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DE RHAM INTERSECTION COHOMOLOGY FOR GENERAL PERVERSITIES

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ABSTRACT. For a stratified pseudomanifold X, we have the de Rham Theorem $\mathbb{H}^*_{\overline{p}}(X) = \mathbb{H}^{\overline{t}-\overline{p}}_*(X)$, for a perversity $\overline{p}$ verifying $\overline{0} \leq \overline{p} \leq \overline{t}$, where $\overline{t}$ denotes the top perversity. We extend this result to any perversity $\overline{p}$. In the direction cohomology $\mapsto$ homology, we obtain the isomorphism

$$\mathbb{H}^*_{\overline{p}}(X) = \mathbb{H}^{\overline{t}-\overline{p}}_*(X, X_{\overline{p}}),$$

where

$$X_{\overline{p}} = \bigcup_{\substack{S \preceq S_1 \\ \overline{p}(S_1)<0}} S = \bigcup_{\overline{p}(S)<0} \overline{S}.$$

In the direction homology $\mapsto$ cohomology, we obtain the isomorphism

$$\mathbb{H}^{\overline{p}}_*(X) = \mathbb{H}^*_{\max(\overline{0},\overline{t}-\overline{p})}(X).$$

In our paper stratified pseudomanifolds with one-codimensional strata are allowed.

Roughly speaking, a stratified pseudomanifold X is a family $\mathcal{S}_X$ of smooth manifolds (strata) assembled in a conical way. A (general) perversity $\overline{p}$ associates an integer to each of the strata of X (see [15]). The classical perversities (see [12], [14], [11], …) are filtration-preserving, that is, they verify:

$$S_1, S_2 \in \mathcal{S}_X \text{ with } \dim S_1 = \dim S_2 \Rightarrow \overline{p}(S_1) = \overline{p}(S_2).$$

The zero-perversity, defined by $\overline{0}(S) = 0$, and the top perversity, defined by $\overline{t}(S) = \operatorname{codim}_X S - 2$, are classical perversities.

The singular intersection homology $\mathbb{H}^{\overline{p}}_*(X)$ was introduced by Goresky-MacPherson in [13] (see also [14]). It is a topological invariant of the stratified pseudomanifold when the perversity satisfies some monotonicity conditions (see [12], [14], …). In particular, we need $\overline{0} \leq \overline{t}$ and therefore X does not possess any one-codimensional strata. Recently, a more general result has been obtained in [11], where one-codimensional strata are allowed. In all these cases, the perversities are classical.

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The de Rham intersection cohomology $\mathbb{H}^{*}_{\overline{p}}(X)$ was also introduced by Goresky-MacPherson (see [7]). It requires the existence of a Thom-Mather neighborhood system. Other versions exist, but in each case an additional notion is needed in order to define this cohomology: a Thom-Mather neighborhood system ([7], [4] ...,) a Riemannian metric ([10], [17], [3], ...), a PL-structure ([1], [9], ...), a blow-up ([2], ...), etc.

The perverse de Rham Theorem

$$\mathbb{H}^{*}_{\overline{p}}(X) = \mathbb{H}_{*}^{\overline{t}-\overline{p}}(X), \tag{1}$$

relates the intersection homology with the intersection cohomology. It was first proved by Brylinski in [7]; later proofs have been given in the above references. The involved perversities are classical perversities verifying some monotonicity conditions. Moreover, the perversity $\overline{p}$ must lie between $\overline{0}$ and $\overline{t}$; this excludes the existence of one-codimensional strata on X.

The first proof of the de Rham Theorem for the general perversities has been given by the author in [18] using the integration $\int$ of differential forms on simplices. Unfortunately, there is a mistake in the statement of Proposition 2.1.4 and Proposition 2.2.5: the hypothesis $\overline{p} \leq \overline{t}$ must be added. As a consequence, the main result[1] of [18] (de Rham Theorem 4.1.5) is valid for a general perversity $\overline{p}$ *verifying the condition* $\overline{0} \leq \overline{p} \leq \overline{t}$. In particular, we have (1) for a general perversity $\overline{p}$ with $\overline{0} \leq \overline{p} \leq \overline{t}$. Notice that one-codimensional strata are not allowed.

In this work we prove a de Rham Theorem for any general perversity $\overline{p}$[2]. The formula (1) changes! We obtain that, in the direction cohomology $\mapsto$ homology, the integration $\int$ induces the isomorphism

$$\mathbb{H}^{*}_{\overline{p}}(X) = \mathbb{H}_{*}^{\overline{t}-\overline{p}}\big(X, X_{\overline{p}}\big),$$

where

$$X_{\overline{p}} = \bigcup_{\substack{S \preceq S_1 \\ \overline{p}(S_1)<0}} S = \bigcup_{\overline{p}(S)<0} \overline{S}$$

(cf. Theorem 3.2.2). In the direction homology $\mapsto$ cohomology, we have the isomorphism

$$\mathbb{H}_{*}^{\overline{p}}(X) = \mathbb{H}^{*}_{\max(\overline{0},\overline{t}-\overline{p})}(X)$$

(cf. Corollary 3.2.5).

We end the work by noticing that the Poincaré duality of [7] and [18] is still valid in our context.

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[1] The statement of the second main result (the Poincaré duality, Theorem 4.2.7) does not need any modification since Propositions 2.1.4 and 2.2.5 are not used for its proof.

[2] The one codimensional strata are finally allowed!

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1. Stratified spaces and unfoldings

We present the geometrical framework of this work, that is, the stratified pseudomanifolds and the unfoldings. For a more complete study of these notions, we refer the reader to, for example, [13] and [18].

In the sequel, any manifold is connected, second countable, Hausdorff, without boundary, and smooth (of class C^∞).

1.1. Stratifications. A *stratification* of a paracompact space X is a locally finite partition $\mathcal{S}_X$ of X into disjoint smooth manifolds, called *strata*, such that

$$S \cap \overline{S'} \neq \emptyset \Longleftrightarrow S \subset \overline{S'};$$

it is denoted by $S \preceq S'$. Notice that $(\mathcal{S}_X, \preceq)$ is a partially ordered set. A subset of X is *saturated* when it is an union of a family of strata.

We say that X is a *stratified space*. The *depth* of X, denoted by $\operatorname{depth} X$, is the length of the maximal chain contained in X. It is always finite because of the locally finiteness of $\mathcal{S}_X$. The minimal (resp. maximal) strata are the closed (resp. open) strata. The open strata are the *regular strata* and the others are the *singular strata*. We shall denote by $\mathcal{S}_X^{\text{sing}}$ the family of singular strata. The union Σ_X of singular strata is the *singular part*, which is a saturated closed subset. The *regular part* $X - \Sigma_X$ is a saturated open dense subset. We require the regular strata to have the same dimension, denoted by $\dim X$.

For each $i \in \{-1, 0, \ldots, \dim X\}$ we consider the saturated subset

$$X_i = \bigcup \{S \in \mathcal{S}_X \mid \dim S \leq i\}.$$

This gives the *filtration* $\mathcal{F}_X$

$$X_{\dim X} \supset X_{\dim X - 1} \supset X_1 \supset X_0 \supset X_{-1} = \emptyset. \tag{2}$$

The main example of a stratified space is given by the following conical construction. Consider a compact stratified space L and let cL be the *cone* of L, that is, $cL = L \times [0,1[\big/ L \times \{0\}$. The points of cL are denoted by $[x,t]$. The *vertex* of the cone is the point $\vartheta = [x,0]$. This cone is naturally endowed with the following stratification:

$$\mathcal{S}_{cL} = \{\{\vartheta\}\} \cup \{S \times]0,1[\mid S \in \mathcal{S}_L\}.$$

For the filtration $\mathcal{F}_{cL}$ we have

$$(cL)_i = \begin{cases} \{\vartheta\} & \text{if } i = 0, \\ cL_{i-1} & \text{if } i > 0. \end{cases}$$

Notice that $\operatorname{depth} cL = \operatorname{depth} L + 1$.

The *canonical stratification* of a manifold X is the family $\mathcal{S}_X$ formed by the connected components of X. The filtration contains just one non-empty element: $X_{\dim X}$.

A continuous map (resp. homeomorphism) $f\colon Y \to X$ between two stratified spaces is a stratified morphism (resp. *isomorphism*) if it sends the strata of Y to the strata of X smoothly (resp. diffeomorphically).

1.2. Stratified pseudomanifolds. A stratified space X is a *stratified pseudomanifold* when it possesses a conical local structure, that is, when for each point x of a singular stratum S of X there exists a stratified isomorphism $\varphi\colon U \longrightarrow \mathbb{R}^n \times cL_S$, where

(a) $U \subset X$ is an open neighborhood of x endowed with the induced stratification,

(b) L_S is a compact stratified space, called *link* of S,

(c) $\mathbb{R}^n \times cL_S$ is endowed with the stratification

$$\{\mathbb{R}^n \times \{\vartheta\}\} \cup \{\mathbb{R}^n \times S' \times]0,1[\mid S' \in \mathcal{S}_{L_S}\},$$

(d) $\varphi(x) = (0, \vartheta)$.

The pair (U, φ) is a *chart* of X containing x. An *atlas* $\mathcal{A}$ is a family of charts covering X. A stratified pseudomanifold is *normal* when all the links are connected. Notice that in this case each link is a connected normal stratified pseudomanifold.

1.3. Unfoldings. Consider a stratified pseudomanifold X. A continuous map $\mathcal{L}\colon \widetilde{X} \to X$, where $\widetilde{X}$ is a (not necessarily connected) manifold, is an *unfolding* if the two following conditions hold:

1. The restriction $\mathcal{L}_X\colon \mathcal{L}_X^{-1}(X - \Sigma_X) \longrightarrow X - \Sigma_X$ is a local diffeomorphism.
2. There exist a family of unfoldings $\{\mathcal{L}_{L_S}\colon \widetilde{L_S} \to L_S\}_{S \in \mathcal{S}_X^{\mathrm{sing}}}$ and an atlas $\mathcal{A}$ of X such that for each chart $(U, \varphi) \in \mathcal{A}$ there exists a commutative diagram

$$\begin{array}{ccc} \mathbb{R}^n \times \widetilde{L_S} \times]-1,1[& \xrightarrow{\widetilde{\varphi}} & \mathcal{L}_X^{-1}(U) \\ {\scriptstyle Q}\big\downarrow & & {\scriptstyle \mathcal{L}_X}\big\downarrow \\ \mathbb{R}^n \times cL_S & \xrightarrow{\varphi} & U \end{array}$$

where

(a) $\widetilde{\varphi}$ is a diffeomorphism,

(b) $Q(x, \widetilde{\zeta}, t) = (x, [\mathcal{L}_{L_S}(\widetilde{\zeta}), |t|])$.

We say that X is an *unfoldable pseudomanifold*. This definition makes sense because it is made by induction on $\operatorname{depth} X$. When $\operatorname{depth} X = 0$, then $\mathcal{L}_X$ is just a local diffeomorphism. For any singular stratum S the restriction $\mathcal{L}_X \colon \mathcal{L}_X^{-1}(S) \to S$ is a fibration with fiber $\widetilde{L}_S$. The *canonical unfolding* of the cone cL_S is the map $\mathcal{L}_{cL_S} \colon \widetilde{cL_S} = \widetilde{L}_S \times]-1,1[\to cL_S$ defined by $\mathcal{L}_{cL_S}(\widetilde{\zeta}, t) = \left[\mathcal{L}_{L_S}(\widetilde{\zeta}), |t|\right]$.

From now on, $(X, \mathcal{S}_X)$ is a stratified pseudomanifold endowed with an unfolding $\mathcal{L}_X \colon \widetilde{X} \to X$.

1.4. Bredon's trick. The typical result we prove in this work looks like the following statement:

> "The differential operator $f \colon A^*(X) \to B^*(X)$, defined between two differential complexes on X, induces an isomorphism in cohomology."

First, we prove this assertion for charts. The passage from local to global can be accomplished using different tools, such as the following:

- Axiomatic presentation of the intersection homology (the most employed are given in [12], [7], [2], ...).
- Uniqueness of the minimal stratification (used in [14]).
- The generalized Mayer-Vietoris principle of [5, Chapter II] (used in [18]).
- Bredon's trick of [6, p. 289].

In this work we choose the last one, which may be the least technical. The exact statement is the following:

1.4.1. LEMMA. *Let Y be a paracompact topological space and let $\mathcal{U}$ be an open covering, closed under finite intersections. Suppose that $Q(U)$ is a statement about open subsets of Y, satisfying the following three properties:*

(BT1) *$Q(U)$ is true for each $U \in \mathcal{U}$;*

(BT2) *$Q(U)$, $Q(V)$ and $Q(U \cap V) \Longrightarrow Q(U \cup V)$, where U and V are open subsets of Y;*

(BT3) *$Q(U_i) \Longrightarrow Q\left(\bigcup_i U_i\right)$, where $\{U_i\}$ is a disjoint family of open subsets of Y.*

Then $Q(Y)$ is true.

This lemma enables us to prove several analogs of 1.4. They will be "two pass" proofs (see [6, p. 291]):

1. The topological space Y is an open subset of a conical chart of the pseudomanifold X and the open covering $\mathcal{U}$ is an adapted one (see (4)).
2. The topological space Y is the pseudomanifold X and the covering $\mathcal{U}$ is $\{$open subsets of a chart of $X\}$.

2. Intersection homology

The intersection homology was introduced by Goresky-MacPherson in [12], [13]. Here we use the singular intersection homology of [14].

2.1. Perversity. Intersection cohomology requires the definition of a perversity parameter $\overline{p}$. It associates an integer to each singular stratum of X, in other words, a *perversity* is a map $\overline{p}\colon \mathcal{S}_X^{\text{sing}} \to \mathbb{Z}$. The *zero perversity* $\overline{0}$ is defined by $\overline{0}(S) = 0$. The *top perversity* $\overline{t}$ is defined by $\overline{t}(S) = \operatorname{codim}_X S - 2$. Notice that the condition $\overline{0} \leq \overline{t}$ implies $\operatorname{codim}_X S \geq 2$ for each singular stratum S, and therefore the one-codimensional strata are not allowed.

The classical perversities (cf. [12], [13], ...), the loose perversities (cf. [14]), and the superperversities (cf. [8], [11], ...), ... are filtration-preserving maps, that is, $\overline{p}(S) = \overline{p}(S')$ if $\dim S = \dim S'$. They also verify a monotonicity condition and, for some of them, the one-codimensional strata are avoided. For such perversities the associated intersection cohomology is a topological invariant.

In our case, the perversities are stratum-preserving without any constraint. Of course, the topological invariance is lost. But we prove that we have a de Rham duality (between the intersection homology and the intersection cohomology) and the Poincaré duality.

We fix a perversity $\overline{p}$. The homologies and the cohomologies we use in this work are with coefficients in $\mathbb{R}$.

2.2. Intersection homology. First approach. A singular simplex $\sigma\colon \Delta \to X$ is a $\overline{p}$-*allowable* simplex if

(All) $\sigma^{-1}(S) \subset (\dim \Delta - 2 - \overline{t}(S) + \overline{p}(S))$–skeleton of Δ, for each singular stratum S.

A singular chain $\xi = \sum_{j=1}^{m} r_j \sigma_j \in S_*(X)$ is $\overline{p}$-*allowable* if each singular simplex σ_j is $\overline{p}$-allowable. The family of $\overline{p}$-allowable chains is a graded vector space, denoted by $AC_*^{\overline{p}}(X)$. The associated differential complex is the complex of $\overline{p}$-*intersection chains*, that is, $SC_*^{\overline{p}}(X) = AC_*^{\overline{p}}(X) \cap \partial^{-1} AC_{*-1}^{\overline{p}}(X)$. Its homology $\mathbb{H}_*^{\overline{p}}(X) = H_*\left(SC_\cdot^{\overline{p}}(X)\right)$ is the $\overline{p}$-*intersection homology* of X. This is the approach of [14]. The barycentric subdivision of an $\overline{p}$-intersection chain is an $\overline{p}$-intersection chain; so, the intersection homology verifies the Mayer-Vietoris property. It also verifies the product formula $\mathbb{H}_*^{\overline{p}}(\mathbb{R} \times X) = \mathbb{H}_*^{\overline{p}}(X)$ (see [12], [13], [18], ...).

The usual local calculation is the following (cf. [13], see also [14]), which corrects Proposition 2.1.4 of [18].

2.2.1. PROPOSITION. *If L is a compact stratified pseudomanifold, then*

$$\mathbb{H}^{\overline{p}}_{i}(cL) = \begin{cases} \mathbb{H}^{\overline{p}}_{i}(L) & \text{if } i \leq \overline{t}(\vartheta) - \overline{p}(\vartheta), \\ 0 & \text{if } 0 \neq i \geq 1 + \overline{t}(\vartheta) - \overline{p}(\vartheta), \\ \mathbb{R} & \text{if } 0 = i \geq 1 + \overline{t}(\vartheta) - \overline{p}(\vartheta). \end{cases}$$

Proof. For $i \leq 1 + \overline{t}(\vartheta) - \overline{p}(\vartheta)$ we have $SC^{\overline{p}}_{i}(cL) = SC^{\overline{p}}_{i}(L \times]0,1[)$, which gives $\mathbb{H}^{\overline{p}}_{i}(cL) = \mathbb{H}^{\overline{p}}_{i}(L)$ if $i \leq \overline{t}(\vartheta) - \overline{p}(\vartheta)$.

For a singular simplex $\sigma\colon \Delta^i \to cL$ we define the *cone* $c\sigma\colon \Delta^{i+1} \to cL$ by

$$c\sigma(x_0, \ldots, x_{i+1}) = (1 - x_{i+1}) \cdot \sigma\left(\frac{x_0}{1 - x_{i+1}}, \ldots, \frac{x_i}{1 - x_{i+1}}\right).$$

Here, we have written $r \cdot [x, s] = [x, rs]$ for a point $[x, s] \in cL$ and a number $r \in [0, 1]$. In the same way, we define the cone $c\xi$ of a singular chain ξ. It defines the linear operator

$$c\colon AC^{\overline{p}}_{\geq 1+\overline{t}(\vartheta)-\overline{p}(\vartheta)}(cL) \longrightarrow AC^{\overline{p}}_{\geq 2+\overline{t}(\vartheta)-\overline{p}(\vartheta)}(cL). \tag{3}$$

Let us prove this property. We take $\sigma \in AC^{\overline{p}}_{\geq 1+\overline{t}(\vartheta)-\overline{p}(\vartheta)}(cL)$ and prove that $c\sigma \in AC^{\overline{p}}_{\geq 2+\overline{t}(\vartheta)-\overline{p}(\vartheta)}(cL)$. Notice first that, for $x_{i+1} \neq 1$, we have that

$$\underbrace{\left(\frac{x_0}{1 - x_{i+1}}, \ldots, \frac{x_i}{1 - x_{i+1}}\right)}_{\tau(x_0, \ldots, x_{i+1})} \in j - \text{skeleton of } \Delta^i$$

implies that $(x_0, \ldots, x_{i+1}) \in (j+1)$–skeleton of Δ^{i+1}. So, the subset $(c\sigma)^{-1}(\vartheta)$ is the union of $\{(0, \ldots, 0, 1)\}$ with the subset

$$\left\{(x_0, \ldots, x_{i+1}) \in \Delta^{i+1} \mid \ \tau(x_0, \ldots, x_{i+1}) \in \sigma^{-1}(\vartheta) \text{ and } x_{i+1} \neq 1\right\},$$

which is included in the $\{(0, \ldots, 0, 1)\} \cup (i - 2 - \overline{t}(\vartheta) + \overline{p}(\vartheta) + 1)$–skeleton of Δ^{i+1} and therefore in the $(i + 1 - 2 - \overline{t}(\vartheta) + \overline{p}(\vartheta))$–skeleton of Δ^{i+1}, since $i \geq 1 + \overline{t}(\vartheta) - \overline{p}(\vartheta)$. For each singular stratum S of L we have that $(c\sigma)^{-1}(S \times]0, 1[)$ is the subset

$$\left\{(x_0, \ldots, x_{i+1}) \in \Delta^{i+1} \mid \tau(x_0, \ldots, x_{i+1}) \in \sigma^{-1}(S \times]0, 1[) \text{ and } x_{i+1} \neq 1\right\}$$

which is included in the $(i+1-2-\overline{t}(\vartheta)+\overline{p}(\vartheta))$–skeleton of Δ^{i+1}. We conclude that $c\sigma \in AC^{\overline{p}}_{\geq 2+\overline{t}(\vartheta)-\overline{p}(\vartheta)}(cL)$. Notice that any singular simplex verifies the formula $\partial c\sigma = c\partial\sigma + (-1)^{i+1}\sigma$, if $i > 0$, and $\partial c\sigma = \vartheta - \sigma$, if $i = 0$.

Consider now a cycle $\xi \in SC^{\overline{p}}_{i}(cL)$ with $i \geq 1 + \overline{t}(\vartheta) - \overline{p}(\vartheta)$ and $i \neq 0$. Since $\xi = (-1)^{i+1}\partial c\xi$, we have $c\xi \in SC^{\overline{p}}_{i+1}(cL)$ and therefore $\mathbb{H}^{\overline{p}}_{i}(cL) = 0$.

For $i = 0 \geq 1 + \overline{t}(\vartheta) - \overline{p}(\vartheta)$ we get that for any point σ of $cL - \{\vartheta\}$ the cone $c\sigma$ is a $\overline{p}$-allowable chain with $\partial c\sigma = \sigma - \vartheta$. This gives $\mathbb{H}^{\overline{p}}_{0}(cL) = \mathbb{R}$. $\square$

In some cases, the intersection homology can be expressed in terms of the usual homology $H_*(-)$ (see [12]).

2.2.2. Proposition. *Let X be a stratified pseudomanifold. Then:*

- $\mathbb{H}^{\overline{p}}_*(X) = H_*(X - \Sigma_X)$ *if* $\overline{p} < \overline{0}$*;*
- $\mathbb{H}^{\overline{q}}_*(X) = H_*(X)$ *if* $\overline{q} \geq \overline{t}$ *and X is normal.*

Proof. We prove, by induction on the depth, that the natural inclusions $I_X : S_*(X - \Sigma_X) \hookrightarrow SC^{\overline{p}}_*(X)$ and $J_X : SC^{\overline{q}}_*(X) \hookrightarrow S_*(X)$ are quasi-isomorphisms (i.e., isomorphisms in cohomology). When the depth of X is 0, then the above inclusions are, in fact, two identities. In the general case, we suppose that the result is true for each link L_S of X and we proceed in two steps.

First step. The operators I_V and J_V are quasi-isomorphisms when V is an open subset of a chart (U, φ) of X.

First of all, we identify the open subset U with the product $\mathbb{R}^n \times cL_S$ through φ. We define a *cube* as a product $]a_1, b_1[\times \cdots \times]a_n, b_n[\subset \mathbb{R}^n$. The *truncated cone* c_tL_S is the quotient $c_tL_S = L_S \times [0, t[/L_S \times \{0\}$. Consider the open covering

$$\mathcal{V} = \begin{array}{c} \Big\{ C \times c_tL_S \subset V \mid C \text{ cube, } t \in]0,1[\Big\} \\ \cup \\ \Big\{ C \times L_s \times]a,b[\subset V \mid C \text{ cube, } a,b \in]0,1[\Big\} \end{array} \tag{4}$$

of V. Notice that this family is closed under finite intersections.

We apply Bredon's trick to the covering $\mathcal{V}$ and the statement

$$Q(W) = \text{“ The operators } I_W \text{ and } J_W \text{ are quasi-isomorphisms”}$$

(cf. Lemma 1.4.1). Let us verify the properties (BT1), (BT2) and (BT3).

(BT1) From the product formula and the induction hypothesis, it suffices to prove that the operators I_{cL_S} and J_{cL_S} are quasi-isomorphisms. This follows from the following relations:

- $\mathbb{H}^{\overline{p}}_*(cL_S) \overset{2.2.1}{=} \mathbb{H}^{\overline{p}}_{\leq \overline{t}(\vartheta) - \overline{p}(\vartheta)}(L_S) \overset{\overline{p}(\vartheta)<0}{=\!=\!=} \mathbb{H}^{\overline{p}}_*(L_S)$ $\overset{ind}{=} H_*(L_S - \Sigma_{L_S}) \overset{prod}{=\!=} H_*(cL_S - \Sigma_{cL_S})$.
- For $\overline{q}(\vartheta) = \overline{t}(\vartheta)$ we have $\mathbb{H}^{\overline{q}}_*(cL_S) \overset{2.2.1}{=} \mathbb{H}^{\overline{q}}_0(L_S) \overset{ind}{=} H_0(L_S) \overset{norm}{=}$ $\mathbb{R} = H_*(cL_S)$.
- For $\overline{q}(\vartheta) > \overline{t}(\vartheta)$ we have $\mathbb{H}^{\overline{q}}_*(cL_S) \overset{2.2.1}{=} \mathbb{R} = H_*(cL_S)$.

(BT2) Mayer-Vietoris.

(BT3) Straightforward.

Second Step. The operators I_X and J_X are quasi-isomorphisms.

Consider the open covering

$$\mathcal{V} = \Big\{ V \text{ open subset of a chart } (U, \varphi) \text{ of } X \Big\}$$

of X. Notice that this family is closed under finite intersections. We apply Bredon's trick to the covering $\mathcal{V}$ and the statement

$$Q(W) = \text{ “The operators } I_W \text{ and } J_W \text{ are quasi-isomorphisms”}$$

(cf. Lemma 1.4.1). Let us verify the properties (BT1), (BT2) and (BT3).

(BT1) First Step.
(BT2) Mayer-Vietoris.
(BT3) Straightforward. □

2.2.3. REMARK. Notice that we can replace the normality of X by the connectedness of the links $\{L_S \mid \overline{q}(S) = \overline{t}(S)\}$.

The following result will be needed in the last section.

2.2.4. COROLLARY. *Let X be a connected normal stratified pseudomanifold. Then, for any perversity $\overline{p}$, we have $\mathbb{H}_0^{\overline{p}}(X) = \mathbb{R}$.*

Proof. We prove this result by induction on the depth. When the depth of X is 0, then $\mathbb{H}^0_{\overline{p}}(X) \overset{\Sigma_X = \emptyset}{=\!=\!=} H_0(X) = \mathbb{R}$. Consider now the general case. Notice that any point $\sigma \in X - \Sigma_X$ is a $\overline{p}$-intersection cycle. So $\mathbb{H}_0^{\overline{p}}(X) \neq 0$. We prove that $[\sigma_1] = [\sigma_2]$ in $\mathbb{H}_0^{\overline{p}}(X)$ for two $\overline{p}$-allowable points. This is the case when $\sigma_1, \sigma_2 \in X - \Sigma_X$, since we know from [16] that $X - \Sigma_X$ is connected. Consider now a $\overline{p}$-intersection cycle $\sigma_1 \in \Sigma_X$ and a chart $\varphi \colon U \to \mathbb{R}^n \times cL_S$ of X containing σ_1. Since

$$\mathbb{H}_0^{\overline{p}}(U) = \mathbb{H}_0^{\overline{p}}(\mathbb{R}^n \times cL_S) \overset{prod}{=\!=} \mathbb{H}_0^{\overline{p}}(cL_S) \overset{2.2.1,\ ind}{=\!=\!=\!=} \mathbb{R},$$

we have $[\sigma_1] = [\sigma_2]$ in $\mathbb{H}_0^{\overline{p}}(X)$ for some $\overline{p}$-intersection point $\sigma_2 \in X - \Sigma_X$. Therefore $\mathbb{H}_0^{\overline{p}}(X) = \mathbb{R}$. □

2.2.5. Relative case. The conical formula given by Proposition 2.2.1 for the intersection homology differs from that of Proposition 3.1.1 for the intersection cohomology, since we do not have $\mathbb{H}^*_{\overline{p}}(cL) = \mathbb{H}_*^{\overline{t}-\overline{p}}(cL)$ when the perversity $\overline{p}$ is not positive. It is natural to think that the closed saturated subset

$$X_{\overline{p}} = \bigcup_{\substack{S \preceq S_1 \\ \overline{p}(S_1)<0}} S = \bigcup_{\overline{p}(S)<0} \overline{S} \overset{loc\ finit}{=\!=\!=\!=} \overline{\bigcup_{\overline{p}(S)<0} S}$$

plays a key role in the de Rham Theorem. This is indeed the case.

The subset $X_{\overline{p}}$ is a stratified pseudomanifold, where the maximal strata may have different dimensions. For any perversity $\overline{q}$ (on X) we have the notion of a $\overline{q}$-allowable chain as in 2.2. We denote by $AC_*^{\overline{q}}(X_{\overline{p}})$ the complex of these $\overline{q}$-allowable chains. Equivalently,

$$AC_*^{\overline{q}}(X_{\overline{p}}) = S_*(X_{\overline{p}}) \cap AC_*^{\overline{q}}(X).$$

In order to recover the de Rham Theorem we introduce the following notion of relative intersection homology. We denote by $SC_*^{\overline{q}}(X, X_{\overline{p}})$ the differential complex

$$\frac{\left(AC_*^{\overline{q}}(X) + AC_*^{\overline{q}+\overline{1}}(X_{\overline{p}})\right) \cap \partial^{-1}\left(AC_{*-1}^{\overline{q}}(X) + AC_{*-1}^{\overline{q}+\overline{1}}(X_{\overline{p}})\right)}{AC_*^{\overline{q}+\overline{1}}(X_{\overline{p}}) \cap \partial^{-1}\left(AC_{*-1}^{\overline{q}+\overline{1}}(X_{\overline{p}})\right)}$$

and by $\mathbb{H}_*^{\overline{q}}(X, X_{\overline{p}})$ its cohomology. Of course, we have $\mathbb{H}_*^{\overline{q}}(X, X_{\overline{p}}) = \mathbb{H}_*^{\overline{q}}(X)$ when $X_{\overline{p}} = \emptyset$, and $\mathbb{H}_*^{\overline{q}}(X, X_{\overline{p}}) = H_*(X, X_{\overline{p}})$ when $\overline{q} \geq \overline{t} + \overline{2}$ (see also (16)).

Since the complexes defining $SC_*^{\overline{q}}(X, X_{\overline{p}})$ verify the Mayer-Vietoris property (they are preserved by the barycentric subdivision), the relative cohomology $\mathbb{H}_*^{\overline{p}}(X, X_{\overline{p}})$ also verifies the Mayer-Vietoris property. For the same reason we have the product formula

$$\mathbb{H}_*^{\overline{p}}(\mathbb{R} \times X, \mathbb{R} \times X_{\overline{p}}) = \mathbb{H}_*^{\overline{p}}(X, X_{\overline{p}}).$$

For the typical local calculation we have the following result.

2.2.6. COROLLARY. *Let L be a compact stratified pseudomanifold. Then, for any perversity $\overline{p}$, we have*

$$\mathbb{H}_i^{\overline{t}-\overline{p}}\left(cL, (cL)_{\overline{p}}\right) = \begin{cases} \mathbb{H}_i^{\overline{t}-\overline{p}}(L, L_{\overline{p}}) & \textit{if } i \leq \overline{p}(\vartheta), \\ 0 & \textit{if } i \geq 1 + \overline{p}(\vartheta). \end{cases} \tag{5}$$

Proof. When $\overline{p} \geq \overline{0}$, then $(cL)_{\overline{p}} = L_{\overline{p}} = \emptyset$ and (5) follows directly from Lemma 2.2.1. Let us suppose $\overline{p} \not\geq \overline{0}$, which gives $(cL)_{\overline{p}} = c\left(L_{\overline{p}}\right) \neq \emptyset$, with $c\emptyset = \{\vartheta\}$. We also use the following equalities:

$$AC_j^{\overline{t}-\overline{p}}(cL) = AC_j^{\overline{t}-\overline{p}}(L \times]0,1[) \quad \text{for } j \leq \overline{p}(\vartheta) + 1, \tag{6}$$

and

$$AC_j^{\overline{t}-\overline{p}+\overline{1}}\left((cL)_{\overline{p}}\right) = AC_j^{\overline{t}-\overline{p}+\overline{1}}\left(L_{\overline{p}} \times]0,1[\right) \quad \text{for } j \leq \overline{p}(\vartheta), \tag{7}$$

We proceed in four steps according to the value of $i \in \mathbb{N}$.

First Step. $i \leq \overline{p}(\vartheta) - 1$.

We have

$$SC_j^{\overline{t}-\overline{p}}\left(cL, (cL)_{\overline{p}}\right) = SC_j^{\overline{t}-\overline{p}}\left(L \times]0,1[, L_{\overline{p}} \times]0,1[\right)$$

for each $j \leq \overline{p}(\vartheta)$ (cf. (6) and (7)) and therefore

$$\mathbb{H}_{i}^{\overline{t}-\overline{p}}\left(cL,(cL)_{\overline{p}}\right) = \mathbb{H}_{i}^{\overline{t}-\overline{p}}\left(L,L_{\overline{p}}\right).$$

Second Step. $i = \overline{p}(\vartheta)$.

The inclusion

$$SC_{*}^{\overline{t}-\overline{p}}\left(L\times]0,1[,L_{\overline{p}}\times]0,1[\right) \hookrightarrow SC_{*}^{\overline{t}-\overline{p}}\left(cL,(cL)_{\overline{p}}\right)$$

induces the epimorphism

$$I\colon \mathbb{H}_{\overline{p}(\vartheta)}^{\overline{t}-\overline{p}}\left(L\times]0,1[,L_{\overline{p}}\times]0,1[\right) \longrightarrow \mathbb{H}_{\overline{p}(\vartheta)}^{\overline{t}-\overline{p}}\left(cL,(cL)_{\overline{p}}\right)$$

(cf. (6) and (7)). It remains to prove that I is a monomorphism. Consider

$$\left[\overline{\alpha+\beta}\right] \in \mathbb{H}_{\overline{p}(\vartheta)}^{\overline{t}-\overline{p}}\left(L\times]0,1[,L_{\overline{p}}\times]0,1[\right)$$

with $I\left(\left[\overline{\alpha+\beta}\right]\right) = 0$. So we have

(a) $\alpha \in AC_{\overline{p}(\vartheta)}^{\overline{t}-\overline{p}}(L\times]0,1[) \subset AC_{\overline{p}(\vartheta)}^{\overline{t}-\overline{p}+\overline{1}}(L\times]0,1[)$,

(b) $\beta \in AC_{\overline{p}(\vartheta)}^{\overline{t}-\overline{p}+\overline{1}}\left(L_{\overline{p}}\times]0,1[\right)$,

and there exist

(c) $A \in AC_{\overline{p}(\vartheta)+1}^{\overline{t}-\overline{p}}(cL) \overset{(6)}{=} AC_{\overline{p}(\vartheta)+1}^{\overline{t}-\overline{p}}(L\times]0,1[)$,

(d) $B \in AC_{\overline{p}(\vartheta)+1}^{\overline{t}-\overline{p}+\overline{1}}\left((cL)_{\overline{p}}\right)$,

(e) $C \in AC_{\overline{p}(\vartheta)}^{\overline{t}-\overline{p}+\overline{1}}\left((cL)_{\overline{p}}\right) \cap \partial^{-1}\left(AC_{\overline{p}(\vartheta)-1}^{\overline{t}-\overline{p}+\overline{1}}\left((cL)_{\overline{p}}\right)\right) \overset{(7)}{=}$

$$AC_{\overline{p}(\vartheta)}^{\overline{t}-\overline{p}+\overline{1}}\left(L_{\overline{p}}\times]0,1[\right) \cap \partial^{-1}\left(AC_{\overline{p}(\vartheta)-1}^{\overline{t}-\overline{p}+\overline{1}}\left(L_{\overline{p}}\times]0,1[\right)\right)$$

with

(f) $\alpha+\beta = \partial A + \partial B + C$.

Since $\partial A \in AC_{\overline{p}(\vartheta)}^{\overline{t}-\overline{p}+\overline{1}}(L\times]0,1[)$ (cf. (c)), the conditions (a), (b), (d), (e) and (f) give

(g) $\partial B \in AC_{\overline{p}(\vartheta)}^{\overline{t}-\overline{p}+\overline{1}}\left(L_{\overline{p}}\times]0,1[\right)$.

We conclude that

$$\partial A = \alpha + (\beta - \partial B - C) \in AC_{\overline{p}(\vartheta)}^{\overline{t}-\overline{p}}(L\times]0,1[) + AC_{\overline{p}(\vartheta)}^{\overline{t}-\overline{p}+\overline{1}}\left(L_{\overline{p}}\times]0,1[\right),$$

which defines the element

$$\overline{A} \in SC_{\overline{p}(\vartheta)+1}^{\overline{t}-\overline{p}}\left(L\times]0,1[,L_{\overline{p}}\times]0,1[\right).$$

If we denote by $\overline{\partial}$ the derivative of $SC_{*}^{\overline{t}-\overline{p}}\big(L\times]0,1[, L_{\overline{p}}\times]0,1[\big)$, we have

$$\Big[\overline{\alpha+\beta}\Big] = \Big[\overline{\partial A+\partial B+C}\Big] = \Big[\overline{\partial}\ \overline{A}+\overline{\partial B+C}\Big] \overset{(e),(g)}{=\!=} \Big[\overline{\partial}\ \overline{A}\Big] = 0.$$

Therefore the operator I is a monomorphism.

Third step. $i \geq 1+\overline{p}(\vartheta)$ *and* $i \neq 0$.

Consider a cycle $\overline{\xi}$ on the complex $SC_{i}^{\overline{t}-\overline{p}}\Big(cL,(cL)_{\overline{p}}\Big)$. Let

$$\xi = \xi_1+\xi_2 \in AC_{i}^{\overline{t}-\overline{p}}(cL) + AC_{i}^{\overline{t}-\overline{p}+\overline{1}}\Big((cL)_{\overline{p}}\Big)$$

with $\partial\xi \in AC_{i-1}^{\overline{t}-\overline{p}+\overline{1}}\Big((cL)_{\overline{p}}\Big)$. Then we have

$$c\xi = c\xi_1+c\xi_2 \in AC_{i+1}^{\overline{t}-\overline{p}}(cL) + AC_{i+1}^{\overline{t}-\overline{p}+\overline{1}}\Big((cL)_{\overline{p}}\Big)$$

and $c\partial\xi \in AC_{i}^{\overline{t}-\overline{p}+\overline{1}}\Big((cL)_{\overline{p}}\Big)$ (cf. (3)). Since

$$\partial\, c\xi = (-1)^{i+1}\xi + c\partial\xi \tag{8}$$

is an element of $AC_{i}^{\overline{t}-\overline{p}}(cL) + AC_{i}^{\overline{t}-\overline{p}+\overline{1}}\Big((cL)_{\overline{p}}\Big)$, the cone $\overline{c\xi}$ belongs to $SC_{i+1}^{\overline{t}-\overline{p}}\Big(cL,(cL)_{\overline{p}}\Big)$. This formula gives $\partial c\partial\xi = (-1)^{i}\partial\xi$ and therefore

$$c\partial\xi \in AC_{i}^{\overline{t}-\overline{p}+\overline{1}}\Big((cL)_{\overline{p}}\Big) \cap \partial^{-1}\left(AC_{i-1}^{\overline{t}-\overline{p}+\overline{1}}\Big((cL)_{\overline{p}}\Big)\right).$$

Applying (8) we obtain that the class $\big[\ \overline{\xi}\ \big]$ vanishes on $\mathbb{H}_{i}^{\overline{t}-\overline{p}}\Big(cL,(cL)_{\overline{p}}\Big)$.

Fourth Step. $i = 0 \geq 1+\overline{p}(\vartheta)$.

For any point $\sigma \in AC_{0}^{\overline{t}-\overline{p}}(cL)$ the cone $c\sigma$ is a $(\overline{t}-\overline{p})$-allowable chain with $\partial c\sigma = \sigma - \vartheta$. Since the point ϑ belongs to the complex $AC_{0}^{\overline{t}-\overline{p}+\overline{1}}\Big((cL)_{\overline{p}}\Big)$, one gets that $\mathbb{H}_{0}^{\overline{t}-\overline{p}}\Big(cL,(cL)_{\overline{p}}\Big) = 0$. □

2.3. Intersection homology. Second approach (see [18]**).** In order to integrate differential forms on allowable simplices, we need to introduce some amount of smoothness on these simplices. Since X is not a manifold, we work in the manifold $\widetilde{X}$. In fact, we consider those allowable simplices which are liftable to smooth simplices in $\widetilde{X}$.

2.3.1. Linear unfolding. The *unfolding* of the standard simplex Δ, relative to the decomposition $\Delta = \Delta_0 * \cdots * \Delta_j$, is the map

$$\mu_\Delta \colon \widetilde{\Delta} = \bar{c}\Delta_0 \times \cdots \times \bar{c}\Delta_{j-1} \times \Delta_j \longrightarrow \Delta$$

defined by

$$\mu_\Delta([x_0,t_0],\dots,[x_{j-1},t_{j-1}],x_j) = t_0x_0 + (1-t_0)t_1x_1 + \cdots + (1-t_0)\cdots(1-t_{j-2})t_{j-1}x_{j-1} + (1-t_0)\cdots(1-t_{j-1})x_j,$$

where $\bar{c}\Delta_i$ denotes the closed cone $\Delta_i \times [0,1]/\Delta_i \times \{0\}$ and $[x_i, t_i]$ a point of it. This map is smooth and its restriction $\mu_\Delta\colon \operatorname{int}(\widetilde{\Delta}) \longrightarrow \operatorname{int}(\Delta)$ is a diffeomorphism ($\operatorname{int}(P) = P - \partial P$ is the *interior* of the polyhedron P). It sends a face U of $\widetilde{\Delta}$ to a face V of Δ and the restriction $\mu_\Delta\colon \operatorname{int}(U) \to \operatorname{int}(V)$ is a submersion.

On the boundary $\partial\widetilde{\Delta}$ we find not only the blow-up $\widetilde{\partial\Delta}$ of the boundary $\partial\Delta$ of Δ, but also the faces

$$F = \bar{c}\Delta_0 \times \cdots \times \bar{c}\Delta_{i-1} \times (\Delta_i \times \{1\}) \times \bar{c}\Delta_{i+1} \times \cdots \times \bar{c}\Delta_{j-1} \times \Delta_j$$

with $i \in \{0,\dots,j-2\}$ or $i = j-1$ and $\dim \Delta_j > 0$, which we call *bad faces.* This gives the decomposition

$$\partial\widetilde{\Delta} = \widetilde{\partial\Delta} + \delta\widetilde{\Delta}. \tag{9}$$

Notice that

$$\dim \mu_\Delta(F) = \dim(\Delta_0 * \cdots * \Delta_i) < \dim\Delta - 1 = \dim F. \tag{10}$$

2.3.2. Liftable simplices. A *liftable simplex* is a singular simplex $\sigma\colon \Delta \to X$ verifying the following two conditions:

(Lif1) *Each pull back* $\sigma^{-1}(X_i)$ *is a face of* Δ.

(Lif2) *There exists a decomposition* $\Delta = \Delta_0 * \cdots * \Delta_j$ *and a smooth map (called* lifting*)* $\widetilde{\sigma}\colon \widetilde{\Delta} \to \widetilde{X}$ *with* $\mathcal{L}_X \circ \widetilde{\sigma} = \sigma \circ \mu_\Delta$.

A singular chain $\xi = \sum_{j=1}^m r_j\sigma_j$ is *liftable* if each singular simplex σ_j is liftable. Since a face of a liftable simplex is again a liftable simplex, the family $L_*(X)$ of liftable chains is a differential complex. We denote by $LC_*^{\overline{p}}(X) = AC_*^{\overline{p}}(X) \cap L_*(X)$ the graded vector space of the $\overline{p}$-allowable liftable chains and by $RC_*^{\overline{p}}(X) = LC_*^{\overline{p}}(X) \cap \partial^{-1}LC_{*-1}^{\overline{p}}(X)$ the associated differential complex.

The barycentric subdivision of an liftable chain is an liftable chain (cf. [3]); so, the homology $H_*\left(RC_\cdot^{\overline{p}}(X)\right)$ verifies the Mayer-Vietoris property. It also verifies the product formula $H_*\left(RC_\cdot^{\overline{p}}(\mathbb{R}\times X)\right) = H_*\left(RC_\cdot^{\overline{p}}(X)\right)$. For the typical local calculation we have the following result, which corrects Proposition 2.2.5 of [18].

2.3.3. PROPOSITION. *Let L be a compact unfoldable pseudomanifold. Consider on cL the canonical induced unfolding. Then*

$$H_i\left(RC^*_{\overline{p}}(cL)\right) = \begin{cases} H_i\left(RC^*_{\overline{p}}(L)\right) & \text{if } i \leq \overline{t}(\vartheta) - \overline{p}(\vartheta), \\ 0 & \text{if } 0 \neq i \geq 1 + \overline{t}(\vartheta) - \overline{p}(\vartheta), \\ \mathbb{R} & \text{if } 0 = i \geq 1 + \overline{t}(\vartheta) - \overline{p}(\vartheta). \end{cases}$$

Proof. We proceed as in Proposition 2.2.1. In fact, it suffices to prove that the cone $c\sigma\colon \overline{c}\Delta \to cL$ of a $\overline{p}$-allowable liftable simplex $\sigma\colon \Delta \to cL$, with $\dim \Delta \geq 1 + \overline{t}(\vartheta) - \overline{p}(\vartheta)$, is a $\overline{p}$-allowable liftable simplex. Let us verify properties $(\text{All})_{c\sigma}$, $(\text{Lif1})_{c\sigma}$ and $(\text{Lif2})_{c\sigma}$.

Put $\overline{c}\Delta = \Delta * \{Q\}$ with $c\sigma(tP + (1-t)Q) = t \cdot \sigma(P)$. We have

$$(c\sigma)^{-1}(cL)_i = \begin{cases} \{Q\} & \text{if } \sigma^{-1}(cL)_i = \emptyset, \\ \overline{c}(\sigma^{-1}(cL)_i) & \text{if } \sigma^{-1}(cL)_i \neq \emptyset, \end{cases}$$

for $i \geq 0$.

We obtain $(\text{Lif1})_{c\sigma}$ from $(\text{Lif1})_{\sigma}$. To prove the property $(\text{All})_{c\sigma}$ we consider a stratum $S \in \mathcal{S}_{cL}$. We have

$$(c\sigma)^{-1}(S) = \begin{cases} \{Q\} & \text{if } S = \{\vartheta\}; \sigma^{-1}(\vartheta) = \emptyset \\ \overline{c}\left(\sigma^{-1}(\vartheta)\right) & \text{if } S = \{\vartheta\}; \sigma^{-1}(\vartheta) \neq \emptyset \\ \emptyset & \text{if } S \neq \{\vartheta\}; \sigma^{-1}(S) = \emptyset \\ \overline{c}\left(\sigma^{-1}(\vartheta)\right) - \{Q\} & \text{if } S \neq \{\vartheta\}; \sigma^{-1}(S) \neq \emptyset \end{cases}$$

$$\overset{(\text{All})_\sigma}{\subset} \begin{cases} 0-\text{skeleton of } \overline{c}\Delta \\ (1+(\dim\Delta - 2 - \overline{t}(\vartheta) + \overline{p}(\vartheta))) - \text{skeleton of } \overline{c}\Delta \\ \emptyset \\ (1+(\dim\Delta - 2 - \overline{t}(S) + \overline{p}(S))) - \text{skeleton of } \overline{c}\Delta \end{cases}$$

$$\subset (\dim \overline{c}\Delta - 2 - \overline{t}(\vartheta) + \overline{p}(\vartheta))) - \text{skeleton of } \overline{c}\Delta,$$

since $\dim \Delta \geq 1 + \overline{t}(\vartheta) - \overline{p}(\vartheta)$. Now we prove $(\text{Lif2})_{c\sigma}$. Consider the decomposition $\Delta = \Delta_0 * \cdots * \Delta_j$ given by σ, and the smooth map $\widetilde{\sigma} = (\widetilde{\sigma}_1, \widetilde{\sigma}_2)\colon \widetilde{\Delta} \to \widetilde{L} \times]-1,1[$ given by $(\text{Lif2})_{\sigma}$. We have the decomposition $\overline{c}\Delta = \{Q\} * \Delta_0 * \cdots * \Delta_j$ whose lifting

$$\mu_{\overline{c}\Delta}\colon \widetilde{\overline{c}\Delta} = \overline{c}\{Q\} \times \widetilde{\Delta} \longrightarrow \overline{c}\Delta$$

is defined by

$$\mu_{\overline{c}\Delta}([Q,t],x) = tQ + (1-t)\mu_\Delta(x).$$

Let $\widetilde{c\sigma}\colon \overline{c}\{Q\} \times \widetilde{\Delta} \longrightarrow \widetilde{L} \times]-1,1[$ be the smooth map defined by

$$\widetilde{c\sigma}([Q,t],x) = (\widetilde{\sigma}_1(x), (1-t) \cdot \widetilde{\sigma}_2(x)).$$

Finally, for each $([Q,t],x) \in \overline{c}\{Q\} \times \widetilde{\Delta}$ we have

$$\begin{aligned} c\sigma\mu_{\overline{c}\Delta}([Q,t],x) &= c\sigma(tQ+(1-t)\mu_\Delta(x)) = (1-t)\cdot\sigma\mu_\Delta(x) \\ &= (1-t)\cdot\mathcal{L}_{cL}\widetilde{\sigma}(x) = (1-t)\left[\mathcal{L}_L\widetilde{\sigma}_1(x), |\widetilde{\sigma}_2(x)|\right] \\ &= \mathcal{L}_{cL}\widetilde{c\sigma}([Q,t],x). \end{aligned}$$

This gives $(\mathrm{Lif2})_{c\sigma}$. □

2.3.4. Relative case. Following 2.2.5 we consider

$$LC_*^{\overline{q}}(X_{\overline{p}}) = S_*(X_{\overline{p}}) \cap LC_*^{\overline{q}}(X).$$

and we define the relative complex $RC_*^{\overline{q}}(X, X_{\overline{p}})$ by

$$\frac{\left(LC_*^{\overline{q}}(X) + LC_*^{\overline{q}+\overline{1}}(X_{\overline{p}})\right) \cap \partial^{-1}\left(LC_{*-1}^{\overline{q}}(X) + LC_{*-1}^{\overline{q}+\overline{1}}(X_{\overline{p}})\right)}{LC_*^{\overline{q}+\overline{1}}(X_{\overline{p}}) \cap \partial^{-1}\left(LC_{*-1}^{\overline{q}+\overline{1}}(X_{\overline{p}})\right)}.$$

We have $LC_*^{\overline{q}}(X, X_{\overline{p}}) = LC_*^{\overline{q}}(X)$ when $X_{\overline{p}} = \emptyset$.

Since the complexes defining $RC_*^{\overline{p}}(X,Z)$ verify the Mayer-Vietoris property (they are preserved by the barycentric subdivision), the relative cohomology $H_*\left(RC_\cdot^{\overline{p}}(X,Z)\right)$ verifies the Mayer-Vietoris property. For the same reason we have the product formula

$$H_*\left(RC_\cdot^{\overline{p}}(\mathbb{R}\times X, \mathbb{R}\times Z)\right) = H_*\left(RC_\cdot^{\overline{p}}(X,Z)\right).$$

For the typical local calculation we have (see [18]):

2.3.5. COROLLARY. *Let L be a compact unfoldable pseudomanifold. Consider on cL the canonical induced unfolding. Then*

$$H_i\left(RC_*^{\overline{t}-\overline{p}}\left(cL,(cL)_{\overline{p}}\right)\right) = \begin{cases} H_i\left(RC_*^{\overline{t}-\overline{p}}(L, L_{\overline{p}})\right) & \text{if } i \leq \overline{p}(\vartheta), \\ 0 & \text{if } i \geq 1+\overline{p}(\vartheta). \end{cases}$$

Proof. The proof is the same as that of Corollary 2.2.6. □

2.4. Comparing the two approaches. When the perversity $\overline{p}$ lies between $\overline{0}$ and $\overline{t}$, then we have the isomorphism $\mathbb{H}_*^{\overline{p}}(X) = H_*\left(RC_\cdot^{\overline{p}}(X)\right)$ (cf. [18]). We are going to check that this property extends to any perversity for the absolute case and the relative case.

2.4.1. PROPOSITION. *For any perversity $\overline{p}$, the inclusion $RC_*^{\overline{p}}(X) \hookrightarrow SC_*^{\overline{p}}(X)$ induces the isomorphism $H_*\left(RC_\cdot^{\overline{p}}(X)\right) = \mathbb{H}_*^{\overline{p}}(X)$.*

Proof. We proceed by induction on the depth. When depth $X = 0$, then $SC_*^{\overline{p}}(X) = RC_*^{\overline{p}}(X) = S_*(X)$. In the general case, we use Bredon's trick (see the proof of Proposition 2.2.2) and we reduce the problem to a chart $X = \mathbb{R}^n \times cL_S$. Here, we apply the product formula and we reduce the problem to $X = cL_S$. We end the proof by applying Propositions 2.2.1, 2.3.3 and the induction hypothesis. □

2.4.2. PROPOSITION. *For any perversity $\overline{p}$, the inclusion*

$$RC_*^{\overline{t}-\overline{p}}(X, X_{\overline{p}}) \hookrightarrow SC_*^{\overline{t}-\overline{p}}(X, X_{\overline{p}})$$

induces the isomorphism

$$H_*\left(RC_\cdot^{\overline{t}-\overline{p}}(X, X_{\overline{p}})\right) = \mathbb{H}_*^{\overline{t}-\overline{p}}(X, X_{\overline{p}}).$$

Proof. We proceed by induction on the depth. If depth $X = 0$, then $SC_*^{\overline{t}-\overline{p}}(X, X_{\overline{p}}) = RC_*^{\overline{t}-\overline{p}}(X, X_{\overline{p}}) = S_*(X)$. In the general case, we use Bredon's trick (see the proof of Proposition 2.2.2) and we reduce the problem to a chart $X = \mathbb{R}^n \times cL_S$ with $X_{\overline{p}} = \mathbb{R}^n \times (cL_S)_{\overline{p}}$. Here, we apply the product formula and we reduce the problem to $(X, X_{\overline{p}}) = \left(cL_S, (cL_S)_{\overline{p}}\right)$. We end the proof by applying Corollaries 2.2.6, 2.3.5 and the induction hypothesis. □

3. Intersection cohomology

The de Rham intersection cohomology was introduced by Brylinski in [7]. In our paper we use the presentation of [18].

3.1. Perverse forms. A *liftable form* is a differential form $\omega \in \Omega^*(X - \Sigma_X)$ possessing a *lifting*, that is, a differential form $\widetilde{\omega} \in \Omega^*\left(\widetilde{X}\right)$ verifying $\widetilde{\omega} = \mathcal{L}_X^*\omega$ on $\mathcal{L}_X^{-1}(X - \Sigma_X)$.

Given two liftable differential forms ω, η, we have the equalities

$$\widetilde{\omega + \eta} = \widetilde{\omega} + \widetilde{\eta}, \quad \widetilde{\omega \wedge \eta} = \widetilde{\omega} \wedge \widetilde{\eta}, \quad \widetilde{d\omega} = d\widetilde{\omega}. \tag{11}$$

We denote by $\Pi^*(X)$ the differential complex of liftable forms.

Recall that, for each singular stratum $S \in \mathcal{S}_X^{\text{sing}}$, the restriction $\mathcal{L}_S \colon \mathcal{L}_S^{-1}(S) \longrightarrow S$ is a fiber bundle. For a differential form $\eta \in \Omega^*\left(\mathcal{L}_S^{-1}(S)\right)$ we define its *vertical degree* as

$$v_S(\eta) = \min \left\{ j \in \mathbb{N} \;\middle|\; \begin{array}{c} i_{\xi_0} \cdots i_{\xi_j}\eta = 0 \\ \text{for each family of vector fields} \\ \xi_0, \ldots \xi_j \text{ tangent to fibers of} \\ \mathcal{L}_S \colon \mathcal{L}_S^{-1}(S) \longrightarrow S \end{array} \right\}$$

(cf. [7], [18]). The *perverse degree* $\|\omega\|_S$ of ω relative to S is the vertical degree of the restriction $\widetilde{\omega}$ relative to $\mathcal{L}_S \colon \mathcal{L}_S^{-1}(S) \longrightarrow S$, that is,

$$\|\omega\|_S = v_S\left(\widetilde{\omega}|_{\mathcal{L}_S^{-1}(S)}\right).$$

The differential complex of $\overline{p}$-intersection differential forms is

$$\Omega^*_{\overline{p}}(X) = \{\omega \in \Pi^*(X) \mid \max(||\omega||_S, ||d\omega||_S) \leq \overline{p}(S)\ \forall S \in \mathcal{S}_X^{\text{sing}}\}.$$

The cohomology $\mathbb{H}^*_{\overline{p}}(X)$ of this complex is the $\overline{p}$-*intersection cohomology* of X. The intersection cohomology verifies two important computational properties: the Mayer-Vietoris property and the product formula $\mathbb{H}^*_{\overline{p}}(\mathbb{R} \times X) = \mathbb{H}^*_{\overline{p}}(X)$. The usual local calculations (see [7], [18]) give:

3.1.1. PROPOSITION. *If L is a compact stratified pseudomanifold, then*

$$\mathbb{H}^i_{\overline{p}}(cL) = \begin{cases} \mathbb{H}^i_{\overline{p}}(L) & \textit{if } i \leq \overline{p}(\vartheta), \\ 0 & \textit{if } i > \overline{p}(\vartheta). \end{cases}$$

3.2. Integration. The relationship between intersection homology and cohomology is established by using integration of differential forms on simplices. Since X is not a manifold, we work on the blow-up $\widetilde{X}$.

Consider a liftable simplex $\varphi \colon \Delta \to X$. We know that there exists a stratum S containing $\sigma(\text{int}(\Delta))$. Since $\mu_\Delta \colon \text{int}(\widetilde{\Delta}) \to \text{int}(\Delta)$ is a diffeomorphism, $\sigma = \mathcal{L}_X \circ \widetilde{\sigma} \circ \mu_\Delta^{-1} \colon \text{int}(\Delta) \longrightarrow S$ is a smooth map.

Consider now a liftable differential form $\omega \in \Pi^*(X)$ and define the *integration* as

$$\int_\sigma \omega = \begin{cases} \int_{\text{int}(\Delta)} \sigma^*\omega & \text{if } S \text{ a regular stratum (i.e., } \sigma(\Delta) \not\subset \Sigma_X), \\ \\ 0 & \text{if } S \text{ a singular stratum (i.e., } \sigma(\Delta) \subset \Sigma_X). \end{cases} \tag{12}$$

This definition makes sense, since

$$\int_{\text{int}(\Delta)} \sigma^*\omega = \int_{\text{int}(\widetilde{\Delta})} \widetilde{\sigma}^*\widetilde{\omega} = \int_{\widetilde{\Delta}} \widetilde{\sigma}^*\widetilde{\omega}. \tag{13}$$

By linearity, we have the linear pairing

$$\int \colon \Pi^*(X) \longrightarrow \text{Hom}\left(L_*(X), \mathbb{R}\right).$$

This operator commutes with the differential d in some cases.

3.2.1. LEMMA. *If $\overline{p}$ is a perversity, then the integration operator*

$$\int \colon \Omega^*_{\overline{p}}(X) \to \text{Hom}\left(RC_*^{\overline{t}-\overline{p}}(X), \mathbb{R}\right)$$

is differential pairing.

Proof. Consider a liftable $\overline{p}$-allowable simplex $\sigma\colon \Delta^i \to X$ with $\sigma(\Delta) \not\subset \Sigma_X$ and $\omega \in \Omega^{i-1}_{\overline{q}}(X)$. It suffices to prove

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega. \tag{14}$$

The boundary of Δ can be written as $\partial\Delta = \partial_1\Delta + \partial_2\Delta$, where $\partial_1\Delta$ (resp. $\partial_2\Delta$) is composed of the faces F of Δ with $\sigma(F) \not\subset \Sigma_X$ (resp. $\sigma(F) \subset \Sigma_X$). This gives the decomposition (see (9))

$$\partial\widetilde{\Delta} = \widetilde{\partial_1\Delta} + \widetilde{\partial_2\Delta} + \delta\widetilde{\Delta}.$$

We have the equalities

$$\int_\sigma d\omega \overset{(13)}{=\!=} \int_{\widetilde{\Delta}} \widetilde{\sigma}^* \widetilde{d\omega} \overset{(11)}{=\!=} \int_{\widetilde{\Delta}} d\widetilde{\sigma}^*\widetilde{\omega} \overset{Stokes}{=\!=\!=} \int_{\partial\widetilde{\Delta}} \widetilde{\sigma}^*\widetilde{\omega}$$

and

$$\int_{\partial\sigma} \omega \overset{(12),(13)}{=\!=\!=} \int_{\widetilde{\partial_1\Delta}} \widetilde{\sigma}^*\widetilde{\omega}.$$

So the equality (14) becomes

$$\int_{\delta\widetilde{\Delta}} \widetilde{\sigma}^*\widetilde{\omega} + \int_{\widetilde{\partial_2\Delta}} \widetilde{\sigma}^*\widetilde{\omega} = 0.$$

The proof will be complete once we show that $\widetilde{\sigma}^*\widetilde{\omega} = 0$ on F, where the face F

- is a bad face, or
- verifies $\sigma(F) \subset \Sigma_X$.

Let C be the face $\mu_\Delta(F)$ of Δ and S the stratum of X containing the subset $\sigma(\text{int}(C))$. Notice that the condition (All) implies

$$\dim C \le \dim F + 1 - 2 - \overline{t}(S) + (\overline{t}(S) - \overline{p}(S)) = \dim F - 1 - \overline{p}(S). \tag{15}$$

We have the following commutative diagram:

$$\begin{array}{ccc} \text{int}(F) & \xrightarrow{\widetilde{\sigma}} & \mathcal{L}_X^{-1}(S) \\ {\scriptstyle \mu_\Delta}\downarrow & & {\scriptstyle \mathcal{L}_X}\downarrow \\ \text{int}(C) & \xrightarrow{\sigma} & S \end{array}$$

It suffices to prove that the vertical degree of $\widetilde{\sigma}^*\widetilde{\omega}$ relative to μ_Δ is strictly lower than the dimension of the fibers of μ_Δ, that is,

$$v_S\left(\widetilde{\sigma}^*\widetilde{\omega}\right) < \dim F - \dim C.$$

We distinguish two cases:

- When S is a regular stratum, the differential form ω is defined on S, and we have
$$\widetilde{\sigma}^*\widetilde{\omega} = \widetilde{\sigma}^*\mathcal{L}_X^*\omega = \mu_\Delta^*\sigma^*\omega,$$

which is a basic form relative to μ_Δ. So, since F is a bad face,

$$v_S\left(\widetilde{\sigma}^*\widetilde{\omega}\right) \leq 0 \overset{(10)}{<} \dim F - \dim C.$$

- When S is a singular stratum, we have

$$v_S\left(\widetilde{\sigma}^*\widetilde{\omega}\right) \leq ||\omega||_S \leq \overline{p}(S) \overset{(15)}{\leq} \dim F - \dim C - 1 < \dim F - \dim C.$$

This ends the proof. □

The above pairing induces the pairing

$$\int : \mathbb{H}^*_{\overline{p}}(X) \longrightarrow \operatorname{Hom}\left(\mathbb{H}^{\overline{t}-\overline{p}}_*(X), \mathbb{R}\right)$$

(cf. Proposition 2.4.1) which is not an isomorphism: For a cone cL we have Proposition 2.2.1 and Proposition 3.1.1. The problem appears when negative perversities are involved. For this reason we consider the relative intersection homology. Since the integration $\int$ vanishes on Σ_X,

$$\int : \Omega^*_{\overline{p}}(X) \longrightarrow \operatorname{Hom}\left(RC^{\overline{t}-\overline{p}}_*\left(X, X_{\overline{p}}\right), \mathbb{R}\right).$$

is a well defined differential operator. We obtain the de Rham duality (in the direction cohomology $\mapsto$ homology):

3.2.2. Theorem. *Let X be an unfoldable pseudomanifold. If $\overline{p}$ is a perversity, then the integration induces the isomorphism*

$$\mathbb{H}^*_{\overline{p}}(X) = \operatorname{Hom}\left(\mathbb{H}^{\overline{t}-\overline{p}}_*\left(X, X_{\overline{p}}\right); \mathbb{R}\right).$$

Proof. Following Proposition 2.4.2 it suffices to prove that the pairing

$$\int : \Omega^*_{\overline{p}}(X) \longrightarrow \operatorname{Hom}\left(RC^{\overline{t}-\overline{p}}_*\left(X, X_{\overline{p}}\right), \mathbb{R}\right).$$

induces an isomorphism in cohomology. We proceed by induction on the depth. If $\operatorname{depth} X = 0$, then $X_{\overline{p}} = \emptyset$ and we have the usual de Rham theorem. In the general case, we use Bredon's trick (see the proof of Proposition 2.2.2) and reduce the problem to a chart $X = \mathbb{R}^n \times cL_S$ with $X_{\overline{p}} = \mathbb{R}^n \times (cL_S)_{\overline{p}}$. Then we apply the product formula and we reduce the problem to $(X, X_{\overline{p}}) = \left(cL_S, (cL_S)_{\overline{p}}\right)$. We end the proof by applying Corollary 2.2.6, Proposition 3.1.1 and the induction hypothesis. □

In particular, we have the deRham isomorphism $\mathbb{H}^*_{\overline{p}}(X) = \mathbb{H}^{\overline{t}-\overline{p}}_*(X)$ when $\overline{p} \geq \overline{0}$.

The intersection cohomology can be expressed in terms of the usual cohomology $H^*(-)$ in some cases (see [7]).

3.2.3. PROPOSITION. *Let X be an unfoldable pseudomanifold. Then we have:*

- $\mathbb{H}^{*}_{\overline{p}}(X) = H^{*}(X - \Sigma_X)$ *if* $\overline{p} > \overline{t}$.
- $\mathbb{H}^{*}_{\overline{q}}(X) = H^{*}\left(X, X_{\overline{q}}\right)$ *if* $\overline{q} \leq \overline{0}$ *and X is normal.*

Proof. By the above theorem it suffices to prove that

$$\mathbb{H}_{*}^{\overline{t}-\overline{p}}(X) = H_{*}(X - \Sigma_X)$$

and

$$\mathbb{H}_{*}^{\overline{t}-\overline{q}}\left(X, X_{\overline{q}}\right) = H_{*}\left(X, X_{\overline{q}}\right). \tag{16}$$

The first assertion follows directly from Proposition 2.2.2. For the second one, we consider the differential morphism A defined between

$$\frac{\left(AC_{*}^{\overline{t}-\overline{q}}(X) + AC_{*}^{\overline{t}-\overline{q}+\overline{1}}\left(X_{\overline{q}}\right)\right) \cap \partial^{-1}\left(AC_{*-1}^{\overline{t}-\overline{q}}(X) + AC_{*-1}^{\overline{t}-\overline{q}+\overline{1}}\left(X_{\overline{q}}\right)\right)}{AC_{*}^{\overline{t}-\overline{q}+\overline{1}}\left(X_{\overline{q}}\right) \cap \partial^{-1}\left(AC_{*-1}^{\overline{t}-\overline{q}+\overline{1}}\left(X_{\overline{q}}\right)\right)}$$

and $S_{*}(X)/S_{*}\left(X_{\overline{q}}\right)$ by $A\{\xi\} = \{\xi\}$. We prove, by induction on the depth, that the morphism A is a quasi-isomorphism. When the depth of X is 0, then A is the identity. In the general case, we use Bredon's trick (see the proof of Proposition 2.2.2) and we reduce the problem to a chart $X = \mathbb{R}^n \times cL_S$ with $X_{\overline{q}} = \mathbb{R}^n \times (cL_S)_{\overline{q}}$. Then we apply the product formula and we reduce the problem to $\left(cL_S, (cL_S)_{\overline{q}}\right)$. We have three cases:

- $\overline{q}(\vartheta) < 0$. Then $(cL_S)_{\overline{q}} = c(L_S)_{\overline{q}} \neq \emptyset$ and we have

$$\mathbb{H}_{*}^{\overline{t}-\overline{q}}\left(cL_S, (cL_S)_{\overline{q}}\right) \overset{2.2.6}{=} 0 = H_{*}\left(cL_S, c(L_S)_{\overline{q}}\right) = H_{*}\left(cL_S, (cL_S)_{\overline{q}}\right).$$

- $\overline{q}(\vartheta) = 0$ and $\overline{q} \neq \overline{0}$ on L_S. Then $(cL_S)_{\overline{q}} = c(L_S)_{\overline{q}} \neq \emptyset$ and we have

$$\begin{aligned}\mathbb{H}_{*}^{\overline{t}-\overline{q}}\left(cL_S, (cL_S)_{\overline{q}}\right) &\overset{2.2.6}{=} \mathbb{H}_{0}^{\overline{t}-\overline{q}}\left(L_S, (L_S)_{\overline{q}}\right) \overset{ind}{=} H_{0}\left(L_S, (L_S)_{\overline{q}}\right) \\ &\overset{norm}{=} 0 = H_{*}\left(cL_S, (cL_S)_{\overline{q}}\right).\end{aligned}$$

- $\overline{q} = 0$. Then $(cL_S)_{\overline{q}} = (L_S)_{\overline{q}} = \emptyset$ and we have

$$\begin{aligned}\mathbb{H}_{*}^{\overline{t}-\overline{q}}\left(cL_S, (cL_S)_{\overline{q}}\right) &\overset{2.2.6}{=} \mathbb{H}_{0}^{\overline{t}-\overline{q}}\left(L_S, (L_S)_{\overline{q}}\right) \overset{ind}{=} H_{0}\left(L_S, (L_S)_{\overline{q}}\right) \\ &\overset{norm}{=} \mathbb{R} = H_{*}\left(cL_S, (cL_S)_{\overline{q}}\right).\end{aligned}$$

This ends the proof. □

3.2.4. REMARK. Notice that we can replace the normality of X by the connectedness of the links $\{L_S \mid \overline{q}(S) = 0\}$. In particular, we have $\mathbb{H}^{*}_{\overline{q}}(X) = H^{*}(X, \Sigma_X)$ when $\overline{q} < \overline{0}$ without the normality condition.

In the direction homology $\mapsto$ cohomology we have the following de Rham theorem:

3.2.5. COROLLARY. *Let X be a normal unfoldable pseudomanifold. If $\overline{p}$ is a perversity, then we have the isomorphism*

$$\mathbb{H}_{*}^{\overline{p}}(X) = \mathbb{H}^{*}_{\max(\overline{0},\overline{t}-\overline{p})}(X),$$

Proof. Since $X_{\max(\overline{0},\overline{t}-\overline{p})} = \emptyset$, the cohomology $\mathbb{H}^{*}_{\max(\overline{0},\overline{t}-\overline{p})}(X)$ is isomorphic to

$$\mathbb{H}_{*}^{\overline{t}-\max(\overline{0},\overline{t}-\overline{p})}(X) = \mathbb{H}_{*}^{\min(\overline{p},\overline{t})}(X)$$

(cf. Theorem 3.2.2). It suffices to prove that the inclusion $SC_{*}^{\min(\overline{p},\overline{t})}(X) \hookrightarrow SC_{*}^{\overline{p}}(X)$ induces an isomorphism in cohomology. We proceed by induction on the depth. When the depth of X is 0, then $SC_{*}^{\min(\overline{p},\overline{t})}(X) = SC_{*}^{\overline{p}}(X) = S_{*}(X)$. In the general case, we use Bredon's trick (see the proof of Proposition 2.2.2) and we reduce the problem to a chart $X = \mathbb{R}^n \times cL_S$. We apply the product formula and we reduce the problem to $X = cL_S$. Now, we have two cases:

- $\overline{t}(\vartheta) < \overline{p}(\vartheta)$. Then
$$\mathbb{H}_{*}^{\min(\overline{p},\overline{t})}(cL_S) \overset{2.2.1}{=} \mathbb{H}_{0}^{\min(\overline{p},\overline{t})}(L_S) \overset{ind}{=} \mathbb{H}_{0}^{\overline{p}}(L_S) \overset{2.2.1,2.2.4}{=\!=\!=\!=} \mathbb{H}_{*}^{\overline{p}}(cL_S).$$
- $\overline{t}(\vartheta) \geq \overline{p}(\vartheta)$. Then
$$\begin{aligned}\mathbb{H}_{*}^{\min(\overline{p},\overline{t})}(cL_S) &\overset{2.2.1}{=} \mathbb{H}_{\leq \overline{t}(\vartheta)-\overline{p}(\vartheta)}^{\min(\overline{p},\overline{t})}(L_S) \overset{ind}{=} \mathbb{H}_{\leq \overline{t}(\vartheta)-\overline{p}(\vartheta)}^{\overline{p}}(L_S)\\ &\overset{2.2.1}{=} \mathbb{H}_{*}^{\overline{p}}(cL_S).\end{aligned}$$

This ends the proof. □

3.2.6. REMARK. Notice that we can replace the normality of X by the connectedness of the links $\{L_S \mid \overline{p}(S) > \overline{t}(S)\}$.

3.3. Poincaré duality. The intersection homology was introduced for the purpose of extending the Poincaré duality to singular manifolds (see [12]). The pairing is given by the intersection of cycles. For manifolds the Poincaré duality also derives from the integration of the wedge product of differential forms. This is also the case for stratified pseudomanifolds.

Let consider a compact and *orientable* stratified pseudomanifold X, that is, the manifold $X - \Sigma_X$ is an orientable manifold. Let m be the dimension of X. It was proved in [7] (see also [18]) that, for a perversity $\overline{p}$, with $\overline{0} \leq \overline{p} \leq \overline{t}$, the pairing $P\colon \Omega_{\overline{p}}^{i}(X) \times \Omega_{\overline{t}-\overline{p}}^{m-i}(X) \longrightarrow \mathbb{R}$, defined by $P(\alpha,\beta) = \int_{X-\Sigma_X} \alpha \wedge \beta$, induces the isomorphism $\mathbb{H}_{\overline{p}}^{*}(X) = \mathbb{H}_{\overline{t}-\overline{p}}^{m-*}(X)$. The same proof works for any perversity. For example, if $\overline{p} < \overline{0}$ or $\overline{p} > \overline{t}$, we obtain the Lefschetz duality $H^{*}(X,\Sigma_X) = H^{m-*}(X - \Sigma_X)$ (cf. Proposition 3.2.3 and Remark 3.2.4).

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