Illinois Journal of Mathematics
Volume 49, Number 3, Fall 2005, Pages 719–736
S 0019-2082

CAPACITY IN SUBANALYTIC GEOMETRY

TOBIAS KAISER

Abstract. In this article we study the capacity of subanalytic sets. First, we show that a subanalytic set and its closure have the same capacity. Using this, we then prove that for subanalytic sets in $\mathbb{R}^2$ the capacity density exists, and for arbitrary dimension we give connections to certain volume densities. Finally, we connect volume densities with fine limit points of subanalytic sets.

Introduction

In this article we begin to show that capacity and energy are tame concepts in subanalytic geometry.

It is well known that the volume of subanalytic sets exhibits a nice behaviour. For example, it was proved by Kurdyka and Raby (see [12]) that the volume density exists for subanalytic sets, and due to the work of Comte, Lion and Rolin (see [3], [4] and [15]), this density is continuous along certain strata and is even definable in the o-minimal structure of bounded analytic functions expanded by the exponential function. Similar nice properties are shared by other "measure quantities". As examples we mention the geodesic distance (see [11]) or the entropy (see [5]). In this paper we begin to develop a similar theory for capacity and energy. These are important concepts from potential theory, inspired by electrostatics, and related to classical boundary problems such as the Dirichlet problem.

Capacity density was introduced in potential theory (see [16] and [19]). There the relation to (Hausdorff) measure is also investigated; see also [18] and [11]. We show that the capacity density exists for subanalytic sets in dimension 2, and in arbitrary dimensions we give connections between the lower, resp. upper, capacity density and the volume density. Moreover, we establish connections between the volume density and the fine topology in the subanalytic case; see [1] for general aspects of the fine topology.

Received February 10, 2004; received in final form July 25, 2005.
2000 *Mathematics Subject Classification.* Primary 31A15, 32B20. Secondary 31C40.
Partially supported by the European RTN Network (Contract no. HPRN-CT-2001-00271).

The paper is organized as follows: We begin by describing (rather briefly) the definition and basic properties of subanalytic sets and (in greater detail) those of capacity. In Section 1 we prove that the capacity of a subanalytic set equals the capacity of its closure. In Section 2, we use this result to show that the capacity density exists for subanalytic sets in dimension 2. The capacity density is defined analogously to the volume density, and as in the case of volume density we reduce the problem to a set with conical structure by considering the tangent cone. This also enables us to show in Section 3 how the lower, resp. upper, capacity density and a certain volume density in codimension 1 are related in arbitrary dimensions. In the final section we use the author's results on the Dirichlet problem in subanalytic geometry (see [8] and [9]) to establish a connection between the volume density in codimension 2 and fine limit points, for which capacity also plays a role via Wiener's criterion.

The author thanks the referee for his very useful comments on the first version of the paper.

0. Basic definitions

Subanalytic sets. A semianalytic set is locally given by a finite number of equalities and inequalities of analytic functions. A subanalytic set is locally given by a projection of a relatively compact semianalytic set. Subanalytic sets have nice geometric properties; see [2], [10] and [14] for the basic definitions and concepts in subanalytic geometry. For example, subanalytic sets allow a good stratification, a fact we will be using throughout this article.

Good stratification (see [10]**).** Let $A \subset \mathbb{R}^n$ be a bounded subanalytic set and let $(B_j \mid j \in J)$ be a decomposition of A into finitely many subanalytic sets. Then there is a decomposition $(\Gamma_i \mid i \in I)$ of A into finitely many subanalytic C^1-manifolds, compatible with the decomposition $(B_j \mid j \in J)$, with the following properties:

(i) For each $i \in I$, after a suitable orthogonal coordinate transformation, there is a subanalytic domain $U_i \subset \mathbb{R}^{\dim \Gamma_i}$ and a subanalytic continuously differentiable Lipschitz function $f_i \colon U_i \to \mathbb{R}^{n-\dim \Gamma_i}$ such that

$$\Gamma_i = \mathrm{Graph}(f_i).$$

(ii) For each $i \in I$ we have

$$\overline{\Gamma_i} \cap A = \bigcup \{\Gamma_j \mid \Gamma_i \cap \overline{\Gamma_j} \neq \emptyset\}.$$

Moreover, given the bounded subanalytic set and its decomposition, we can find a good stratification such that the Lipschitz constants of the involved Lipschitz functions f_i are arbitrarily small.

Energy and capacity. These basic concepts can be found in any book about potential theory, for example, in [1] or [7].

Let ν be a non-trivial positive Borel measure in $\mathbb{R}^n, n \geq 2$, with compact support. Then the Newton potential of ν is defined as follows:

$$U^\nu \colon \mathbb{R}^n \longrightarrow \begin{cases}]-\infty,\infty] & \text{if } n=2,\\ ]0,\infty] & \text{if } n\geq 3,\end{cases}$$

$$x \longmapsto \begin{cases} -\displaystyle\int_{\mathbb{R}^2} \log|x-y|d\nu(y) & \text{if } n=2,\\ \displaystyle\int_{\mathbb{R}^n} \frac{d\nu(y)}{|x-y|^{n-2}} & \text{if } n\geq 3.\end{cases}$$

A Newton potential U^ν is superharmonic on $\mathbb{R}^n$ and harmonic outside $\operatorname{supp}(\nu)$, the support of ν. The energy

$$I(\nu) \in \begin{cases}]-\infty,\infty] & \text{if } n=2,\\ ]0,\infty] & \text{if } n\geq 3,\end{cases}$$

of ν is defined as

$$I(\nu) \colon = \begin{cases} -\displaystyle\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y|d\nu(y)d\nu(x) & \text{if } n=2,\\ \displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{d\nu(y)d\nu(x)}{|x-y|^{n-2}} & \text{if } n\geq 3.\end{cases}$$

Now let K be a compact subset of $\mathbb{R}^n, n \geq 2$, and let $\mathcal{P}(K)$ be the set of all positive Borel measures with support contained in K and with mass 1. Then there is exactly one measure $\mu \in \mathcal{P}(K)$ with

$$I(\mu) = \inf\{I(\nu) \mid \nu \in \mathcal{P}(K)\}.$$

μ is called the equilibrium measure of K. The capacity $c(K) \in [0,\infty[$ is then defined as

$$c(K) = \begin{cases} e^{-I(\mu)} & \text{if } n=2,\\ \dfrac{1}{I(\mu)} & \text{if } n\geq 3.\end{cases}$$

The equilibrium measure μ of a compact set K of positive capacity has the property that its Newton potential is constant on $\operatorname{supp}(\mu)$ outside a set of capacity 0. (On this exceptional set the Newton potential can be smaller than this constant.) The capacity of a compact set K can be also described in the following way:

- If $n \geq 3$, then

$$c(K) = \sup\Big\{\nu(K) \mid \nu \text{ positive Borel measure with support contained in } K \text{ and } U^\nu \leq 1 \text{ on } K\Big\}.$$

- If $n=2$ and $\operatorname{diam} K<1$, where diam is the diameter, then

$$c(K)=\sup\Big\{e^{-1/\nu(K)} | \nu \text{ positive Borel measure with support contained in } K \text{ and } U^\nu\leq 1 \text{ on } K\Big\}.$$

Now for a Borel set $E\subset\mathbb{R}^n, n\geq 2$, the capacity of E is defined as

$$c(E)=\sup\{c(K)\mid K\subset E \text{ compact}\}.$$

EXAMPLE. We denote the open unit ball in $\mathbb{R}^n$, $n\geq 2$, by $B_1(0)$ and the closed unit ball in $\mathbb{R}^n$, $n\geq 2$, by $\overline{B}_1(0)$. Then $c(B_1(0))=c(\overline{B}_1(0))=1$.

We list some basic facts about the measure quantity capacity:

- **Monotonicity:** If $E_1\subset E_2$ are Borel sets, then $c(E_1)\leq c(E_2)$.
- **Subadditivity:** If $E_m, m\in\mathbb{N}$, are Borel sets in $\mathbb{R}^n, n\geq 3$, then

$$c\left(\bigcup_{m\in\mathbb{N}}E_m\right)\leq\sum_{m\in\mathbb{N}}c(E_m).$$

- **Continuity from below:** If $(E_m\mid m\in\mathbb{N})$ is an increasing sequence of Borel sets, then

$$c\left(\bigcup_{m\in\mathbb{N}}E_m\right)=\lim_{m\to\infty}c(E_m).$$

For compact sets more is true:

- **Continuity from above:** If $(K_m\mid m\in\mathbb{N})$ is a decreasing sequence of compact sets, then

$$c\left(\bigcap_{m\in\mathbb{N}}K_m\right)=\lim_{m\to\infty}c(K_m).$$

- **"Capacity lives on the boundary":** If K is compact, then $c(K)=c(\partial K)$ where $\partial K:=K\setminus\overset{\circ}{K}$.

Finally, we quote the behaviour of capacity under elementary maps:

- Capacity is invariant under translations and orthogonal coordinate transformations.
- Dilatation with a factor $r>0$ has the following consequences: If $E\subset\mathbb{R}^n, n\geq 2$, is a Borel set, then

$$c(rE)=\begin{cases} rc(E) & \text{if } n=2,\\ r^{n-2}c(E) & \text{if } n\geq 3.\end{cases}$$

1. Capacity of the closure of a subanalytic set

In general it is not true that the capacity of a (bounded) set and the capacity of its closure are the same. A counterexample is $E\colon = \mathbb{Q} \cap [0,1] \subset \mathbb{R}^2$. Here $c(E) = 0$ since E is countable, whereas $c([0,1]) = 1/4$. Analogous examples can obviously be found in any dimension. But such phenomena do not occur in subanalytic geometry:

Theorem 1. *The capacity of a subanalytic set in $\mathbb{R}^n, n \geq 2$, and the capacity of its closure are the same.*

Before proving this theorem we make the following definition.

Definition 1. *Let $A \subset \mathbb{R}^n$ be a set and let $\delta > 0$. Then we define*

$$A_\delta\colon = \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, A) < \delta\}.$$

Proof of Theorem 1. Let $E \subset \mathbb{R}^n, n \geq 2$, be a subanalytic set. By "continuity from below" it is enough to consider the case when E is bounded. (Otherwise we consider the intersection of E with arbitrarily large balls.) Using a dilatation we may also assume that $\operatorname{diam} E < 1$. We have to show:

(a) $c(E) \leq c(\overline{E})$: This is clear from the previous section.

(b) $c(E) \geq c(\overline{E})$: If $\overline{E}$ is polar, i.e., $c(\overline{E}) = 0$, then this is obviously true, so we may assume $c(\overline{E}) > 0$. Since $\operatorname{diam} E < 1$, we have $c(\overline{E}) < 1$. Let μ be the equilibrium measure of $\overline{E}$. We define

$$\nu\colon = \begin{cases} -\dfrac{1}{\log c(\overline{E})}\ \mu & \text{if } n = 2, \\ c(\overline{E})\ \mu & \text{if } n \geq 3. \end{cases}$$

Then $U^\nu \leq 1$ on $\overline{E}$ (on $\mathbb{R}^n$) and $\operatorname{supp} \nu \subset \overline{E} \setminus \overset{\circ}{E}$. Let $((C_i)_{i \in I}, (D_j)_{j \in J})$ be a good stratification of $\overline{E}$ such that $E = \bigcup_{j \in J} D_j$. We choose a stratification such that the Lipschitz constants of the involved functions are bounded by $1/2$. We set $I'\colon = \{i \in I \mid \dim C_i = n-1\}$ and $J'\colon = \{j \in J \mid \dim D_j \leq n-1\}$. For $\Lambda\colon = \bigcup_{i \notin I'} C_i$ we have $c(\Lambda) = 0$ (see [1]), and hence for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$(*) \qquad \nu(\overline{E} \setminus \Lambda_\delta) \geq \nu(\overline{E}) - \varepsilon = \begin{cases} -\dfrac{1}{\log c(\overline{E})}\ - \varepsilon & \text{if } n = 2, \\ c(\overline{E})\ - \varepsilon & \text{if } n \geq 3. \end{cases}$$

We set $\nu_\delta\colon = \nu\big|_{\overline{E} \setminus \Lambda_\delta}$ and we choose some $0 < \varepsilon' < \varepsilon$ such that

- (i) for each $i \in I'$ there is a translation τ_i of length $\frac{1}{2}(\varepsilon')^2$ with $\tau_i(C_i \setminus \Lambda_\delta) \subset E$,
- (ii) for each $i_1 \neq i_2 \in I'$ we have $\operatorname{dist}(C_{i_1} \setminus \Lambda_\delta,\ C_{i_2} \setminus \Lambda_\delta) \geq \varepsilon'$,
- (iii) for each $i \in I'$, $j \in J'$ we have $\operatorname{dist}(C_i \setminus \Lambda_\delta,\ D_j) \geq \varepsilon'$.

(i) can be seen as follows: Let $i' \in I'$ be fixed. Since $((C_i)_{i\in I}, (D_j)_{j\in J})$ is a good stratification of $\overline{E}$, there is, after a suitable orthogonal coordinate transformation, a subanalytic domain $U \subset \mathbb{R}^{n-1}$ and a subanalytic continuously differentiable Lipschitz function $f\colon U \to \mathbb{R}$ with Lipschitz constant bounded by $1/2$ such that

$$C_{i'} = \operatorname{graph}(f).$$

Moreover, there is some $j \in J$ with $C_{i'} \subset \overline{D_j} \setminus D_j$. The stratum $D_j \subset E$ is open, and we may assume that D_j lies "above" $C_{i'}$. For each $x \in U$ let

$$\sigma_x \colon = \sup\{\sigma > 0 \mid (x, f(x) + \sigma) \in D_j\}.$$

Given $x \in U$ we show that there is some $r > 0$ such that $\sigma_y \geq (\sigma_x + r)/2$ for $y \in B_r(x) \cap U$: We have $(x, f(x) + \sigma_x/2) \in D_j$. Since D_j is open, there is some $r > 0$ with

$$\Big\{(x', x'') \in \mathbb{R}^{n-1} \times \mathbb{R} \mid |x' - x| \leq r,\ |x'' - (f(x) + \frac{\sigma_x}{2})| \leq r\Big\} \subset D_j.$$

Let $\hat{\sigma} \colon = (\sigma_x + r)/2$. We have for $y \in B_r(x) \cap U$

$$\begin{aligned} |f(y) + \hat{\sigma} - (f(x) + \frac{\sigma_x}{2})\ | &\leq |f(y) - f(x)| + |\hat{\sigma} - \frac{\sigma_x}{2}| \\ &\leq \frac{1}{2}|y - x| + \frac{r}{2} \leq r, \end{aligned}$$

and hence $(y, f(y) + \hat{\sigma}) \in D_j$. Let now $K \colon = \{x \in U \mid (x, f(x)) \notin \Lambda_\delta\}$. Then K is compact. With the previous claim we see that there is some $\overline{\sigma} > 0$ such that $(x, f(x) + \overline{\sigma}) \in D_j$ for all $x \in U$. Hence we obtain (i).

Next, we define

$$\sigma_\delta \colon = \nu_\delta\ \big|_E + \sum_{i \in I'} \nu_\delta\big|_{C_i \setminus \Lambda_\delta} \circ \tau_i^{-1},$$

which is a positive Borel measure with $\operatorname{supp}(\sigma_\delta) \subset E$. Potentials are harmonic outside the support of the measure. Thus, applying Harnack's inequality to a ball of radius ε' (see [1, p. 13]), we get using (i)–(iii) that for $x \in \operatorname{supp}(\sigma_\delta)$

$$U^{\sigma_\delta}(x) \leq \frac{1 + \varepsilon'}{(1 - \varepsilon')^{n-1}},$$

and hence by $(*)$ and the characterization of capacity in Section 0 that

$$c(E) \geq \begin{cases} \big(c(\overline{E})\big)^{\frac{1}{1+\varepsilon \log c(\overline{E})} \cdot \frac{1+\varepsilon'}{(1-\varepsilon')^{n-1}}} & \text{if } n = 2, \\ \big(c(\overline{E}) - \varepsilon\big) \cdot \dfrac{(1 - \varepsilon')^{n-1}}{1 + \varepsilon'} & \text{if } n \geq 3. \end{cases}$$

Letting $\varepsilon \searrow 0$ gives now the theorem. □

2. Capacity density of a subanalytic set

Using Theorem 1 we prove in Theorem 2 below that the capacity density exists for subanalytic sets in dimension 2.

DEFINITION 2. *Let $E \subset \mathbb{R}^n, n \geq 2$, be a Borel set. We define for $x \in \mathbb{R}^n$ the lower capacity density of E at x by*

$$\gamma_*(E,x):=\varliminf_{r\to 0}\frac{c\,(E\cap B_r(x))}{c\,(B_r(x))} = \begin{cases} \varliminf_{r\to 0}\dfrac{c\,(E\cap B_r(x))}{r} & \textit{if } n=2,\\ \varliminf_{r\to 0}\dfrac{c\,(E\cap B_r(x))}{r^{n-2}} & \textit{if } n\geq 3.\end{cases}$$

and the upper capacity density of E at x by

$$\gamma^*(E,x):=\varlimsup_{r\to 0}\frac{c\,(E\cap B_r(x))}{c\,(B_r(x))} = \begin{cases} \varlimsup_{r\to 0}\dfrac{c\,(E\cap B_r(x))}{r} & \textit{if } n=2,\\ \varlimsup_{r\to 0}\dfrac{c\,(E\cap B_r(x))}{r^{n-2}} & \textit{if } n\geq 3.\end{cases}$$

If $\gamma_*(E,x) = \gamma^*(E,x)$, then we define the capacity density of E at x as $\gamma(E,x) := \gamma_*(E,x) = \gamma^*(E,x)$.

EXAMPLE 1. Let $E \subset \mathbb{R}^n$ be a Borel set with conical structure at 0; that is, for every $y \in E$ and $t > 0$ also $ty \in E$. Then $\gamma(E,0)$ exists and we have $\gamma(E,0) = c(E \cap B_1(0))$.

Proof. This is clear because

$$c(E\cap B_r(0)) = \begin{cases} r\;c(E\cap B_1(0)) & \text{if } n=2,\\ r^{n-2}\;c(E\cap B_r(0)) & \text{if } n\geq 3.\end{cases} \qquad \square$$

We can now state the following result in dimension 2, which is analogous to the existence of the volume density proved in [12]. We use the ideas of the proof given there, but the situation here is much more complicated, mainly because capacity is not continuous for decreasing sequences of open subsets. We intend to extend this result to any dimension in a subsequent paper.

THEOREM 2. *Let $E \subset \mathbb{R}^2$ be a subanalytic set. Then for each $x \in \mathbb{R}^2$ the capacity density $\gamma(E,x)$ exists.*

We prove Theorem 2 by reducing the problem to a set with conical structure, the tangent cone; compare [12].

Definition 3. *Let $E \subset \mathbb{R}^n, n \geq 1$, be a set and let $x \in \mathbb{R}^n$. Then the tangent cone of E at x is defined by*

$$C_x(E) := \Big\{ y \in \mathbb{R}^n \mid \forall \varepsilon > 0 \, \exists z \in E \, \exists \lambda > 0 \\ |z - x| < \varepsilon \, \wedge \, |\lambda(z - x) - y| < \varepsilon \Big\}.$$

Remark 1. Let $E \subset \mathbb{R}^n$, $n \geq 1$, and $x \in E$. Then $C_x(E)$ is a set with conical structure and $C_x(E) = C_x(\overline{E})$.

Remark 2. If $E \subset \mathbb{R}^n$, $n \geq 1$, is subanalytic, then for each $x \in \mathbb{R}^n$ $C_x(E)$ is subanalytic and $\dim C_x(E) \leq \dim_x E$.

Proof of Theorem 2. We show for each $x \in \mathbb{R}^2$ that $\gamma(E, x) = \gamma(C_x(E), x)$. Since capacity is invariant under translations we may assume that $x = 0$. First we show that $\gamma^*(E, 0) \leq \gamma(C_0(E), 0)$. For $r > 0$ let C_r be the conical set generated by $E \cap B_r(0)$, i.e.,

$$C_r := \{ y \in \mathbb{R}^n \mid \exists \lambda > 0 \quad \lambda y \in E \cap B_r(0) \}.$$

The function $r \mapsto \; c\,(C_r \cap B_1(0))$ is decreasing, so $\lim_{r \to 0} \; c\,(C_r \cap B_1(0))$ exists. Hence we get by Theorem 1, Example 1 and the properties of capacity from Section 0 that

$$\begin{aligned} \gamma^*(E, 0) &= \varlimsup_{r \to 0} \frac{c(E \cap B_r(0))}{r} \leq \varlimsup_{r \to 0} \frac{c(C_r \cap B_r(0))}{r} \\ &= \varlimsup_{r \to 0} c\,(C_r \cap B_1(0)) = \lim_{r \to 0} \; c\,(\overline{C}_r \cap \overline{B}_1(0)) \\ &= c\,\left(\bigcap_{r > 0} \overline{C}_r \cap \overline{B}_1(0) \right) = c\,(C_0(E) \cap \overline{B}_1(0)) \\ &= \gamma\,(C_0(E), 0). \end{aligned}$$

Thus, to prove the theorem, we have to show that $\gamma_*(E, 0) \geq \gamma\,(C_0(E), 0)$.

By Theorem 1 we may assume that E is closed. There is a small $\delta > 0$ with $\delta < 1$ such that $\partial E \cap \overline{B}_\delta(0)$ consists of N half-branches Γ_i, $1 \leq i \leq N$, with $\Gamma_i \cap \Gamma_j = \{0\}$, $i \neq j$. By shrinking δ if necessary we can assume that each Γ_i, $1 \leq i \leq N$, can be parametrized in the distance from the origin. Let $\lambda_i \colon [0, \delta[\to \mathbb{R}^2$, $1 \leq i \leq N$, be the parametrizations. For $0 < r < \delta$ we order $\partial B_r(0)$ counterclockwise with starting point $\lambda_1(r)$, i.e., we define $\arg_r \lambda_1(r) = 0$. (This is not the usual order on $\partial B_r(0)$, and the order $\arg_r$ depends on r.) By renumbering the indices we can assume that

$$0 = \arg_r \lambda_1(r) < \arg_r \lambda_2(r) < \cdots < \arg_r \lambda_N(r) < 2\pi \quad \text{for } 0 < r < \delta.$$

For $i \in \{1, \ldots, N\}$ let $i_- := i - 1 \mod N$ and $i_+ := i + 1 \mod N$. For $i \in \{1, \ldots, N\}$ we set

$$A_i^+ := \Big\{ x \in \overline{B}_\delta(0) \setminus \{0\} \mid \arg_{|x|} \, \lambda_i(|x|) < \arg_{|x|}(x) < \arg_{|x|} \, \lambda_{i_+}(|x|) \Big\}$$

and

$$A_i^-: = \left\{ x \in \overline{B}_\delta(0) \setminus \{0\} \mid \arg_{|x|}\ \lambda_{i_-}(|x|) < \arg_{|x|}(x) < \arg_{|x|}\ \lambda_i(|x|) \right\}.$$

Note that $A_i^+ = A_{i_+}^-$ and $A_i^- = A_{i_-}^+$. By shrinking δ if necessary we can assume that $A_i^* \subset E$ or $A_i^* \cap E = \emptyset$, $1 \le i \le N$, with $* \in \{+, -\}$. Finally, for $i \in \{1, \ldots, N\}$ we define

$$\alpha_i^+: = \sphericalangle\ (\dot{\lambda}_i(0),\ \dot{\lambda}_{i_+}(0)),$$
$$\alpha_i^-: = \sphericalangle\ (\dot{\lambda}_{i_-}(0),\ \dot{\lambda}_i(0)).$$

All angles are computed counterclockwise. Now we are making the following

Reductions I. Let $i \in \{1, \ldots, N\}$.

(1) If $A_i^- \not\subset E$ and $A_i^+ \not\subset E$ and $\alpha_i^- = 0$ or $\alpha_i^+ = 0$, then we can cancel λ_i.

(2) If $\alpha_i^+ = 0$ and $A_i^+ \subset E$, then we can cancel λ_i and A_i^+.

The subanalytic set obtained after the cancellation is contained in the original one and has the same tangent cone, so it is enough to consider the "reduced" situation, where E allows no cancellations as described in Reductions I.

Next, we straighten the "isolated half-branches". Let

$$I': = \left\{ i \in \{1, \ldots, N\} \mid \dim_{\lambda_i(r)}\ E = 1 \quad \text{for } 0 < r < \delta \right\}.$$

For $i \in I'$ we replace $\lambda_i(r)$ by $r\,\dot{\lambda}_i(0)$ and thus obtain a subanalytic set E_1, in which the half-branches Γ_i are replaced by the tangent half-lines $G_i: = \{r\,\dot{\lambda}_i(0) \mid 0 \le r \le \delta\}$, $i \in I'$. We show that $\gamma_*(E, 0) \ge \gamma_*(E_1, 0)$.

Proof. For $0 < r < \delta$ let μ_r be the equilibrium measure of $E_1 \cap \overline{B}_r(0)$. We define

$$\nu_r: = -\frac{1}{\log\, c\,(E_1 \cap \overline{B}_r(0))} \cdot \mu_r.$$

Then $U^r \le 1$ on $E_1 \cap \overline{B}_r(0)$ and

$$\nu_r(E_1 \cap \overline{B}_r(0)) = -\frac{1}{\log\, c\,(E_1 \cap \overline{B}_r(0))}.$$

For $i \in I'$ we define the subanalytic function $f_i \colon G_i \to \Gamma_i$ as follows: Let $x \in G_i$. Then $f_i(x)$ is the unique $y \in \Gamma_i$ with $|y| = |x|$. We have the following situation at E (after Reductions I): If $i \in I'$, then Γ_i is an isolated half-branch by definition and α_{i_+}, α_{i_-} are not zero. Since Γ_i is tangent to G_i at 0, for $i \in I'$, we obtain (after shrinking δ if necessary) a subanalytic function

$$L \colon]0, \delta[\longrightarrow \mathbb{R}_{>0} \quad \text{with} \quad \lim_{r \to 0} L(r) = 1$$

such that for $0 < r < \delta$

(α) $|f_i(x) - f_j(y)| \ge L(r)|x - y|$ for $x \in G_i \cap \overline{B}_r(0)$, $y \in G_j \cap \overline{B}_r(0)$ and $i, j \in I'$ with $i \ne j$.

(β) $|f_i(x)-y| \geq L(r)|x-y|$ for $x \in G_i \cap \overline{B}_r(0)$, $y \in E_1 \setminus \bigcup_{i\in I'} G_i$ and $i \in I'$.

We define for $0<r<\delta$ the measure

$$\sigma_r :\; = \nu_r\big|_{E_1\setminus \bigcup_{i\in I'} G_i} + \sum_{i\in I'} \nu_r\big|_{G_i} \circ f_i^{-1}.$$

Then $\operatorname{supp}(\sigma_r) \subset E \cap \overline{B}_r(0)$ and

$$\sigma_r(E\cap \overline{B}_r(0)) = -\frac{1}{\log\, c\,(E_1 \cap \overline{B}_r(0))}.$$

Now we want to estimate U^{σ_r} on $E\cap \overline{B}_r(0)$. Let $x \in E \cap \overline{B}_r(0)$.

Case 1: $x \in \Gamma_i$ for some $i \in I'$. Let $y \in G_i \cap \overline{B}_r(0)$ with $x = f_i(y)$. We get

$$\begin{aligned}
U^{\sigma_r}(x) &= \int_{\mathbb{R}^2} \log \frac{1}{|x-z|}\, d\sigma_r(z)\\
&= \int_{E_1\setminus \bigcup_{j\in I'} G_j} \log \frac{1}{|f_i(y)-z|} d\nu_r(z)\\
&\qquad\qquad + \sum_{j\in I'} \int_{G_j} \log \frac{1}{|f_i(y)-f_j(w)|} d\nu_r(w)\\
&\overset{(\alpha),(\beta)}{\leq} \int_{E'} \log \frac{1}{|y-z|} d\nu_r(z) - \log L(r)\cdot \nu_r(\mathbb{R}^2)\\
&= U^{\sigma_r}(y) + \frac{\log L(r)}{\log\, c\,(E_1\cap \overline{B}_r(0))}\\
&\leq 1 + \frac{\log L(r)}{\log\, c\,(E_1\cap \overline{B}_r(0))}.
\end{aligned}$$

Case 2: $x \notin \bigcup_{i\in I'} \Gamma_i$. With a similar computation we get the same result.

$$U^{\sigma_r}(x) \leq 1 + \frac{\log L(r)}{\log\, c\,(E_1\cap \overline{B}_r(0))}.$$

Using the characterization of capacity from Section 0 we get

$$\begin{aligned}
c\,(E\cap \overline{B}_r(0)) &\geq e^{\log\, c\,(E_1\cap \overline{B}_r(0))\left(1+\frac{\log L(r)}{\log\, c\,(E_1\cap \overline{B}_r(0))}\right)}\\
&= c\,(E_1\cap \overline{B}_r(0)) L(r).
\end{aligned}$$

Finally we conclude that

$$\gamma_*(E,0) \geq \gamma_*(E_1,0) \lim_{r\to 0} L(r) = \gamma_*(E_1,0).$$

Hence, since $C_0(E) = C_0(E_1)$, we can assume that the "isolated half-branches" of E are half-lines.

Reductions II. If $0 < \alpha_i^+ < 2\pi$ and $A_i^+ \subset E$, we replace A_i^+ by $A_i^+ \cap C_0(A_i^+)$.

The subanalytic set obtained after the cancellation is contained in the original one and has the same tangent cone, so it is enough to give a proof in this reduced situation: If $\Gamma_i \subset E$, $i \in I$, is an isolated half-branch, then Γ_i is a half-line, and if for $i \in I$ with $0 < \alpha_{i_+} < 2\pi$ and $A_{i_+} \subset E$, then $A_{i_+} \subset C_0(A_{i_+})$. Now for $0 < r < \delta$ we define F_r as the convex hull of the set

$$\{0\} \cup \{\lambda_i(r) \mid i \in I\} \cup \Big\{x \in \partial B_r(0) \mid \exists\, i \in I \text{ with } A_i^+ \subset E \\ \text{and } \arg_r \lambda_i(r) < \arg_r(x) < \arg_r \lambda_{i_+}(r)\Big\}.$$

Then by the "reduced" situation we have $F_r \subset E \cap \overline{B}_r(0)$, after shrinking δ if necessary. Let $\rho_t \colon \mathbb{R}^2 \to \mathbb{R}^2$, $x \mapsto tx$, be the dilatation with factor $t > 0$. Then we have by construction

$$\overline{\bigcup_{r>0} \rho_{1/r}(F_r)} = C_0(E).$$

Finally, using Theorem 1, Example 1 and the basic properties of capacity from Section 0, we conclude that

$$\begin{aligned}
\gamma_*(E,0) &= \varliminf_{r\to 0} \frac{c\,(E \cap \overline{B}_r(0))}{r} \geq \varliminf_{r\to 0} \frac{c\,(F_r)}{r} \\
&= \varliminf_{r\to 0} c\,(\rho_{1/r}(F_r)) = c\left(\bigcup_{r>0} \rho_{1/r}(F_r)\right) \\
&= c\left(\overline{\bigcup_{r>0} \rho_{1/r}(F_r)}\right) = c\,(C_0(E)) \\
&= \gamma(C_0(E),0).
\end{aligned}$$

$\square$

3. Capacity density and volume density

In the previous section we proved that for a subanalytic set in $\mathbb{R}^2$ the capacity density exists at every point. We followed there the ideas of the proof for the existence of the volume density in [12], i.e., we reduced the problem to the tangent cone. There are in general connections between the two densities based on the fact that the capacity "lives" on the boundary. These connections are special for the subanalytic case; for the general connections in potential theory see, for example, [6] and [8].

THEOREM 3. *Let $E \subset \mathbb{R}^n$, $n \geq 2$, be a subanalytic set. Then we have for $x \in \mathbb{R}^n$*

$$\gamma^*(E,x) > 0 \Longrightarrow \sup_{\substack{\Gamma \text{ suban.} \\ \dim \Gamma = n-1}} \Theta_{n-1}(E \cap \Gamma) > 0,$$

$$\gamma_*(E,x) > 0 \Longleftarrow \sup_{\substack{\Gamma \text{ suban.} \\ \dim \Gamma = n-1}} \Theta_{n-1}(E \cap \Gamma) > 0.$$

Thereby, given a subanalytic set $E \subset \mathbb{R}^n, n \geq 2$, and $x \in \mathbb{R}^n$, we have

$$\Theta_k(E,x) = \lim_{r \to 0} \frac{\mathrm{vol}_k(E \cap B_r(x))}{\mathrm{vol}_k(B_r(0))},$$

with $\mathrm{vol}_k(-)$ *the k-dimensional Hausdorff measure.*

Proof. "$\Longrightarrow$:" Suppose that $\gamma^*(E,x) > 0$. Then, as in the previous section we get $\gamma(C_x(E),x) > 0$. Hence $\dim C_x(E) \geq n-1$ and, as a consequence $\dim_x E \geq n-1$.

Case 1: $\dim_x E = n-1$. Choose $r > 0$ such that $\dim(E \cap B_r(x)) = n-1$. Take $\Gamma\colon = E \cap B_r$. Then $C_x(E) = C_x(\Gamma)$ and so

$$\Theta_{n-1}(\Gamma,x) \geq \Theta_{n-1}(C_x(E),x) > 0.$$

Case 2: $\dim_x E = n$. If $\dim C_x(E \setminus \mathring{E}) \geq n-1$, we can apply the first case to the set $E \setminus E^0$ since $\dim_x E \setminus E^0 = n-1$. Otherwise we have $\dim C_x(\mathring{E}) \geq n-1$ and therefore may assume that E is open. Moreover, we may assume by restriction to a half-space that E is contained in a half-space with x on the boundary. Then $\dim \partial C_x(E) = n-1$. We then define for $\sigma > 0$

$$U\colon = \{y \in E \mid \mathrm{dist}(y, \overline{E} \setminus E) \geq |y-x|^\sigma\}.$$

For large enough $\sigma > 0$ we have $C_x(U) = C_x(E)$. With $\Gamma\colon = \overline{U} \setminus (U \cup \{x\}) \subset E$ we get because of $C_x(\Gamma) = \partial C_x(E)$ that

$$\Theta_{n-1}(\Gamma,x) = \Theta_{n-1}(C_x(\Gamma),x) = \Theta_{n-1}(\partial C_x(E),x) > 0.$$

"$\Longleftarrow$:" Suppose that there is a subanalytic set $\Gamma \subset E$ with $\dim \Gamma = n-1$ and $\Theta_{n-1}(\Gamma,x) > 0$. Because of the translation invariance of capacity and volume we may assume that $x = 0$. Then by our good stratification we get, after an appropriate orthogonal coordinate transformation, the following situation: There is a subanalytic domain $U \subset \mathbb{R}^{n-1}$ containing a cone K with vertex in 0 and a subanalytic Lipschitz function $f\colon U \to \mathbb{R}$ such that $\mathrm{Graph}(f) \subset E$. The existence of such a cone follows from the inequality $\Theta_{n-1}(\Gamma,0) > 0$ (see [12]). Here a cone is defined as follows.

Definition 4. *A cone $K \subset \mathbb{R}^n, n \geq 1$, with vertex in $x \in \mathbb{R}^n$ and central vector $v \in \mathbb{R}^n \setminus \{0\}$ is a set*

$$K\colon = \left\{ y \in \mathbb{R}^n \setminus \{x\} \mid \frac{\langle y-x, v\rangle}{|y-x|\,|v|} > \alpha \right\} \cap B_r(x)$$

with some $\alpha > 0$ and some $r > 0$.

By the same argument as in the proof of Theorem 2 there are $C, C' > 0$ such that for small $r > 0$

$$\begin{aligned} c\,(\mathrm{Graph}(f) \cap \overline{B}_r(0)) &\geq C\,c\,(K \cap B_r(0)) \\ &\geq C' \cdot \begin{cases} r & \text{if } n = 2, \\ r^{n-2} & \text{if } n \geq 3. \end{cases} \end{aligned}$$

This shows that $\gamma_*(E, 0) > 0$. □

In dimension $n-1$ we have the following nice correspondence.

Corollary 4. *Let $E \subset \mathbb{R}^n$, $n \geq 2$, be a subanalytic set. Let $x \in \mathbb{R}^n$ with $\dim_x E = n-1$. Then we have*

$$\gamma^*(E, x) > 0 \Longrightarrow \Theta_{n-1}(E, x) > 0,$$
$$\gamma_*(E, x) > 0 \Longleftarrow \Theta_{n-1}(E, x) > 0.$$

The connection with the volume density in dimension n, however, works only in one direction:

Remark 3. Let $E \subset \mathbb{R}^n, n \geq 2$, be a subanalytic set. For $x \in \mathbb{R}^n$ we have

$$\Theta_n(E, x) > 0 \implies \gamma_*(E, x) > 0.$$

The implication $\gamma^*(E, x) > 0 \Rightarrow \Theta_n(E, x) > 0$ is false, in general; a counterexample is obtained by taking

$$E\colon = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \leq |(x_1, \ldots, x_{n-1})|^2 \right\}$$

and letting x be the origin. In this case, $\gamma^*(E, 0) > 0$ since $\dim C_0(E) = n-1$, but for the same reason $\Theta_n(E, 0) = 0$.

Remark 4. In contrast to the case of volume density, the multiplicity plays no role for the capacity density. For example, if

$$E = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^3 = x_2^2\},$$

then

$$\gamma(E, 0) = \gamma(C_0(E), 0),$$

whereas

$$\Theta_1(E, 0) = 2\,\Theta_1(C_0(E), 0).$$

Here $C_0(E) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = 0, x_1 \geq 0\}$ and 2 is the multiplicity of E along $C_0(E)$.

4. Fine topology and volume density

The class of superharmonic (resp. subharmonic) functions forms a natural generalization of the class of harmonic functions. For example, Newton potentials are only harmonic outside the support of the measure, but they are superharmonic on $\mathbb{R}^n$. In contrast to harmonic functions, superharmonic functions need not to be continuous. Therefore in potential theory there is the well known fine topology, the coarsest topology on $\mathbb{R}^n, n \geq 2$, such that every superharmonic function in $\mathbb{R}^n$ is continuous with respect to the topology. The fine topology is (obviously) finer than the Euclidean topology. A limit point in the fine topology is called a fine limit point.

In this section we characterize fine limit points of subanalytic sets by a certain volume density in codimension 2, using the author's work on the Dirichlet problem for subanalytic domains (see [8] and [9]) and the following facts (see [1]).

The theorem below states classical equivalent conditions (including the capacity) for a boundary point to be not a fine limit point of a given Borel set.

THEOREM 5. *Let $E \subset \mathbb{R}^n$, $n \geq 2$, be a Borel set. Let $x \in \overline{E} \setminus E$. Then the following are equivalent:*

(a) *E is thin at x, i.e., x is not a fine limit point of E.*

(b) *There is some $r > 0$ and some superharmonic function u on $B_r(x)$ such that*
$$\varliminf_{\substack{y \in E \\ y \to x}} u(y) > u(x).$$

(c) *There is some $r > 0$ and some superharmonic function u on $B_r(x)$ such that*
$$\lim_{\substack{y \in E \\ y \to x}} u(y) = \infty \quad \text{and} \quad u(x) < \infty.$$

(d) *If $n = 2$, then*
$$\int_0^{1/2} -\frac{1}{\log c\,(E \cap \overline{B}_r(x))} \cdot \frac{1}{r}\, dr = \infty,$$
and if $n \geq 3$, then
$$\int_0^1 \frac{c\,(E \cap \overline{B}_r(x))}{r^{n-1}}\, dr = \infty.$$

The last condition is known as Wiener's criterion (see [1]). Using our results about the capacity of subanalytic sets we can show the following:

Corollary 6. *Let $E \subset \mathbb{R}^n$, $n \geq 2$, be a subanalytic set. Let $x \in \overline{E} \setminus E$. Then x is a fine limit point of E if and only if it is a fine limit point of $\overline{E} \setminus \{x\}$.*

Proof. This is a simple consequence of Wiener's criterion and Theorem 1, using the fact that for a subanalytic set E and $x \in \mathbb{R}^n$

$$\overline{E \cap B_r(x)} = \overline{E} \cap \overline{B}_r(x)$$

for all small $r > 0$. □

Therefore we can concentrate on closed subanalytic sets; for closed sets there is another nice description of thinness (see [1]):

Theorem 7. *Let E be a closed subset of $\mathbb{R}^n$, $n \geq 2$. Let $x \in \partial E$. Then the following are equivalent:*

(a) *x is a fine limit point of $E \setminus \{x\}$.*

(b) *x is a regular boundary point of the open set $\Omega :=\mathbb{R}^n \setminus E$. Here regular means regular with respect to the Dirichlet problem.*

In the author's thesis [8], resp. in his article [9], the regular boundary points of a subanalytic domain were investigated. With the above equivalence we can now establish a geometric description of the limit points of a (closed) subanalytic set. We have to distinguish the cases $n = 2$ and $n \geq 3$.

Case $n = 2$. We use the following theorem.

Theorem 8. *Let $E \subset \mathbb{R}^2$ be a subanalytic set. Let $x \in \overline{E} \setminus E$. Then x is a fine limit point of E.*

Proof. See [8, p. 22]. □

Corollary 9. *Let $E \subset \mathbb{R}^2$ be a subanalytic set. Then $\overline{E} = \overline{E}^{\text{fine}}$.*

Case $n \geq 3$. In this case the situation is more complicated. We start with a notation:

Notation. Let $E \subset \mathbb{R}^n$ be subanalytic. We set

$$E' :=\{x \in E \mid \dim_x E \geq n-1\}.$$

For the purpose of investigating fine limit points the sets E and E' are equivalent:

Lemma 10. *Let E be a subanalytic subset of $\mathbb{R}^n$, $n \geq 2$. Let $x \in \overline{E} \setminus E$. Then x is a fine limit point of E if and only if x is a fine limit point of E'.*

Proof. This is an immediate consequence of Wiener's criterion and the fact that an embedded submanifold of codimension greater than one has capacity 0. □

Using Theorem 7 and the results of [9] we get:

THEOREM 11. *Let E be a subanalytic subset of $\mathbb{R}^n$, $n \geq 3$. Let $x \in \overline{E} \setminus E$. Then the following are equivalent:*

(a) *x is a fine limit point of E.*
(b) *There is a cone K with vertex in x and an affine space L of codimension 2, containing x and the central vector of K, such that the projection of $E' \cap K$ onto L contains a cone (in L) with vertex in x.*

With this theorem limit points can also be expressed in terms of a certain volume density in codimension 2:

THEOREM 12. *Let E be a subanalytic subset of $\mathbb{R}^n, n \geq 3$. Let $x \in \overline{E} \setminus E$. Then the following are equivalent:*

(a) *x is a fine limit point of E.*
(b) $\sup\limits_{\substack{\Gamma \subset E \text{ suban.} \\ \dim \Gamma = n-2}} \Theta_{n-2}\,(E' \cap \Gamma, x) > 0.$

Proof. We show that condition (b) of Theorem 11 and condition (b) of Theorem 12 are equivalent. The equivalence follows from the "good stratification" property and the following fact:

Let $U \subset \mathbb{R}^n$ be an open subanalytic set and let $x \in \overline{U}$. Then

$$\Theta_n(U, x) > 0 \Longleftrightarrow U \text{ contains a cone with vertex in } x.$$

This relation can be deduced from the proof of the existence of the volume density in [12]. □

The above geometric description also gives information about definability of the set of fine limit points:

COROLLARY 13. *Let $E \subset \mathbb{R}^n, n \geq 3$, be a subanalytic set. Then we have:*

(a) *$\overline{E}^{\text{fine}}$ is again subanalytic.*
(b) $\dim(\overline{E} \setminus \overline{E}^{\text{fine}}) \leq n - 3$.

Proof. (a) This follows since condition (b) of Theorem 11 is a "definable" condition.

(b) We obtain from Corollary 6 and Theorem 7 with $\Omega :\,= \mathbb{R}^n \setminus \overline{E}$ that $\overline{E} \setminus \overline{E}^{\text{fine}} = \{x \in \overline{E} \setminus E \mid x \text{ is not a regular boundary point of } \Omega\}$. But by [9] the dimension of this set is smaller than or equal to $n - 3$. □

References

[1] D. H. Armitage and S. J. Gardiner, *Classical potential theory*, Springer Monographs in Mathematics, Springer-Verlag London Ltd., London, 2001. MR 1801253 (2001m:31001)

[2] E. Bierstone and P. D. Milman, *Subanalytic geometry*, Model theory, algebra, and geometry, Math. Sci. Res. Inst. Publ., vol. 39, Cambridge Univ. Press, Cambridge, 2000, pp. 151–172. MR 1773706 (2001g:14086)

[3] G. Comte, *Équisingularité réelle: nombres de Lelong et images polaires*, Ann. Sci. École Norm. Sup. (4) **33** (2000), 757–788. MR 1832990 (2002d:32040)

[4] G. Comte, J.-M. Lion, and J.-P. Rolin, *Nature loganalytique du volume des sous-analytiques*, Illinois J. Math. **44** (2000), 884–888. MR 1804313 (2001m:32016)

[5] G. Comte and Y. Yomdin, *Tame geometry with application in smooth analysis*, Lecture Notes in Mathematics, vol. 1834, Springer-Verlag, Berlin, 2004. MR 2041428 (2005i:14076)

[6] S. J. Gardiner, *A fine limit property of functions superharmonic outside a manifold*, Compositio Math. **83** (1992), 239–249. MR 1174425 (93i:31001)

[7] L. L. Helms, *Introduction to potential theory*, Pure and Applied Mathematics, Vol. XXII, Wiley-Interscience, New York-London-Sydney, 1969. MR 0261018 (41 #5638)

[8] T. Kaiser, *Dirichletregularität in polynomial beschränkten o-minimalen Strukturen auf* $\mathbb{R}$, Dissertation, Regensburger Mathematische Schriften, vol. 31, Universität Regensburg Fakultät für Mathematik, Regensburg, 2001. MR 1859561 (2002k:31015)

[9] T. Kaiser, *Dirichlet-regularity in polynomially bounded o-minimal structures on the real field*, preprint, 2003.

[10] K. Kurdyka, *On a subanalytic stratification satisfying a Whitney property with exponent* 1, Real algebraic geometry (Rennes, 1991), Lecture Notes in Math., vol. 1524, Springer, Berlin, 1992, pp. 316–322. MR 1226263 (94f:32016)

[11] K. Kurdyka and P. Orro, *Distance géodésique sur un sous-analytique*, Rev. Mat. Univ. Complut. Madrid **10** (1997), 173–182. MR 1485298 (98m:32008)

[12] K. Kurdyka and G. Raby, *Densité des ensembles sous-analytiques*, Ann. Inst. Fourier (Grenoble) **39** (1989), 753–771. MR 1030848 (90k:32026)

[13] S. Łojasiewicz, *Ensembles semianalytiques*, Institut des Hautes Etudes Scientifiques Bures-sur-Yvette (Seine-et-Oise), France, 1965.

[14] ———, *On semi-analytic and subanalytic geometry*, Panoramas of mathematics (Warsaw, 1992/1994), Banach Center Publ., vol. 34, Polish Acad. Sci., Warsaw, 1995, pp. 89–104. MR 1374342 (96m:32004)

[15] J.-M. Lion and J.-P. Rolin, *Intégration des fonctions sous-analytiques et volumes des sous-ensembles sous-analytiques*, Ann. Inst. Fourier (Grenoble) **48** (1998), 755–767. MR 1644093 (2000i:32011)

[16] O. Martio, *Capacity in measure densities*, Ann. Acad. Sci. Fenn. Ser. A I Math. **4** (1979), 109–118. MR 0538093 (81b:31007)

[17] O. Martio and J. Sarvas, *Density conditions in the n-capacity*, Indiana Univ. Math. J. **26** (1977), 761–776. MR 0477038 (57 #16582)

[18] P. Mattila and P.V. Paramonov, *On density properties of the Riesz capacities and the analytic capacity* γ_+, Proc. Steklov Inst. Math. **235** (2001), 136–149. MR 1886579 (2003a:31003)

[19] J. Väisälä, *Capacity and measure*, Michigan Math. J. **22** (1975), 1–3. MR 0372225 (51 #8441)

[20] J. Wermer, *Potential theory*, Lecture Notes in Mathematics, vol. 408, Springer, Berlin, 1981. MR 634962 (82k:31001)

[21] Y. Xu, *The capacity density and the Hausdorff dimension of fractal sets*, Chinese Ann. Math. Ser. B **16** (1995), 43–50. MR 1338927 (96f:28012)

Department of Mathematics, University of Regensburg, 93040 Regensburg, Germany

E-mail address: tobias.kaiser@mathematik.uni-regensburg.de