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OPEN MANIFOLDS WITH ASYMPTOTICALLY NONNEGATIVE CURVATURE

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Abstract. In this paper we establish a generalization of Toponogov's theorem for manifolds with asymptotically nonnegative sectional curvature, and we give a pinching condition under which asymptotically nonnegative curved manifolds are diffeomorphic to $\mathbb{R}^n$.

1. Introduction

The main theme in Riemannian geometry is the relationship between the curvature and the topology of a manifold. Toponogov's comparison theorem and the Gromov-Bishop volume comparison theorem play important roles in this context. Several results have been established which compare the volumes of concentric balls or triangles with those in simply connected manifolds with constant curvature. It seems natural to try to extend these results by considering model spaces with non-constant curvature.

U. Abresch [A] established a triangle comparison theorem for manifolds with asymptotically nonnegative sectional curvature: A complete Riemannian n-dimensional manifold M has asymptotically nonnegative sectional curvature if there exists a point p, called base point, and a monotone decreasing function

$$\lambda : [0, +\infty) \to [0, +\infty)$$

satisfying

$$\int_0^{+\infty} s\lambda(s)ds = b_0 < +\infty, \tag{1}$$

such that, for all points $x \in M$ and every plane P of the tangent space T_xM at x, $-\lambda(d(p,x)) \leq K_M(x,P)$, where $K_M(x,P)$ is the sectional curvature at x with respect to P.

Unfortunately, the comparison does not involve the angle at the base point p. The goal of this paper is to fill this gap. We also prove some pinching results.

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The volume comparison theorem proved in [Zh], and more generally in [B], says that the function $r \to \operatorname{vol} B(p,r)/\operatorname{vol} B(\overline{p},r)$ is monotone decreasing, where $B(\overline{p},r)$ denotes the open geodesic ball in the simply connected noncompact manifold $\overline{M}$ with sectional curvature $-\lambda\,(d(\overline{p},\overline{x}))$ at the point $\overline{x}$.

Set

$$\alpha_p = \lim_{r\to+\infty} \frac{\operatorname{vol} B(p,r)}{\operatorname{vol} B(\overline{p},r)} \text{ and } \alpha(M) = \inf_{p\in M} \alpha_p.$$

We say that (M,g) has large volume growth if $\alpha(M) > 0$.

Let Σ be a closed subset of the unit tangent sphere U_pM at p and $B_\Sigma(p,r)$ the set of points $x \in B(x,r)$ such that there is a minimizing geodesic γ from p to x with $\gamma'(0) \in \Sigma$. For all r let $\Sigma(r)$ be the set of all vectors v in Σ such that the geodesic $\gamma(t) = \exp_p tv$ is minimizing at least on $[0,r]$.

2. Main results

THEOREM 2.1. *Let $\Delta = (p_0,p_1,p_2)$ be a geodesic triangle in M with vertices p_0, p_1, p_2, corresponding edges γ_0, γ_1, γ_2, and angles α_0, α_1, α_2. Assume that γ_0, γ_1, γ_2 are minimal geodesics. Let $\overline{\Delta} = (\overline{p}_0,\overline{p}_1,\overline{p}_2)$ be a geodesic triangle in $M^2(-\lambda)$ with* $\operatorname{length}(\overline{\gamma}_i) = \operatorname{length}(\gamma_i)$ *for $i = 0,1,2$. Then*

$$\overline{\alpha}_0 \le \alpha_0, \quad \overline{\alpha}_1 \le \alpha_0, \quad \overline{\alpha}_2 \le \alpha_2,$$

where $M^2(-\lambda)$ is a simply connected surface of revolution with curvature $-\lambda\,(d(\overline{p}_0,.))$.

Set $d_p(x) = d(p,x)$. This defines a smooth function on $M\backslash\{p \cup C_p\}$, where C_p is the cut locus of p.

Grove and Shiohama [GS] observed that in the presence of a lower curvature bound Toponogov's theorem can be used to derive geometric information from the existence of critical points. A point $x \neq p$ is a critical point of d_p if for any v in the tangent space T_xM there is a minimal geodesic γ from x to p forming an angle $\angle(v,\gamma'(0)) \le \pi/2$ with $\gamma'(0)$.

By the Isotopy Lemma, the absence of critical points (other than p) implies that the manifold is diffeomorphic to $\mathbb{R}^n$. Marenich and Toponogov [MT] proved that an n-dimensional complete manifold with nonnegative sectional curvature and large volume growth is diffeomorphic to $\mathbb{R}^n$. For manifolds with nonnegative Ricci curvature and large volume growth Z. Shen [S] showed that if

$$\frac{\operatorname{vol} B(p,r)}{\omega_n r^n} = \alpha(M) + o\left(\frac{1}{r^{n-1}}\right),$$

then M has finite topological type provided that either the conjugate radius satisfies $\operatorname{conj}_M > c > 0$ or the sectional curvature satisfies $K_M \ge -k$. Cheeger and Colding [CC] proved that there exists a constant $\delta(n) > 0$, depending on n, such that if $\alpha(M) > 1 - \delta(n)$, then M is diffeomorphic to $\mathbb{R}^n$. C. Xia [X1] showed that if, in addition, M has radial sectional curvature bounded from below by a constant $-k$ and if

$$\frac{\operatorname{vol} B(p,r)}{\omega_n r^n} < \alpha(M)\left[1+2^{-n}\left(\frac{1}{8\sqrt{k}r}\ln(\frac{2}{1+e^{-2\sqrt{k}r}})\right)^{n-1}\right],$$

then M is diffeomorphic to $\mathbb{R}^n$. Xia also studied manifolds with sectional curvature bounded below by a negative constant $-k$ and large volume growth; see [X2], [X3], [X4], etc.

From Theorem 2.1 we deduce:

THEOREM 2.2. *Let M be a Riemannian manifold with asymptotically nonnegative sectional curvature with base point p. For all $c \geq \omega_0$, if*

$$\operatorname{vol}(B(p,r)) \geq cr^n \quad \forall r > 0, \tag{2}$$

then M is diffeomorphic to $\mathbb{R}^n$, where

$$\omega_0 = \frac{1}{2}\omega_n\left(1+\frac{2b_0e^{b_0}}{1+b_0e^{b_0}}\right)e^{(n-1)b_0},$$

and ω_n is the volume of the n-dimensional unit Euclidian ball.

Let R_p denotes the set of all rays issuing from p. For $r > 0$, let $S(p,r)$ be the geodesic sphere of radius r with center p. Set

$$H(p,r) = \max_{x\in S(p,r)} d(x, R_p).$$

In [X4] Xia proved the following result:

THEOREM (Lemma 3.1, [X4]). *For any $r_0 > 0$ and integer $n \geq 2$ there exists a positive constant δ, depending on r_0 and n, such that any complete Riemannian n-manifold M with sectional curvature $K_M \geq -1$, $\operatorname{Ric}_M \geq 0$, critical radius at some point p $\operatorname{crit}_p \geq r_0$, and $H(p,r) \leq \delta r^{1/n}$ for all $r \geq r_0$, is diffeomorphic to $\mathbb{R}^n$.*

Xia also proved that, for any $r_0 > 0$ and $n \geq 2$ there exists a constant $\epsilon(n, r_0)$, depending on r_0 and n, such that if $\operatorname{Ric}_M \geq 0$, $K_M \geq -1$, $\operatorname{crit}_p \geq r_0$, $\alpha_M > 0$ and

$$\operatorname{vol} B(p,r) \leq \omega_n\alpha_M\left(1+\epsilon r^{-(n-1)^2/n}\right)r^n$$

for some p, then M is diffeomorphic to $\mathbb{R}^n$ (Theorem 3.2 in [X4]).

For manifolds with asymptotically nonnegative sectional curvature we have:

THEOREM 2.3. *Let M be a complete open n-dimensional Riemannian manifold with asymptotically nonnegative sectional curvature. There exists a positive constant $\epsilon_0 = \epsilon_0(b_0, n)$ depending on b_0 and n such that if*

$$0 < \alpha_p = \lim_{r\to+\infty}\frac{\operatorname{vol} B(p,r)}{\operatorname{vol} B(\overline{p},r)},$$

and

(3) $$\operatorname{vol}(B(p,r)) < \alpha_p (1+\epsilon_0) \operatorname{vol} B(\overline{p}, r) \quad \forall r > 0,$$

then M is diffeomorphic to $\mathbb{R}^n$.

3. Proofs

Proof of Theorem 2.1. Notice that, for $z \in M$ and x a fixed point in M, we have by triangle inequality

$$K_M(z,P) \geq -\lambda(d(p,z)) \geq -\lambda\left(|d(p,x) - d(x,z)|\right).$$

Set $d(p,x) = r, \quad d(x,z) = t$ and $\lambda_r(t) = \lambda\left(|r-t|\right)$. Let y be the solution of the system

$$\begin{cases} y''(t) = \lambda_r(t) \cdot y(t), \\ y(0) = 0, \quad y'(0) = 1. \end{cases}$$

Let $\overline{M}(-\lambda_r)$ be the open simply connected manifold such that there exists a point $\overline{p}$ in $\overline{M}$ such that for all $\overline{z} \in \overline{M}$ we have $K_{\overline{M}}(\overline{z},) = -\lambda_r(d(\overline{p},\overline{z}))$.

To prove his triangle comparison theorem, Abresch [A] used models with non-positive curvature. For any continuous function $k : [0,\infty) \to [0,\infty)$ there exists a unique simply connected surface of revolution $M^2(-k)$ with curvature $-k(d(.,p_0)$. Abresch considered some approximating functions $k_\epsilon : [0,\infty) \to [0,\infty)$ and proved following two lemmas:

Lemma 3.1. *Given the triangles $\Delta = (\overline{p}_0, \overline{p}_1, \overline{p}_2)$ in $M^2(-k)$ and $\Delta^\epsilon = ((\overline{p}_0^\epsilon, \overline{p}_1^\epsilon, \overline{p}_2^\epsilon)$ in $M^2(-k_\epsilon)$, let γ_i (respectively γ_i^ϵ) be the geodesic joining $\overline{p}_{i+1}$ and $\overline{p}_{i+2}$ (respectively $\overline{p}_{i+1}^\epsilon$ and $\overline{p}_{i+2}^\epsilon$) ($i = j$ modulo 3) with lengths $l_i^\epsilon = l_i$, $i = 0,1,2$. Then their angles depend continuously on ϵ, i.e.,*

$$\lim_{\epsilon \to 0} \angle\, \overline{p}_i^\epsilon = \angle\, \overline{p}_i.$$

Lemma 3.2. *Let $\Delta = (p_0, p_1, p_2)$ be a triangle in a Riemannian manifold M. Suppose that the edges γ_1 and γ_2 are minimal geodesics and γ_0 is a geodesic. If M is of sectional curvature bounded below by $-k(d(x,p_0)) \leq 0$ for all x in M, and if $l_i = \bar{l}_i$ for all i, then $\angle\, \overline{p}_1 \leq \angle\, p_1$ and $\angle\, \overline{p}_2 \leq \angle\, p_2$. If $l_i = \bar{l}_i$ for $i = 0,1$ and $\angle\, p_2 \leq \angle\, \overline{p}_2$, then $l_2 \leq \bar{l}_2$.*

To prove Theorem 2.1 it suffices to establish the first inequality.

Consider the model space $M^2(-\lambda_r)$, $\lambda_r(t) = \lambda\left(|r-t|\right) = \lambda_r(d(p_1,.)$ ($r = d(p,p_1)$) and a triangle $\tilde{\Delta} = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_0)$ in $M^2(-\lambda_r)$ with length $l_i = L(\tilde{\gamma}_i) = L(\gamma_i)$ for $i = 0,1,2$, where $\tilde{\gamma}_i$ and γ_i are geodesics in $M^2(-\lambda_r)$ and M, respectively. Denote by $\tilde{\alpha}_i^r$ its angle at $\overline{p}_i$. From Lemma 3.2 we conclude that

$$\tilde{\alpha}_0^r \leq \alpha_0, \quad \tilde{\alpha}_2^r \leq \alpha_2.$$

Fix $\delta > 0$ small enough and let $0 < r' \leq l_2 - \delta$. By the parallel transport of $\gamma_0'(0)$ along the geodesic γ_2^-, the inverse of γ_2 in M, we define a minimal geodesic $\theta_{r'}$ from $\gamma_2^-(r')$ to a point $p_{2,r'}$ of γ_1. Thus $(\gamma_2^-(r'), \theta_{r'}(s_{r'}), p_0)$ is a geodesic triangle in M. In the same way as above, we have a triangle in $M^2(-\lambda_r)$ $(r = l_2 - r')$ and deduce that the angle $\tilde{\alpha}_{0,r'}$ at $\tilde{p}_{0,r'}$ of the triangle $(\tilde{\gamma}_2^-(r'), \tilde{\theta}_{r'}(s), \tilde{p}_0)$ is less than or equal to α_0.

Since λ_r converges to $\lambda(d(p_0,.)-\delta)$ when r' converges to $l_2-\delta$, we conclude by Lemma 3.1 that $\tilde{\alpha}_{0,r'}$ converges to $\overline{\alpha}_0^\delta$, the angle at $\overline{p}_0^\delta$. Hence for all $\delta > 0$, $\overline{\alpha}_0^\delta \leq \alpha_0$, and the conclusion follows. □

Proof of Theorem 2.2. In [B] (Theorem 1) we showed the following result:

Lemma 3.3. *For all $x \neq p$ in M such that $d(p,x) = r$ we have*

$$t \leq y(t) \leq \begin{cases} e^{b_0} \cdot t & \text{if } t \leq r, \\ e^{2b_0}\,(1 + 2\lambda_1(0)r) \cdot t & \text{if } t > r. \end{cases}$$

For $x = p$ we have

$$t \leq y(t) \leq e^{b_0} t, \tag{4}$$

where

$$\lambda_1(0) = \int_0^{+\infty} \lambda(t)dt.$$

Let $J(t)$ be the exponential Jacobian in polar coordinates. Then the function J/y is nonincreasing (see [B] for all $x \in M$ or [Zh] for the base point), and using the fact that the space of curvature $-\lambda_r$ and dimension n, $\overline{M}(-\lambda_r)$, is a complete simply connected manifold with nonpositive sectional curvature we have:

Lemma 3.4. *Let M be an n-dimensional Riemannian manifold with asymptotically nonnegative (sectional or Ricci) curvature. Then for all $x \in M$ and all numbers R, R' such that $0 < R' \leq R$ we have*

$$\begin{aligned} \frac{\operatorname{vol}(B(x,R))}{\operatorname{vol}(B(x,R'))} &\leq \frac{\operatorname{vol}(B(\overline{x},R))}{\operatorname{vol}(B(\overline{x},R'))} \\ &\leq \begin{cases} e^{(n-1)b_0}\left(\dfrac{R}{R'}\right)^n & \text{if } 0 < R < r, \\ e^{(n-1)b_0}\left(\dfrac{R+r}{R'}\right)^n & \text{if } R \geq r, \end{cases} \end{aligned}$$

where $B(\overline{x}, s)$ is the ball in $\overline{M}(-\lambda_r)$ with center $\overline{x}$ and radius s.

Let $R_{p,t}$ be the set of unit initial tangent vectors to the geodesics starting from p which are minimized at least to t, and let $R_{p,t}^c$ be the complement of this set. Then

$$\lim_{t \to +\infty} R_{p,t} = R_p.$$

We have the following lemma:

LEMMA 3.5. *Let M be a complete Riemannian manifold with asymptotically nonnegative sectional curvature and base point p. If for some $c > 0$*

$$\operatorname{vol}(B(p,r)) \geq cr^n \quad \forall r > 0, \tag{5}$$

then

$$\operatorname{vol}(R_p) \geq \frac{nc}{e^{(n-1)b_0}}. \tag{6}$$

Proof. Let $t > 0$. Then for $r > t$ we have

$$\begin{aligned}
\operatorname{vol} B(p,r) &= \int_{S^{n-1}/R^c_{p,t}} \int_0^{\min[\operatorname{cut}(\theta),r]} J(s)\,ds\,d\theta \\
&\quad + \int_{R^c_{p,t}} \int_0^{\min[\operatorname{cut}(\theta),r]} J(s)\,ds\,d\theta \\
&\leq \int_{S^{n-1}/R^c_{p,t}} \int_0^{\min[\operatorname{cut}(\theta),r]} e^{(n-1)b_0} s^{n-1}\,ds\,d\theta \\
&\quad + e^{(n-1)b_0} \int_{R^c_{p,t}} \int_0^{\min[\operatorname{cut}(\theta),r]} s^{n-1}\,ds\,d\theta \\
&\leq e^{(n-1)b_0} \frac{1}{n} \left[r^n \operatorname{vol}\left(S^{n-1}/R^c_{p,t}\right) + t^n \operatorname{vol}(R^c_{p,t}) \right].
\end{aligned}$$

Using the inequality (5) and dividing by r^n, we obtain

$$c \leq \frac{e^{(n-1)b_0}}{n} \left(\operatorname{vol}(S^{n-1}) - \operatorname{vol}(R^c_{p,t})\right) + \frac{t^n}{nr^n} \operatorname{vol}(R^c_{p,t}).$$

Let $r \to +\infty$. Then for all $t > 0$ we get

$$nc \leq e^{(n-1)b_0} \left(\operatorname{vol}(S^{n-1}) - \operatorname{vol}(R^c_{p,t})\right).$$

Hence

$$\lim_{t \to +\infty} \operatorname{vol}(R^c_{p,t}) = \operatorname{vol}(R^c_p) \leq \operatorname{vol}(S^{n-1}) - \frac{nc}{e^{(n-1)b_0}},$$

which implies

$$\operatorname{vol}(R_p) \geq \frac{nc}{e^{(n-1)b_0}}. \qquad \square$$

Let p_1 be a point in M, $p_1 \neq p_0$. Consider a triangle $\Delta = (p_0, p_1, p_2)$ as in Theorem 2.3 and let $\overline{\Delta} = (\overline{p}_0, \overline{p}_1, \overline{p}_2)$ be the corresponding triangle in $M(-\lambda)$. We have the following lemma:

LEMMA 3.6. *Let $\overline{\Delta} = (\overline{p}_0, \overline{p}_1, \overline{p}_2)$ be a triangle in $M^2(-\lambda)$ such that $L(\overline{\gamma}_i) = L(\gamma_i)$ for all i. Then*

$$\overline{\alpha}_0 + \overline{\alpha}_1 + \overline{\alpha}_2 - \pi \geq -\overline{\alpha}_0 b_0 e^{b_0}.$$

Proof. By the Gauss-Bonnet Theorem we have

$$\begin{aligned}\overline{\alpha}_0 + \overline{\alpha}_1 + \overline{\alpha}_2 - \pi &\geq \int_{\overline{\Delta}} K_{M^2} d\Omega \geq \int_{\angle \overline{p}_0} K_{M^2} d\Omega \\ &\geq -\overline{\alpha}_0 \int_0^{+\infty} \lambda(t) y(t) dt.\end{aligned}$$

Hence by inequality (4) we have

$$\overline{\alpha}_0 + \overline{\alpha}_1 + \overline{\alpha}_2 - \pi \geq -\overline{\alpha}_0 b_0 e^{b_0}. \qquad \square$$

Let x be a point in M and γ be a geodesic joining x and p. By inequality (6) there exists a ray γ_1 issuing from p such that

$$\alpha_0 = \angle(\gamma_1'(0), -\gamma'(r)) \leq \pi \left(1 - \frac{c}{\omega_n e^{(n-1)b_0}}\right).$$

Hence by Theorem 2.1 we have

$$\overline{\alpha}_0 \leq \alpha_0 \leq \pi \left(1 - \frac{c}{\omega_n e^{(n-1)b_0}}\right).$$

Let γ_2 be a geodesic joining x to a point $\gamma_1(t_0)$ of γ_1. Then

$$\begin{aligned}\overline{\alpha}_1 &\geq \pi - \overline{\alpha}_0(1 + b_0 e^{b_0}) - \overline{\alpha}_2 \\ &\geq \pi - \pi \left(1 - \frac{c}{\omega_n e^{(n-1)b_0}}\right)(1 + b_0 e^{b_0}) - \overline{\alpha}_2 \\ &\geq \frac{\pi}{2}\left(1 + \frac{b_0 e^{b_0}}{1 + b_0 e^{b_0}}\right) - \overline{\alpha}_2.\end{aligned}$$

Take t_0 sufficiently large so that $\overline{\alpha}_2$ is small enough. We deduce

$$\alpha_1 \geq \overline{\alpha}_1 > \frac{\pi}{2}$$

and the conclusion follows. $\square$

Proof of Theorem 2.3. First, we establish some lemmas.

LEMMA 3.7. *Let M be an n-dimensional Riemannian open manifold with asymptotically nonnegative sectional curvature. There exists a positive constant $\delta_0 = \delta_0(b_0)$ depending on b_0 such that if*

$$H(p, r) < \delta_0 r \quad \forall r > 0,$$

then M is diffeomorphic to $\mathbb{R}^n$.

Proof. It suffices to prove that d_p has no critical point $x \neq p$. Let x be a point in M, $x \neq p$. Set $r = d(p, x)$. Since M is asymptotically nonnegative curved, there exists a positive, nonincreasing function λ such that $K_M(x) \geq -\lambda(d(p, x))$. It is easy to see that $\lambda(t) \leq 2b_0/t^2$ for all $t > 0$, where b_0 is defined as in (1). Set $s = d(x, R_p)$. We have $s \leq r$. Since R_p is closed, there exists a ray γ issuing from p and a point m on γ such that $s = d(x, \gamma) = d(x, m)$.

Set $q = \gamma(t_0)$, $t_0 \geq 2r$. Let σ_1 be a minimal geodesic joining x to p, and let σ be a minimal geodesic from x to q. Let X_0 be the positive solution of the equation

$$\cosh^2 2X - \cosh 3X = 0. \tag{7}$$

Take δ small enough. Set $\tilde{p} = \sigma_1(\delta r)$ and $\tilde{q} = \sigma(\delta r)$. The triangle $(\tilde{p}, \tilde{q}, x)$ is contained in $M \backslash B(p, r(1-3\delta)$. For all y in $M \backslash B(p, r(1-3\delta)$ we have

$$K_M(y) \geq \frac{-2b_0}{r^2(1-3\delta)^2}.$$

Applying Toponogov's theorem to the triangle $(\tilde{p}, \tilde{q}, x)$, we obtain

$$\cosh \frac{(2b_0)^{1/2}}{r(1-3\delta)} d(\tilde{p}, \tilde{q}) \leq \cosh^2 \frac{(2b_0)^{1/2}}{1-3\delta}\delta - \sinh^2 \frac{(2b_0)^{1/2}}{1-3\delta}\delta \cos\theta, \tag{8}$$

where θ is the angle at x of the triangle $(\tilde{p}, \tilde{q}, x)$.

By triangle inequality we have

$$\begin{aligned}
d((\tilde{p}, \tilde{q}) &\geq d(p,q) - d((p, \tilde{p}) - d(q, \tilde{q}) \\
&\geq d(p,q) - (d(p,x) - d(x,\tilde{p})) - (d(q,x) - d(x,\tilde{q})) \\
&\geq d(p,m) + d(m,q) + d(\tilde{p},x) + d(x,\tilde{q}) - d(p,x) - d(x,q) \\
&\geq 2\delta r + d(p,m) + d(m,q) - d(p,m) - d(m,x) - d(q,m) - d(m,x) \\
&\geq 2\delta r - 2d(x,m).
\end{aligned}$$

Suppose that $d(x,m) < \delta r/4$. Then from (8) we get

$$\sinh^2 \frac{(2b_0)^{1/2}}{(1-3\delta)}\delta \cos\theta < \cosh^2 \frac{(2b_0)^{1/2}}{1-3\delta}\delta - \cosh \frac{(2b_0)^{1/2}}{1-3\delta}\frac{3}{2}\delta.$$

We take δ such that

$$\frac{(2b_0)^{1/2}}{1-3\delta}\delta = 2X_0,$$

and let

$$\delta_0 = \frac{X_0}{(b_0/2)^{1/2} + 3X_0}.$$

Then $\theta > \pi/2$. Therefore x is not a critical point of d_p and the conclusion follows from the Isotopy Lemma. $\square$

The following lemma is a generalized comparison theorem for manifolds with asymptotically nonnegative Ricci curvature proved in [B] or [Zh].

LEMMA 3.8. *Let M be a complete dimensional Riemannian manifold with* $\mathrm{Ric}_M \geq -(n-1)\lambda\left(d(p,x)\right)$. *Let $\Sigma \subset U_pM$ be a closed subset. Then the function*

$$r \to \frac{\mathrm{vol}\left(B_\Sigma(p,r)\right)}{\mathrm{vol}\, B(\overline{p},r)}$$

is monotone decreasing.

Thus we have the following result:

LEMMA 3.9. *Let (M,g) be a complete n-dimensional manifold with* $\mathrm{Ric}_M(x) \geq -(n-1)\lambda\left(d(p,x)\right)$. *Then the function*

$$r \to \frac{\mathrm{vol}\left(B_{\Sigma(r)}(p,r)\right)}{\mathrm{vol}\, B(\overline{p},r)}$$

is monotone decreasing.

Proof. For $0 < r_1 \leq r_2$ we have $R_{p,r_2} \subset R_{p,r_1}$, and by Lemma 3.8 we have

$$\frac{\mathrm{vol}\, B_{R_{p,r_2}}(p,r_2)}{\mathrm{vol}\, B(p,r_2} \leq \frac{\mathrm{vol}\, B_{R_{p,r_2}}(p,r_1)}{\mathrm{vol}\, B(p,r_1)} \leq \frac{\mathrm{vol}\, B_{R_{p,r_1}}(p,r_1)}{\mathrm{vol}\, B(p,r_1)}. \qquad \square$$

LEMMA 3.10. *Let (M,g) be a complete n-dimensional manifold with* $\mathrm{Ric}_M(x) \geq -(n-1)\lambda\left(d(p,x)\right)$. *Then*

$$\frac{\mathrm{vol}\left(B_{R_{p,r}}(p,r)\right)}{\mathrm{vol}\, B(\overline{p},r)} \geq \alpha_p.$$

Proof. By Lemma 3.9 we have

$$\mathrm{vol}\, B_{R_{p,r}}(p,2r) \leq \frac{\mathrm{vol}\, B(\overline{p},2r)}{\mathrm{vol}\, B(\overline{p},r)}\, \mathrm{vol}\, B_{R_{p,r}}(p,r).$$

In addition, we have

$$B(p,2r)\backslash B(p,r) \subset B_{R_{p,r}}(p,2r)\backslash B_{R_{p,r}}(p,r).$$

Hence

$$\begin{aligned} \mathrm{vol}\, B(p,2r) - \mathrm{vol}\, B(p,r) &\leq \mathrm{vol}\, B_{R_{p,r}}(p,2r) - \mathrm{vol}\, B_{R_{p,r}}(p,r) \\ &\leq \left(\frac{\mathrm{vol}\, B(\overline{p},2r)}{\mathrm{vol}\, B(\overline{p},r)} - 1\right) \mathrm{vol}\, B_{R_{p,r}}(p,r) \end{aligned}$$

and therefore

$$\frac{\mathrm{vol}\, B(p,2r) - \mathrm{vol}\, B(p,r)}{\mathrm{vol}\, B(\overline{p},2r) - \mathrm{vol}\, B(\overline{p},r)} \leq \frac{\mathrm{vol}\, B_{R_{p,r}}(p,r)}{\mathrm{vol}\, B(\overline{p},r)}. \tag{9}$$

Notice that

$$\begin{aligned}\alpha_p &= \lim_{r\to+\infty} \frac{\operatorname{vol} B(p,2r)}{\operatorname{vol} B(\overline{p},2r)} \\ &= \lim_{r\to+\infty} \left[\left(\alpha_p - \frac{\operatorname{vol} B(p,r)}{\operatorname{vol} B(\overline{p},r)}\right) \cdot \frac{\operatorname{vol} B(\overline{p},r)}{\operatorname{vol} B(\overline{p},2r)}\right] \\ &\quad + \lim_{r\to+\infty} \frac{\operatorname{vol} B(p,2r)}{\operatorname{vol} B(\overline{p},2r)}.\end{aligned}$$

Thus

$$\alpha_p = \lim_{r\to+\infty} \left[\frac{\alpha_p \operatorname{vol} B(\overline{p},r) - \operatorname{vol} B(p,r) + \operatorname{vol} B(p,2r)}{\operatorname{vol} B(\overline{p},2r)}\right].$$

Hence, for all $\epsilon > 0$ and r large enough we have

$$\alpha_p(\operatorname{vol} B(\overline{p},2r) - \operatorname{vol} B(\overline{p},r)) - \operatorname{vol} B(p,2r) + \operatorname{vol} B(p,r) < \epsilon \cdot \operatorname{vol} B(\overline{p},2r),$$

that is,

$$\alpha_p \leq \frac{\operatorname{vol} B(p,2r) - \operatorname{vol} B(p,r)}{\operatorname{vol} B(\overline{p},2r) - \operatorname{vol} B(\overline{p},r)} + \epsilon \cdot \frac{\operatorname{vol} B(\overline{p},2r)}{\operatorname{vol} B(\overline{p},2r) - \operatorname{vol} B(\overline{p},r)}.$$

Therefore

$$\alpha_p \leq \frac{\operatorname{vol} B(p,2r) - \operatorname{vol} B(p,r)}{\operatorname{vol} B(\overline{p},2r) - \operatorname{vol} B(\overline{p},r)} + \frac{\epsilon}{1 - \frac{\operatorname{vol} B(\overline{p},r)}{\operatorname{vol} B(\overline{p},2r)}}.$$

We claim that

$$\operatorname{vol} B(\overline{p},2r) - \operatorname{vol} B(\overline{p},r) \geq \frac{\operatorname{vol} B(\overline{p},2r)}{2e^{(n-1)b_0}}.$$

Indeed, let $y(t)$ be the function given by the Jacobi equation

$$y''(t) = \lambda(t)y(t)$$

in the space $\overline{M}$ with sectional curvature $K(\overline{x}) = -\lambda\left(d(\overline{p},\overline{x})\right)$ at a point $\overline{x}$. We have by Lemma 3.3

$$t \leq y(t) \leq e^{b_0} \cdot t. \tag{10}$$

Hence

$$\begin{aligned}
\operatorname{vol} B(\overline{p}, 2r) - \operatorname{vol} B(\overline{p}, r) &= \int_{S^{n-1}} \int_0^{\min(\operatorname{cut}(u), 2r)} J(t)\, dt\, du \\
&\quad - \int_{S^{n-1}} \int_0^{\min(\operatorname{cut}(u), r)} J(t)\, dt\, du \\
&= \int_{S^{n-1}} \int_0^{2r} J(t)\, dt\, du - \int_{S^{n-1}} \int_0^{r} J(t)\, dt\, du \\
&= \int_{S^{n-1}} \int_r^{2r} y^{n-1}(t)\, dt\, du \\
&\geq \int_{S^{n-1}} \int_r^{2r} t^{n-1}\, dt\, du \\
&\geq \frac{\operatorname{vol}(S^{n-1})}{n}(2^n - 1) r^n \geq \omega_n 2^{n-1} r^n \\
&\geq \frac{\operatorname{vol} B(\overline{p}, 2r)}{2e^{(n-1)b_0}},
\end{aligned}$$

where ω_n denotes the volume of the unit Euclidian ball and S^{n-1} the unit Euclidian sphere. This proves the above claim.

Thus

$$1 - \frac{\operatorname{vol} B(\overline{p}, r)}{\operatorname{vol} B(\overline{p}, 2r)} \geq \frac{1}{2e^{(n-1)b_0}}.$$

Therefore

$$\alpha_p \leq \frac{\operatorname{vol} B(p, 2r) - \operatorname{vol} B(p, r)}{\operatorname{vol} B(\overline{p}, 2r) - \operatorname{vol} B(\overline{p}, r)} + \epsilon',$$

and the conclusion follows from (9). □

Next, we prove Theorem 2.3. Notice that

$$\lim_{r \to +\infty} \frac{\operatorname{vol} B_{R_p}(p, r)}{\operatorname{vol} B(\overline{p}, r)} = \lim_{r \to +\infty} \frac{\operatorname{vol} B_{R_{p,r}}(p, r)}{\operatorname{vol} B(\overline{p}, r)} \geq \alpha_p.$$

Hence by Lemma 3.8 we deduce

$$\operatorname{vol} B_{R_p}(p, r) \geq \alpha_p \operatorname{vol} B(\overline{p}, r). \tag{11}$$

Now, let $x \in S(p, r)$ and set $s = d(x, R_p)$. Note that $s \leq r$. Thus we have a disjoint union

$$B(x, s) \cup B_{R_p}(p, 2r) \subset B(p, 2r).$$

This implies

$$\operatorname{vol} B(x, s) + \operatorname{vol} B_{R_p}(p, 2r) \leq \operatorname{vol} B(p, 2r). \tag{12}$$

Applying the volume comparison theorem (Lemma 3.4) and the fact that

$$B(p, r) \subset B(x, r + s),$$

we obtain

$$\begin{aligned}\frac{\operatorname{vol} B(p,r)}{\operatorname{vol} B(x,s)} \leq \frac{\operatorname{vol} B(x,r+s)}{\operatorname{vol} B(x,s)} &\leq \frac{\operatorname{vol} B(\overline{x},r+s)}{\operatorname{vol} B(\overline{x},s)} \\ &\leq e^{(n-1)b_0}\left(\frac{2r+s}{s}\right)^n \leq e^{(n-1)b_0}3^n\left(\frac{r}{s}\right)^n.\end{aligned}$$

Hence

$$\operatorname{vol} B(x,s) \geq \frac{\operatorname{vol} B(p,r)}{3^n e^{(n-1)b_0} r^n} \cdot s^n. \tag{13}$$

It follows from (10) that

$$\omega_n r^n \leq \operatorname{vol} B(\overline{p},r) \leq \omega_n e^{(n-1)b_0} r^n \tag{14}$$

Substituting (14) in (13), we obtain

$$\begin{aligned}\operatorname{vol} B(x,s) &\geq \frac{\omega_n}{3^n e^{(n-1)b_0}}.\frac{\operatorname{vol} B(p,r)}{\operatorname{vol} B(\overline{p},r)} \cdot s^n \\ &\geq \frac{\omega_n}{3^n} e^{-(n-1)b_0}.\alpha_p \cdot s^n.\end{aligned} \tag{15}$$

From (11), (12) and (15) we deduce

$$\frac{\omega_n}{3^n} e^{-(n-1)b_0}.\alpha_p \cdot s^n + \alpha_p \operatorname{vol} B(\overline{p},2r) \leq \operatorname{vol} B(p,2r).$$

Thus

$$s^n \leq \frac{3^n e^{(n-1)b_0}}{\alpha_p \omega_n}\left[\frac{\operatorname{vol} B(p,2r)}{\operatorname{vol} B(\overline{p},2r)} - \alpha_p\right] \operatorname{vol} B(\overline{p},2r).$$

Hence using the inequalities (14) and the above relation we obtain

$$s^n \leq \frac{6^n e^{2(n-1)b_0}}{\alpha_p}\left[\frac{\operatorname{vol} B(p,2r)}{\operatorname{vol} B(\overline{p},2r)} - \alpha_p\right] r^n.$$

Taking in the relation (3)

$$\epsilon(n,b_0) = \left(\frac{\delta_0}{6}\right)^n \cdot e^{-2(n-1)b_0},$$

the conclusion follows from the Lemma 3.7. □

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