposition 2.5. □

To prove the equivalence part of the statement, we show that the parametrizations (4.1) and (4.2) correspond by $\mathcal{L}$-transformations (resp., conformal transformations) and stereographic projections. We use Proposition 2.5 several times without further reference. If $a = 0$ in (4.2), we can make $a \neq 0$ by a conformal transformation. In fact, we may assume that $c \neq 0$. Otherwise, $v \neq 0$ and a translation T_v gives $c \neq 0$. Now an inversion yields $a \neq 0$. By a homothety H_a followed by a translation T_v of (4.2), we have that

$$h'_t = h' - \left(\|h'\|^2 + c_1\right)\frac{h' + \delta + t}{\|h' + \delta + t\|^2}, \tag{4.3}$$

where $h' = ah + v$, $h'_t = ah_t + v$ and $c_1 = ca - \|v\|^2$.

CLAIM 4.4. *We may assume that $h' + \delta$ in* (4.3) *is an immersion.*

The conformal map $C = T_{-(c_1+\|q\|^2)^{-1}q} \circ i \circ T_q$ for $q \in \mathbb{R}^{n+p}$ takes (4.3) into

$$\bar{h}_t = \bar{h} - (\|\bar{h}\|^2 + c_1(c_1 + \|q\|^2)^{-2})\frac{\bar{h} + (c_1 + \|q\|^2)^{-1}\bar{\delta} + t}{\|\bar{h} + (c_1 + \|q\|^2)^{-1}\bar{\delta} + t\|^2},$$

where $\bar{h} = C(h')$ and $\bar{\delta} = \mathcal{P}_C\delta = \delta - 2\langle\delta, h' + q\rangle i(h' + q)$. At each point, we obtain using (2.3) for $P_0 = -q$ that $A^{\bar{h}}_{\bar{\delta}} = \|h' + q\|^2 A^{h'}_{\delta} + 2\langle\delta, h' + q\rangle I$. Thus,

$$I - (c_1 + \|q\|^2)^{-1}A^{\bar{h}}_{\bar{\delta}} = -(c_1 + \|q\|^2)^{-1}\|h' + q\|^2(A^{h'}_{\delta} - \sigma(q)I),$$

where $\sigma(q) = \|h'+q\|^{-2}(c_1 - 2\langle\delta, h'+q\rangle + \|q\|^2)$. The proof of the claim follows from the fact that σ is a nonconstant continuous function of q. Otherwise, we would have $\|h'\|^2 + c_1 = 0$, which is in contradiction with (4.3) being a parametrization.

A parallel translation $L_{\hat{\delta}}(f)$ (see (3.15)) yields

$$h''_t = h'' - \left(\|h''\|^2 + c_2\right)\frac{h'' + t}{\|h'' + t\|^2}, \tag{4.4}$$

where $h'' = h' + \delta$, $h''_t = h'_t + \delta_t$ and $c_2 = c_1 + \|\delta\|^2$. Now a homothety $H_{|c_2|^{-1/2}}$, if necessary, and Lemma 4.3 yield

$$g_t = g - \left(\|g\|^2 + \epsilon\right)\frac{g + t}{\|g + t\|^2}, \quad \epsilon = \epsilon(w) = 0, \pm 1, \tag{4.5}$$

where $g = |c_2|^{-1/2}h''$ and $g_t = |c_2|^{-1/2}h''_t$ when $c_2 \neq 0$.

It remains to show that the stereographic projection

$$S = T_{\epsilon e_0}H_{1+\epsilon^2}\, i\, T_{-\epsilon^2 e_0}\colon \mathbb{R}^N \to \mathbb{Q}^N_\epsilon,$$

where $e_0 = (1,0)$, takes the normal form (4.5) to a generalized cylinder in $\mathbb{Q}_\epsilon^N$. Notice that $S = i$ if $\epsilon = 0$. For each $t \in \mathcal{N}^{\parallel}$, the corresponding parallel normal vector field $\hat{t} \in \mathcal{V}''$ of $k = S(g) = \epsilon e_0 + (1+\epsilon^2)(\|g\|^2+\epsilon)^{-1}(g-\epsilon^2 e_0)$ is $\hat{t} = t - 2\langle t, \tilde{g}\rangle i(\tilde{g})$, where $\tilde{g} = g - \epsilon^2 e_0$. Thus $k_{\hat{t}} = S(g_t)$ is given by

$$k_{\hat{t}} = \epsilon e_0 + (1+\epsilon^2)\frac{\|\tilde{g}+\epsilon^2 e_0 + t\|^2\tilde{g} - \|\tilde{g}\|^2(\tilde{g}+\epsilon^2 e_0+t)}{\epsilon + \|\hat{t}\|^2} = k - (1+\epsilon^2)\frac{\epsilon k + \hat{t}}{\epsilon + \|\hat{t}\|^2},$$

which is, precisely, a generalized cylinder in $\mathbb{Q}_\epsilon^N$ over k determined by $\mathcal{V}$. □

Corollary 4.5. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ be an isometric immersion carrying a Dupin principal normal with integrable conullity. The following assertions are equivalent:*

(a) *f is $\mathcal{L}$-equivalent to (the stereographic projection of) a generalized cylinder.*
(b) *There is a pair of immersed conullity leaves of f that are $\mathcal{L}$-equivalent.*
(c) *Any pair of immersed conullity leaves of f are $\mathcal{L}$-equivalent.*

Remark. Theorems 3.3 and 4.2 were used in [Fl] to characterize doubly (conformally) ruled submanifolds in space forms, that is, submanifolds that have a pair of transversal umbilical foliations. It was shown that, if nontrivial, the intersection of both foliations is itself an umbilical foliation and the normal component of its mean curvature vector is always a Dupin principal normal of the submanifold with integrable conullity. If the codimension is big enough, it was also shown that the submanifold is then (conformally equivalent to) a generalized cylinder.

The concepts of rotation submanifold and tube admit the following extensions for an Euclidean submanifold $g\colon S \to \mathbb{R}^N$ with a parallel flat normal subbundle $\mathcal{N}$.

(i) The *generalized rotation submanifold* $\psi\colon \mathcal{N} \to \mathbb{R}^N$ over g determined by $\mathcal{N}$ and $e \in \mathbb{R}^N$ is given by

$$\psi(\gamma) = g(x) - 2\langle g(x), e\rangle \frac{e+\gamma}{\|e+\gamma\|^2}, \quad x = \pi(\gamma).$$

(ii) The *generalized tube* $\psi\colon \mathcal{N}_1 \to \mathbb{R}^N$ over g determined by $\mathcal{N}$ and $a \in \mathbb{R}^*$ is given by

$$\psi(\gamma) = g(x) + a\gamma, \quad x = \pi(\gamma),$$

where $\mathcal{N}_1 \subset \mathcal{N}$ denotes the unit sphere subbundle.

Proposition 4.6. *The stereographic projection on $\mathbb{R}^N$ of a generalized cylinder in $\mathbb{Q}_\epsilon^N$ over h determined by $\mathcal{V}$ for $\epsilon \neq 0$ is $\mathcal{L}$-equivalent to one of the following submanifolds:*

(a) *A generalized tube if $\epsilon = 1$ and $\mathcal{V}$ is not maximal.*

(b) *A generalized rotation submanifold if* $\epsilon = -1$.

Conversely, any generalized tube or rotation submanifold is $\mathcal{L}$*-equivalent to the stereographic projection of a generalized cylinder in* $\mathbb{Q}_\epsilon^N$ *for* $\epsilon = 1$ *or* $\epsilon = -1$*, respectively.*

Proof. From the proof of Theorem 4.2, we know that the stereographic projection of a generalized cylinder in $\mathbb{Q}_\epsilon^N$ has the form (4.5), with $t \in \tilde{\mathcal{V}} = S(\mathcal{V})$. If $\epsilon = -1$, take $e \in \mathbb{R}^N$ such that $\|e\|^2 = 1$. By a translation T_{-e} in (4.5) we have that

$$g'_t = g' - \left(\|g'\|^2 + 2\langle g', e\rangle\right) \frac{g' + e + t}{\|g' + e + t\|^2},$$

where $g' = g - e$ and $g'_t = g_t - e$. Composing with an inversion i yields

$$i(g'_t) = i(g') - 2\left(\langle i(g'), e\rangle + 1/2\right) \frac{e + t}{\|e + t\|^2}.$$

After a translation $T_{e/2}$, we obtain that $h_t = i(g'_t) + e/2$ is a generalized rotation submanifold over $h = i(g') + e/2$.

For $\epsilon = 1$ we need an alternative description of a generalized tube. Take a parallel $\nu \in \mathcal{N}_1$ and set $\tilde{\mathcal{N}} = \mathcal{N} \cap \{\nu\}^\perp$. The generalized tube $f\colon \tilde{\mathcal{N}} \to \mathbb{R}^N$ over h determined by $\mathcal{N}$ is given by

$$f(\gamma') = h - \nu + 2\frac{\nu + \gamma'}{\|\nu + \gamma'\|^2}.$$

Since $\mathcal{V}$ is not maximal, there is a unit parallel $\xi' \in \tilde{\mathcal{V}}^\perp$. As in the proof of Claim 4.4, we use the conformal map C for some $q \in \mathbb{R}^{n+p}$ to replace $\epsilon = 1$ in (4.5) by $(1 + \|q\|^2)^{-2}$ and to obtain that $C(g) + \xi$ is an immersion, where $\xi = (1 + \|q\|^2)^{-1}\mathcal{P}_C\xi'$. Then, the parallel translation $L_{\hat{\xi}}$ of (4.5) yields

$$g'_t = g' - \|g'\|^2 \frac{g' - \xi + t}{\|g' - \xi + t\|^2},$$

where $g' = C(g) + \xi$ and $g'_t = C(g_t) + \xi_t$. We obtain a generalized tube by composing with the inversion i. ☐

Remark. The three classes in part (b) of Theorem 4.2 for distinct ϵ do not have to be disjoint if h is not conformally substantial. For example, tubes may also be rotational submanifolds. In fact, one can see that an element belonging to any two classes also belongs to the third.

As an application of Theorem 4.2, we are now able to give a short proof of the main result in [DFT1] with the additional assumption that the submanifold is locally conformally substantial, thus showing the advantage one may have in working with a parametric description instead of the fundamental equations of the submanifold.

THEOREM 4.7. *Let $f\colon M^n \to \mathbb{R}^N$ be a locally conformally substantial submanifold with a Dupin principal normal of multiplicity k such that its conullity is totally umbilical in M^n. If $k = n-1$, assume further that the integral curves of the conullity are circles in M^n. Then $f(M)$ is conformally congruent to an open subset of one of the following submanifolds:*

(a) *$M^n = L^{n-k} \times \mathbb{R}^k$, and $f = (g, id)$ for a submanifold $g\colon L^{n-k} \to \mathbb{R}^{N-k}$;*

(b) *$M^n = CL^{n-k} \times \mathbb{R}^{k-1}$, and $f = (Cg, id)$, where Cg is the cone over a spherical submanifold $g\colon L^{n-k} \to \mathbb{S}^{N-k} \subset \mathbb{R}^{N-k+1}$;*

(c) *$M^n = L^{n-k} \times_\rho \mathbb{S}^k$, and $f = (g, \rho i)$ for a submanifold $g\colon L^{n-k} \to \mathbb{R}^{N-k-1}$, the inclusion $i\colon \mathbb{S}^k \to \mathbb{R}^{k+1}$ and a function $\rho \in C^\infty_+(L)$.*

Proof. We use the parametrization (3.1) and claim that w is conformally trivial. Since the nullity distribution is $\mathcal{U}(h_t(x)) = \mathcal{P}_{[(\varphi,\beta+t)]}\mathcal{N}(x)$, the conullity distribution is totally umbilical in the manifold if and only if for each $\delta \in \mathcal{N}$ there exists $\kappa_t^\delta \in \mathbb{R}$ such that

$$\langle A^{h_t}_{\delta_t} X, Y\rangle = \kappa_t^\delta \langle X, Y\rangle$$

for all $X, Y \in \mathcal{U}^\perp(t) = T_{h_t}L$. At the leaf parametrized by h ($\|t\| \to \infty$), we have that $A^h_\delta = \kappa^\delta I$ for all $\delta \in \mathcal{N}$. We obtain from Theorem 2.1(i) that

$$\kappa_t^\delta (I - 2\varphi\nu_t\Phi_t) = \kappa^\delta I + 2\langle \beta_t, \delta_t\rangle \nu_t \Phi_t.$$

We easily conclude that $\Phi = aI$ for some $a \in \mathbb{R}$, and the proof of the claim follows using that $\mathcal{F}_* = f_* \circ \Phi$.

Case $\kappa^\delta = 0$ for all $\delta \in \mathcal{N}$: Observe that the conullity being totally umbilical in the manifold is a conformally invariant property. Since $\mathcal{N}$ is parallel and totally geodesic by assumption, we conclude that h reduces codimension to $N-k$. Thus, up to translation and homothety, we have an orthogonal splitting $\mathbb{R}^N = \mathbb{R}^{N-k} \oplus \mathbb{R}^k$ such that $h \subset \mathbb{R}^{N-k}$, $\mathcal{N} = \mathbb{R}^k$ and (4.5) takes the form

$$h_t = (h, 0) - \left(\|h\|^2 + \epsilon\right) \frac{(h,t)}{\|(h,t)\|^2}, \quad e = 0, \pm 1.$$

Choose $e \in \mathbb{R}^N$ such that $e = 0, e_N, e_1$ for $\epsilon = 0, 1, -1$, respectively. It is easy to see, by composing with the conformal transformation $T_{-e/2}\, i\, T_e$, that we obtain cases (a), (b), (c) in the statement for $\epsilon = 0, 1, -1$, respectively.

Case $\kappa \neq 0$: Now h is totally umbilical with respect to the subbundle $\mathcal{N}$. It is a standard fact (cf. [Ya]) that $h(L)$ is contained in a sphere $\mathbb{S}^{N-k} \subset \mathbb{R}^{N-k+1}$, which we may assume to be centered at the origin, and that $\mathcal{N} = \operatorname{span}\{\xi\} \oplus \mathcal{N}'$, where ξ is the position vector of $\mathbb{S}^{N-k}$ in $\mathbb{R}^{N-k+1}$ and $\mathcal{N}' = \mathbb{R}^{k-1}$ is the orthogonal complement of $\mathbb{R}^{N-k+1}$ in $\mathbb{R}^N$. Now, an inversion with respect to a sphere centered at a point in $\mathbb{S}^{N-k}$ reduces this case to the first one.

Observe now that the three types cannot be glued together by the last part of Theorem 4.2, since h is conformally substantial. □

5. Weakly reducible Dupin submanifolds

Our main goal in this section is to describe how to construct locally all weakly reducible k-Dupin submanifolds as defined below. As a consequence, we obtain an explicit coordinate description of a recursive procedure to construct all the holonomic k-Dupin submanifolds. Several related results on k-Dupin submanifolds are also given.

It is a well-known fact (see [Re1]) that at each point $x \in M^n$ of an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ with flat normal bundle there exist an integer $k(x)$ and unique principal normals $\eta_1, \ldots, \eta_k \in T_x^\perp M$ such that the tangent space splits orthogonally as

$$T_x M = \mathcal{E}_{\eta_1}(x) \oplus \cdots \oplus \mathcal{E}_{\eta_k}(x). \tag{5.1}$$

We call f *proper* if $k = k(x)$ is constant on M^n. In this case, each η_j is smooth and the dimension of $\mathcal{E}_{\eta_j}$ is constant. Hence, $\mathcal{E}^f = (\mathcal{E}_{\eta_1}, \ldots, \mathcal{E}_{\eta_k})$ is an orthogonal k–net on M^n, that is, an orthogonal decomposition of TM into k integrable subbundles (cf. [RS]).

Definition. An isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ with flat normal bundle is a *k–Dupin submanifold* if it is proper and any one of its principal normals $\eta_1, \ldots, \eta_k$ is Dupin.

We now introduce the main concept of this section.

Definition. A k–Dupin submanifold is *weakly reducible* if it has a principal normal with integrable conullity.

Given a k–Dupin submanifold $f\colon M^n \to \mathbb{R}^{n+p}$, we call a Codazzi tensor Φ on M^n a *Dupin tensor adapted to* $\mathcal{E}^f$ if there exist $\phi_1, \ldots, \phi_k \in C^\infty(M)$ such that each function ϕ_j is constant along $\mathcal{E}_{\eta_j}$ and $\Phi = \sum_{j=1}^k \phi_j P_{\mathcal{E}_{\eta_j}}$, where $P_{\mathcal{E}_{\eta_j}}$ denotes the orthogonal projection of TM onto $\mathcal{E}_{\eta_j}$.

Definition. Given a parallel flat normal subbundle $\mathcal{N}$ on a k–Dupin submanifold $f\colon M^n \to \mathbb{R}^{n+p}$, the $\mathcal{N}$-Ribaucour transform $\mathcal{R}_w^{\mathcal{N}}(f)$ of f determined by $w \in \mathcal{S}_{\mathcal{N}}$ is of *Dupin type* if Φ_w is a Dupin tensor adapted to $\mathcal{E}^f$.

Finally, given $w = [(\varphi, \beta)] \in \mathcal{S}_{\mathcal{N}}$, we call $\mathcal{R}_w^{\mathcal{N}}(f)$ *regular* if $\mathcal{N} \neq 0$ and the vector fields $\bar{\beta} = -\varphi^{-1}\beta_{\mathcal{N}^\perp}, (\eta_1)_{\mathcal{N}^\perp}, \ldots, (\eta_k)_{\mathcal{N}^\perp}$, are everywhere distinct. Notice that regularity implies that $\mathcal{E}(w) = 0$.

We are now in a position to prove the main result in this section. The assumption of regularity in the direct statement is only needed to assure that the number of principal normals of the submanifold generated by the $\mathcal{N}$-Ribaucour transformation of a $(k-1)$-Dupin submanifold is nowhere less than k.

THEOREM 5.1. *Let $h\colon L^{n-s}\to\mathbb{R}^{n+p}$ be a simply connected $(k-1)$–Dupin submanifold with a parallel flat normal subbundle $\mathcal{N}$ of rank s. Then any regular $\mathcal{N}$-Ribaucour transform of Dupin type of h is an n–dimensional weakly reducible k–Dupin submanifold in an open neighborhood of h.*

Conversely, let $f\colon M^n\to\mathbb{R}^{n+p}$ be a weakly reducible k–Dupin submanifold. Then there exists a $(k-1)$-Dupin submanifold $h\colon L^{n-s}\to\mathbb{R}^{n+p}$ and a parallel flat normal subbundle $\mathcal{N}$ of h with rank s such that f is locally a regular $\mathcal{N}$-Ribaucour transform of h of Dupin type.

Proof. Let $f=\mathcal{R}^{\mathcal{N}}_w(h)$ be a regular $\mathcal{N}$-Ribaucour transform of Dupin type of h. By Proposition 3.1(iv), the principal normals of f are $\mathcal{P}_t\bar\beta$ and

$$\bar\eta_j=(\lambda_j^t)^{-1}\mathcal{P}_t(\eta_j-2\varphi\nu_t\rho_j^t\bar\beta)_{\mathcal{N}^\perp},\tag{5.2}$$

where $w=[(\varphi,\beta)]$, $\eta_1,\dots,\eta_{k-1}$ are the principal normals of h, the ρ_j^t are the eigenvalues of Φ_t and the $\lambda_j^t=1-2\varphi\nu_t\rho_j^t$ are the ones of D_t. Since $\lim_{t\to\infty}\bar\eta_j=(\eta_j)_{\mathcal{N}^\perp}$, we have from the regularity assumption that $\bar\eta_1,\dots,\bar\eta_{k-1}$ and $\mathcal{P}_t\bar\beta$ are pairwise distinct on an open neighborhood U of the section at infinity of $\mathcal{N}$, and that $\mathcal{E}_{\bar\eta_j}=\mathcal{E}_{\eta_j}$. A long but straightforward computation shows that

$$(\lambda_j^t)^2\nabla^\perp_{X_j}\bar\eta_j=\mathcal{P}_t(\lambda_j^t\nabla^\perp_{X_j}\eta_j-2\varphi\nu_tX_j(\rho_j)(\bar\beta-\eta_j)_{\mathcal{N}^\perp}),\tag{5.3}$$

where $X_j\in\mathcal{E}_{\eta_j}$. We have that $\nabla^\perp_{X_j}\eta_j=0$ because h is a $(k-1)$–Dupin submanifold, and that $X_j(\rho_j)=0$ because Φ_w is a Dupin tensor. It follows from (5.3) that $\nabla^\perp_{X_j}\bar\eta_j=0$, and hence $\bar\eta_j=0$ is a Dupin principal normal for $1\le j\le k-1$. Moreover, $\mathcal{P}_t\bar\beta$ is also a Dupin principal normal and has integrable conullity by Proposition 3.2. Therefore $f|_U$ is an n–dimensional weakly reducible k–Dupin submanifold.

Conversely, let $f\colon M^n\to\mathbb{R}^{n+p}$ be a k–Dupin submanifold and let η_k be a principal normal of f such that $\mathcal{E}^\perp_{\eta_k}$ is integrable. For a leaf L^{n-s} of $\mathcal{E}^\perp_{\eta_k}$, it is easy to see that $h=f|_{L^{n-s}}$ has flat normal bundle with principal normals $\bar\eta_1,\dots,\bar\eta_{k-1}$ given by $\bar\eta_j=\eta_j+H_j$, where H_j is the mean curvature vector of $\mathcal{E}_{\eta_j}$. Since η_j is parallel in the normal connection of f, we have that

$$\tilde\nabla_{X_j}\bar\eta_j=-\|\eta_j\|^2X_j+\nabla_{X_j}H_j+\alpha_f(X_j,H_j).\tag{5.4}$$

The right hand side of (5.4) has no $\mathcal{E}^\perp_{\eta_j}$-component since the last term vanishes and $\mathcal{E}_{\eta_j}$ is spherical. Hence h is a $(k-1)$–Dupin submanifold. By Theorem 3.3, we have that $\mathcal{N}=\mathcal{E}_{\eta_k}|_{L^{n-s}}$ is a parallel flat normal subbundle of h and that $f(M)$ is locally an open neighborhood of a regular $\mathcal{N}$-Ribaucour transform of h. It follows from (5.3) that $X_j(\rho_j)=0$, $1\le j\le k-1$, that is, the $\mathcal{N}$-Ribaucour transform is of Dupin type. ☐

Given a k–Dupin submanifold f and $[(\varphi,\beta)]\in\mathcal{S}_0(f)$, we say that a Ribaucour transform $\mathcal{R}_{[(\varphi,\beta)]}(f)$ is *regular* if $\lambda_j^{-1}(\eta_j-\bar\beta)$, $1\le j\le k$, are everywhere nonzero and pairwise distinct, where the λ_j are the eigenvalues of $D_{[(\varphi,\beta)]}$ and $\bar{\beta} = -\varphi^{-1}\beta$. As a consequence of the proof of Theorem 5.1, we obtain the following characterization of the Ribaucour transformations that preserve the class of k–Dupin submanifolds.

Corollary 5.2. *A regular Ribaucour transform of a k–Dupin submanifold is also a k–Dupin submanifold if and only if it is of Dupin type.*

Proof. We have from (5.2) that $\bar{\eta}_i - \bar{\eta}_j = \mathcal{P}(\lambda_i^{-1}(\eta_i - \bar{\beta}) - \lambda_j^{-1}(\eta_j - \bar{\beta})_{\mathcal{N}^\perp})$, and the result follows. □

Corollary 5.2 generalizes Theorem 2.8 in [CFT], where it was proved for holonomic Dupin hypersurfaces. In particular, it shows that the class of k-Dupin submanifolds is invariant under $\mathcal{L}$-transformations. In view of Theorem 4.2, we have also the following consequence of Theorem 5.1.

Corollary 5.3. *A submanifold that is $\mathcal{L}$-equivalent to (the stereographic projection of) a generalized cylinder over a submanifold $h\colon\ L^{n-s} \to \mathbb{Q}_\epsilon^{n+p}$ is a k–Dupin submanifold if and only if h is a $(k-1)$–Dupin submanifold and the regularity condition is satisfied.*

Definition. A k–Dupin submanifold is *$\mathcal{L}$-reducible* if it is $\mathcal{L}$-equivalent to (the stereographic projection of) a generalized cylinder over a (k–1)–Dupin submanifold in $\mathbb{Q}_\epsilon^{n+p}$.

By Proposition 4.6, the class of $\mathcal{L}$-reducible k–Dupin submanifolds includes the ones that are $\mathcal{L}$-equivalent to those obtained as in any one of Pinkall's examples by starting with a (k–1)–Dupin submanifold of arbitrary codimension, which we call *reducible*. Clearly, for Dupin hypersurfaces this coincides with the usual notion of reducibility. Thus, we have the following implications for k–Dupin submanifolds; the validity of their converses is discussed at the end of this section:

$$\text{Reducible} \Longrightarrow \mathcal{L}\text{-reducible} \Longrightarrow \text{Weakly reducible.} \tag{5.5}$$

One main application of Theorem 5.1 is for the class of holonomic k–Dupin submanifolds. Observe that starting in Theorem 5.1 with a holonomic (k–1)–Dupin submanifold yields a holonomic k–Dupin submanifold, for we have seen that $\mathcal{E}_{\bar{\eta}_j} = \mathcal{E}_{\eta_j}$. Conversely, holonomic k–Dupin submanifolds are constructed from holonomic $(k-1)$–Dupin submanifolds. Therefore, Theorem 5.1 provides the inductive step for a recursive procedure to construct all holonomic Dupin submanifolds. We derive next an explicit coordinate description of this construction.

For our construction we have to use a principal system of coordinates on a holonomic k–Dupin submanifold which we call a *natural coordinate system.* By that we mean that the coordinates for each spherical leaf of $\mathcal{E}_{\eta_j}$ for $1 \le j \le k$

are conformal. In fact, the recursive construction given by Theorem 5.1 yields such coordinates. To see this, observe that the parametrization of the leaves of $\mathcal{E}_{\eta_k}$ for the generated principal normal η_k is conformal by Proposition 3.1(v). Since $D_t|_{\mathcal{E}_{\eta_i}} = \lambda_i^t I$, $1 \le i \le k-1$, the parametrization of the spherical leaves of $\mathcal{E}_{\eta_i}$ remains conformal under the transformation.

Let $h\colon L^{n-s} \to \mathbb{R}^{n+p}$ be a holonomic $(k-1)$–Dupin submanifold endowed with a natural coordinate system $(u_1, \ldots, u_{n-s})$. For the statements of the next results, we agree that $1 \le i, j, \ell \le n-s$, $1 \le m \le k-1$ and $1 \le r \le p$. For each index i, let i' with $1 \le i' \le k-1$ be such that $\partial/\partial u_i \in \mathcal{E}_{\eta_{i'}}$. Set $v_{i'} = \|\partial/\partial u_i\|$ and $h_{jm} = v_{j'}^{-1}\partial v_m/\partial u_j$. Given a parallel orthonormal normal frame $\xi_1, \ldots, \xi_{s+p}$, we define $V_{i'}^r$ by

$$A_{\xi_r}\partial/\partial u_i = v_{i'}^{-1} V_{i'}^r\, \partial/\partial u_i.$$

We call (v, h, V), where $v = (v_1, \ldots, v_{k-1})$, $h = (h_{im})$ and $V = (V_m^r)$, the *triple associated to* h with respect to the coordinates $(u_1, \ldots, u_{n-s})$ and the normal frame $\xi_1, \ldots, \xi_{s+p}$. We first prove the following fact.

LEMMA 5.4. *The triple* (v, h, V) *satisfies the completely integrable system of partial differential equations*

$$\text{(I)}\quad \begin{cases} \text{(i) } \dfrac{\partial v_m}{\partial u_j} = h_{jm} v_{j'}, \quad \text{(ii) } \dfrac{\partial h_{ij'}}{\partial u_i} + \dfrac{\partial h_{ji'}}{\partial u_j} + \displaystyle\sum_\ell h_{\ell i'} h_{\ell j'} + \sum_r V_{i'}^r V_{j'}^r = 0, \\ \text{(iii) } \dfrac{\partial h_{im}}{\partial u_j} = h_{ij'} h_{jm}, \quad \text{(iv) } \dfrac{\partial V_m^r}{\partial u_j} = h_{jm} V_{j'}^r, \end{cases}$$

where $\ell' \neq i' \neq j' \neq \ell'$ *in* (ii) *and* $i' \neq m$ *in* (iii). *Conversely, let* (v, h, V) *be a solution of* (I) *on a simply connected open subset* $U \subset \mathbb{R}^{n-s}$ *such that* $v_m \neq 0$ *everywhere. Then there exists a* $(k-1)$*–Dupin submanifold* $h\colon U \to \mathbb{R}^{n+p}$ *such that the standard coordinates* $(u_1, \ldots, u_{n-s})$ *are natural coordinates for* h *and* (v, h, V) *is the triple associated to* h *with respect to these coordinates and some parallel orthonormal frame.*

Proof. Equations (i) are merely the definition of h_{jm}. From Lemma 1 in [DT1] we have

$$\nabla_{\partial/\partial u_i} v_{j'}^{-1}\partial/\partial u_j = v_{i'}^{-1} h_{ji'} \partial/\partial u_i \quad \text{for all } i \neq j. \tag{5.6}$$

Using this, the remaining equations, except for (iii) when $j' = m \neq i'$ and (iv) when $m = j'$, follow by computing the Gauss and Codazzi equations of h. In order to prove that (iii) also holds for $j' = m \neq i'$, let H_m be the mean curvature vector of $\mathcal{E}_{\eta_m}$. Then we obtain using (5.6) that

$$\begin{aligned}\langle \nabla_{\partial/\partial u_j} H_m, \partial/\partial u_i\rangle &= v_{i'}\partial\langle H_m, v_{i'}^{-1}\partial/\partial u_i\rangle/\partial u_j = -v_{i'}\partial(v_m^{-1}h_{im})/\partial u_j \\ &= -v_{i'}v_m^{-1}(\partial h_{im}/\partial u_j - h_{ij'}h_{jm}),\end{aligned}$$

and our claim follows from the fact that $\mathcal{E}_{\eta_m}$ is spherical. Finally, (iv) for $m = j'$ follows from $0 = \partial(V^r_{j'} v^{-1}_{j'})/\partial u_j = v^{-1}_{j'} \left(\partial V^r_{j'}/\partial u_j - V^r_{j'} h_{jj'}\right)$, where we have used (i) for $m = j'$.

Conversely, we have from Proposition 3 in [DT1] that there exists a holonomic submanifold $h\colon U \to \mathbb{R}^{n+p}$ such that (v, h, V) is the triple associated to h with respect to the standard coordinates $(u_1, \ldots, u_{n-s})$ and some parallel orthonormal frame. Since $\partial(v^{-1}_{j'} V^r_{j'})/\partial u_j = 0$ from (iv), it follows that h is a $(k-1)$–Dupin submanifold and that the standard coordinates are natural. □

For our coordinate description of the holonomic Dupin submanifolds we also need the following fact.

Lemma 5.5. *Let $h\colon L^{n-s} \to \mathbb{R}^{n+p}$ be a holonomic $(k-1)$–Dupin submanifold, and let (v, h, V) be the triple associated to h with respect to natural coordinates $(u_1, \ldots, u_{n-s})$ and some parallel orthonormal normal frame. Then the system of total partial differential equations*

$$\partial B_m/\partial u_j = h_{jm} B_{j'} \tag{5.7}$$

is completely integrable. Moreover, if $(B_1, \ldots, B_{k-1})$ is a solution of (5.7), then the system of total partial differential equations

$$\begin{cases} \text{(i)}\ \dfrac{\partial \varphi}{\partial u_i} = v_{i'}\gamma_i, \quad \text{(ii)}\ \dfrac{\partial \gamma_j}{\partial u_i} = h_{ji'}\gamma_i, \quad i \neq j, \\ \text{(iii)}\ \dfrac{\partial \gamma_i}{\partial u_i} = B_{i'} - \displaystyle\sum_{j, j' \neq i'} h_{ji'}\gamma_i + \sum_r \beta_r V^r_{i'}, \quad \text{(iv)}\ \dfrac{\partial \beta_r}{\partial u_i} = -V^r_{i'}\gamma_i, \end{cases} \tag{5.8}$$

is also completely integrable.

Proof. An easy computation shows that the compatibility conditions of (5.7) follow from (I)(iii). The compatibility conditions of (5.8) can be verified by a straightforward computation using (I) and (5.7). □

To simplify the statement of the next result, we call a solution (φ, γ, β) of (5.8) *generic* if the vectors $-\sum_{r=s+1}^{s+p} \varphi^{-1}\beta_r \xi_r$ and $\sum_{r=s+1}^{s+p} v^{-1}_m V^r_m \xi_r$, $1 \leq m \leq k-1$, are everywhere pairwise distinct.

Theorem 5.6. *Let $h\colon L^{n-s} \to \mathbb{R}^{n+p}$ be a holonomic $(k-1)$–Dupin submanifold endowed with natural coordinates. If (φ, γ, β) is a generic solution of (5.8), then the map $f\colon L^{n-s} \times \mathbb{R}^s \to \mathbb{R}^{n+p}$ given by (3.18) is, at regular points, a holonomic k–Dupin submanifold.*

Conversely, if $f\colon M^n \to \mathbb{R}^{n+p}$ is a holonomic k–Dupin submanifold and η_ℓ is any of its principal normals, then $h = f|_L$ is a holonomic $(k-1)$–Dupin submanifold for any leaf L^{n-s} of its conullity, and there exists a solution (φ, γ, β) of (5.8) such that f can be parametrized by (3.18).

Proof. It is easily seen that (φ,γ,β) being a solution of (5.8) and B_1, ..., B_{k-1} a solution of (5.7) is equivalent to the tensor $\Phi = \operatorname{Hess}\varphi - A_\beta$ being a Dupin tensor adapted to $\mathcal{E}_h$. Therefore, f parametrizes the $\mathcal{N}$-Ribaucour transform $\mathcal{R}_w^{\mathcal{N}}(h)$ of Dupin type of h determined by $w = [(\varphi,\beta)] \in \mathcal{S}_{\mathcal{N}}$, where $\beta = \sum_r \beta_r\xi_r$ and $\mathcal{N}$ is the parallel flat normal subbundle of h spanned by $\xi_1,\ldots,\xi_s$. Moreover, the solution (φ,γ,β) of (5.8) being generic is equivalent to the $\mathcal{N}$-Ribaucour transform $\mathcal{R}_w^{\mathcal{N}}(h)$ being regular. The result now follows from Theorem 5.1. □

In order to derive a sufficient condition for a k–Dupin submanifold to be holonomic, we define the *local conformal codimension* of an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ as

$$c_\ell(f) = \min\{c(f|_U) : U \subset M^n \text{ is open}\}.$$

Recall that f is called 1-*regular* if the first normal spaces

$$N_1^f(x) = \operatorname{span}\{\alpha_f(X,Y) : X,Y \in T_xM\}$$

have constant dimension.

PROPOSITION 5.7. *If f is a 1-regular k–Dupin submanifold, then $c(f) \leq k-1$. Moreover, if $c_\ell(f) = k-1$, then f is holonomic.*

Proposition 5.7 is an easy consequence of the following results.

LEMMA 5.8. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ be a proper isometric immersion with flat normal bundle and principal normals $\eta_1,\ldots,\eta_k$. Then η_ℓ has integrable conullity if the vectors $\eta_i - \eta_\ell$ and $\eta_j - \eta_\ell$ are everywhere linearly independent for any pair of indices $1 \leq i \neq j \leq k$ with $i,j \neq \ell$.*

Proof. The Codazzi equation implies that

$$\langle \nabla_{X_i}X_j, X_\ell\rangle(\eta_j - \eta_\ell) = \langle \nabla_{X_j}X_i, X_\ell\rangle(\eta_i - \eta_\ell) \tag{5.9}$$

for all unit vectors $X_i \in \mathcal{E}_{\eta_i}$, $X_j \in \mathcal{E}_{\eta_j}$ and $X_\ell \in \mathcal{E}_{\eta_\ell}$. □

LEMMA 5.9. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ be a proper isometric immersion with flat normal bundle and principal normals $\eta_1,\ldots,\eta_k$. At $x \in M^n$ set*

$$S_f(x) = \operatorname{span}\{\eta_i(x) - \eta_j(x) : 1 \leq i,j \leq k\}.$$

Then $\dim S_f(x) \leq k-1$, and f is holonomic if equality holds everywhere.

Proof. The first assertion follows from $S_f = \operatorname{span}\{\eta_j - \eta_\ell,\ 1 \leq j \leq k\}$ for any fixed $1 \leq \ell \leq k$. If $\dim S_f(x) = k-1$ everywhere, then Lemma 5.8 implies that the conullity $\mathcal{E}_{\eta_\ell}^\perp$ is integrable for any $1 \leq \ell \leq k$, and the second assertion is a consequence of Theorem 1 in [RS]. □

LEMMA 5.10. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ be a 1-regular connected k–Dupin submanifold with* $\dim N_1^f = s$. *Then $f(M)$ is substantially contained in an affine subspace $\mathbb{R}^{n+s}$. If $S_f(x)$ has constant dimension, then either $S_f = N_1^f$ everywhere or* $\dim S_f = s-1$. *Moreover,* $\dim S_f = s-1 \geq 0$ *if and only if $f(M)$ is contained in a sphere $\mathbb{S}^{n+s-1} \subset \mathbb{R}^{n+s}$. In particular, we have* $c(f) = \dim S_f$.

Proof. The Codazzi equation yields

$$\nabla^{\perp}_{X_j}\eta_i = \langle \nabla_{X_i}X_i, X_j\rangle(\eta_i - \eta_j) \quad \text{if} \quad i \neq j, \tag{5.10}$$

where $X_i \in \mathcal{E}_{\eta_i}$ and $X_j \in \mathcal{E}_{\eta_j}$ are unit vectors fields. Since f is Dupin, the normal vector subbundle N_1^f is parallel in the normal connection, and the first assertion follows.

At any point we have that

$$\dim N_1^f(x) - 1 \leq \dim S_f(x) \leq \dim N_1^f(x). \tag{5.11}$$

Assume that $\dim S_f = s - 1 \geq 0$. Our claim is trivial for $s = 1$, thus suppose that $s \geq 2$. The principal curvatures corresponding to a normal vector field η are $\langle \eta, \eta_j\rangle$ for $1 \leq j \leq k$. Hence, a smooth unit vector field ξ spanning the orthogonal complement of S_f in N_1^f is an umbilical vector field. For $i \neq j$ we have from (5.10) that

$$\begin{aligned} 0 &= \langle \nabla^{\perp}_{X_j}\eta_i, \xi\rangle = X_j\langle \eta_i, \xi\rangle - \langle \eta_i, \nabla^{\perp}_{X_j}\xi\rangle = X_j\langle \eta_j, \xi\rangle - \langle \eta_i, \nabla^{\perp}_{X_j}\xi\rangle \\ &= \langle \eta_j - \eta_i, \nabla^{\perp}_{X_j}\xi\rangle. \end{aligned}$$

Thus $\nabla^{\perp}_{X_j}\xi \in N_1^f$ is orthogonal to S_f, and hence must vanish. Therefore, ξ is parallel in the normal connection, and the last assertion follows. □

Proof of Proposition 5.7. The first claim is an easy consequence of Lemma 5.10 because, if $\dim N_1 = k$, then (5.11) implies that $\dim S_f = k-1$ everywhere. By Lemma 5.10 the hypothesis on $c_\ell(f)$ now forces S_f to have constant dimension $k-1$, and the second claim follows from Lemma 5.9. □

The next result shows that a k–Dupin submanifold must be weakly reducible if its conformal codimension is sufficiently high.

PROPOSITION 5.11. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ be a locally weakly irreducible k-Dupin submanifold. Then* $c(f) \leq (2/3)k - 1$ *on each connected component of an open dense subset of M^n.*

Proof. On an open subset $U \subset M^n$ where f is 1-regular and S_f has constant dimension, using Lemma 5.8 we have that for each $1 \leq \ell \leq k$ there is an (affine) line L_ℓ which contains η_ℓ and at least two more principal normals. The estimate for $c(f|_U)$ now follows easily from Lemma 5.10 since S_f is the affine space generated by these lines. □

The following example shows that the estimate in the last result is sharp.

Example. Take the product immersion of ℓ copies of an irreducible isoparametric hypersurface $M^n \subset \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ with three distinct principal curvatures. This is a conformally substantial weakly irreducible $n\ell$-dimensional submanifold in a sphere of dimension $(n+2)\,\ell - 1$ for which equality in the estimate holds.

We now discuss whether the converses hold in (5.5). First we show that the converse is false in the first implication, even for hypersurfaces.

Proposition 5.12. *There exist k-Dupin hypersurfaces with $k \geq 4$ that are $\mathcal{L}$-reducible but not reducible.*

Proof. Equation (3.15) shows that if a k–Dupin hypersurface is weakly reducible with respect to a principal curvature λ, and $\tilde{\lambda}$ is the corresponding principal curvature of an $\mathcal{L}$-transform of it, then the conullity leaves of λ and $\tilde{\lambda}$ correspond under the $\mathcal{L}$-transformation. Since the conformal codimension of the conullity leaves of a principal curvature generated by any one of Pinkall's examples is one and the conformal codimension is invariant under $\mathcal{L}$-transformations by Proposition 2.3, it follows that all the conullity leaves of a principal curvature of a reducible Dupin hypersurface have conformal codimension one. Thus, a tube over a weakly irreducible $(k-1)$–Dupin submanifold $h : L^{n-s} \to \mathbb{R}^{n+1}$ with $k \geq 4$ and $c(h) \geq 2$ is an $\mathcal{L}$-reducible k–Dupin hypersurface that is not reducible. The following well-known fact shows that any irreducible (as a Riemannian manifold) isoparametric submanifold with conformal codimension at least two can be taken as such an h. $\square$

Proposition 5.13. *Any locally irreducible (as a Riemannian manifold) isoparametric submanifold is weakly irreducible.*

Proof. Let $\eta_1, \ldots, \eta_k$ denote the principal normals of an isoparametric submanifold. For any principal normal η_ℓ, the Codazzi equation (5.10) and the fact that η_ℓ is parallel in the normal connection imply that $\mathcal{E}_{\eta_\ell}$ is totally geodesic. On the other hand, if $\mathcal{E}_{\eta_\ell}^{\perp}$ is integrable, then the expressions under parenthesis in (5.9) coincide. Since $k \geq 3$ by the assumption, it follows that both must vanish. Thus $\mathcal{E}_{\eta_\ell}^{\perp}$ is also totally geodesic, and the de Rham Theorem yields a contradiction. $\square$

For 3–Dupin hypersurfaces, however, the three notions of reducibility do coincide. In fact, a weakly reducible 3–Dupin hypersurface can not be Lie equivalent to an isoparametric hypersurface in the sphere by Proposition 5.13, hence the main result of Cecil and Jensen in [CJ] implies that it must be reducible.

We do not have an explicit example showing that the converse is false also in the second implication in (5.5). However, we prove the following result.

PROPOSITION 5.14. *For any $k \geq 4$ there exists a holonomic k–Dupin submanifold (hence weakly reducible with respect to every principal normal) that is not $\mathcal{L}$-reducible with respect to some principal normal.*

By Theorem 5.1, in order to prove Proposition 5.14 it suffices to show that for any $k \geq 4$ there exists a $(k-1)$–Dupin submanifold $h\colon M^n \to \mathbb{R}^{n+p}$ that carries a nontrivial Dupin tensor adapted to $\mathcal{E}^h$. In fact, we prove that any holonomic k–Dupin submanifold h that satisfies $c(h) \leq k-2$ has this property. This is done by comparing the dimension of the vector space of $\mathcal{L}$-trivial tensors on M^n with that of Dupin tensors on M^n that are (locally) adapted to $\mathcal{E}^h$. The former is clearly equal to $c(h)+1$ for any k–Dupin submanifold h. The latter is computed next for holonomic submanifolds.

PROPOSITION 5.15. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ be a holonomic k–Dupin submanifold. For any $p_0 \in M^n$ there exist an open neighborhood U of p_0 and a unique Dupin tensor Φ on U adapted to $\mathcal{E}^f$ such that $\Phi(p_0) = \sum_{m=1}^k \phi_m^0 P_{\mathcal{E}_{\eta_m}}(p_0)$ for given $(\phi_1^0,\ldots,\phi_k^0) \in \mathbb{R}^k$. In particular, the vector space of Dupin tensors on U adapted to f has dimension k.*

Proof. Let $U \subset M^n$ be a simply connected neighborhood of p_0 endowed with natural coordinates and let $\phi_1,\ldots,\phi_k$ be smooth functions on U. It is easily checked that the tensor $\Phi = \sum_{m=1}^k \phi_m P_{\mathcal{E}_{\eta_m}}$ is a Dupin tensor on U if and only if the functions $B_m = v_m\phi_m$ satisfy system (5.7). The result then follows from the first assertion of Lemma 5.5. □

We conclude the paper with some consequences of our previous results for 3–Dupin and 4–Dupin submanifolds.

PROPOSITION 5.16. *Any nonholonomic* 3*–Dupin submanifold is Lie equivalent to the stereographic projection of an isoparametric hypersurface in the sphere.*

Proof. Since any 2–Dupin submanifold is holonomic, it follows from Theorem 5.1 that a nonholonomic 3–Dupin submanifold must be weakly irreducible. Moreover, it must also have local conformal codimension one by Proposition 5.7. Therefore, it is (locally) irreducible, and hence Lie equivalent to the stereographic projection of an isoparametric hypersurface in the sphere by the result of Cecil and Jensen [CJ]. □

For 4–Dupin submanifolds the situation is far more complex, even globally. As mentioned in the introduction, there are examples of compact 4–Dupin hypersurfaces that are neither weakly reducible nor Lie equivalent to isoparametric hypersurfaces. However, we have the following result for the weakly reducible case.

THEOREM 5.17. *Any weakly reducible nonholonomic 4–Dupin submanifold is $\mathcal{L}$-equivalent to the stereographic projection of a generalized cylinder over a hypersurface that is Lie equivalent to an isoparametric hypersurface in the sphere.*

Proof. Let $h : L^{n-s} \to \mathbb{R}^{n+p}$ be a 3–Dupin submanifold such that $f = \mathcal{R}^{\mathcal{N}}_w(h)$ is not holonomic. The Codazzi equation for the Dupin tensor Φ_w in terms of its eigenvalues is

$$\begin{cases} \text{(i)} \;\; X_j\phi_i + \langle \nabla_{X_i} X_j, X_i\rangle(\phi_i - \phi_j) = 0, \\ \text{(ii)} \;\; \langle \nabla_{X_i} X_j, X_k\rangle(\phi_j - \phi_k) = \langle \nabla_{X_j} X_i, X_k\rangle(\phi_i - \phi_k), \end{cases} \tag{5.12}$$

where $X_\ell \in \mathcal{E}_{\eta_\ell}$ and $1 \leq i \neq j \neq k \neq i \leq 3$. Since (5.12) also holds for any shape operator $A \neq 0$ in the direction of a parallel normal vector field, it follows easily from (5.12)(ii) and the fact that not all functions $\langle \nabla_{X_i} X_j, X_k\rangle$ can vanish that $\Phi_w = aI + bA$ for some smooth functions a and b. We obtain from (5.12)(i) that $a, b \in \mathbb{R}$. Therefore f is $\mathcal{L}$-equivalent to the stereographic projection of a generalized cylinder over a 3–Dupin submanifold by Theorem 4.2. If such a submanifold were holonomic, the same would be true for f. The conclusion now follows from Proposition 5.16. □

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