MULTILINEAR MOTIVIC POLYLOGARITHMS

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ABSTRACT. We explicitly describe a candidate for the regulator map from $H^1_{\mathcal{M}}\big(\operatorname{Spec}\mathbb{C}, \mathbb{Z}(n)\big)$ into $\mathbb{R}$ using analogues of polylogarithms. When $n = 2$, the above procedure agrees with the one in the author's paper [14], which was shown to be compatible with Bloch's dilogarithm.

1. Introduction

In [3] Beilinson constructed a regulator map from the motivic cohomology $H^p_{\mathcal{M}}\big(X, \mathbb{Q}(q)\big)$, which is defined using the γ-filtration of K-theory, to the Deligne-Beilinson cohomology group $H^p_{\mathcal{D}}\big(X, \mathbb{R}(q)\big)$ for a scheme X over $\operatorname{Spec}(\mathbb{C})$.

Bloch [4] introduced a single-valued analogue D_2 of the dilogarithm function to describe the regulator map on $K_3(\mathbb{C})$ explicitly. Zagier [18] constructed similar single-valued analogues of classical polylogarithm functions and conjectured that the Borel regulator maps can be described with his polylogarithm functions in a certain way. Goncharov, in [5] and [7], constructed a regulator map for his motivic complex whose homology groups are conjectured to be isomorphic to the consecutive quotients of the γ-filtration of K-theory after tensoring with $\mathbb{Q}$. His definition of a motivic complex is motivated by properties of polylogarithms, but its legitimacy as a motivic complex is mostly conjectural. The formulas for these polylogarithms, which are single-valued functions related to the classical higher logarithms $\operatorname{Li}_n(z) = \sum_{k=1}^{\infty} z^k/k^n$, are given in [18] by Zagier. In [6], Goncharov constructed a regulator map for Bloch's higher Chow groups via his Chow polylogarithms, which resemble our formulas in a sense. While this paper was being prepared for publication, another paper [8] of Goncharov, based on a similar philosophy, has been published.

In a different approach, Hain and MacPherson [11] proposed their version of higher logarithms as multi-valued functions on a certain complex manifold,

motivated by classical properties and functional equations of the dilogarithm. Their higher logarithms satisfy functional equations immediately once their existence is established, but their relation to the regulator map has not been given.

The main purpose of this paper is to give an explicit description of a candidate for the regulator map from $H^1_{\mathcal{M}}\big(\operatorname{Spec}\mathbb{C},\ \mathbb{Z}(n)\big)$ into $\mathbb{R}$ based on analogues of polylogarithms. The motivic cohomology we use throughout this paper is the one defined via the motivic complex proposed by Goodwillie and Lichtenbaum as in [9]. We briefly recall the definition. For a ring R, let $\mathcal{P}(R, \mathbb{G}_m^t)$ be the exact category each of whose objects $(P, \theta_1, \ldots, \theta_t)$ consists of a finitely generated projective R-module P and commuting automorphisms $\theta_1, \ldots, \theta_t$ of P. A morphism from $(P, \theta_1, \ldots, \theta_t)$ to $(P', \theta'_1, \ldots, \theta'_t)$ in this category is a homomorphism $f : P \to P'$ of R-modules such that $f\theta_i = \theta'_i f$ for each i. Let $K_0(R, \mathbb{G}_m^t)$ be the Grothendieck group of this category and let $K_0(R, \mathbb{G}_m^{\wedge t})$ be the quotient of $K_0(R, \mathbb{G}_m^t)$ by the subgroup generated by those objects $(P, \theta_1, \ldots, \theta_t)$, where $\theta_i = 1$ for some i.

For each $d \geq 0$, let $R\Delta^d$ be the R-algebra

$$R\Delta^d = R[T_0, \ldots, T_d]/(T_0 + \cdots + T_d - 1),$$

which is isomorphic to a polynomial ring with d indeterminates over R. We denote by **Ord** the category of finite nonempty ordered sets and by $[d]$, where d is a nonnegative integer, the object $\{0 < 1 < \cdots < d\}$. Given a map $\varphi : [d] \to [e]$ in **Ord**, the map $\varphi^* : R\Delta^e \to R\Delta^d$ is defined by $\varphi^*(T_j) = \sum_{\varphi(i)=j} T_i$. The map φ^* gives us a simplicial ring $R\Delta^\bullet$.

By applying the functor $K_0(-, \mathbb{G}_m^{\wedge t})$, we get the simplicial abelian group

$$d \mapsto K_0(R\Delta^d, \mathbb{G}_m^{\wedge t}).$$

The associated (normalized) chain complex via the Dold-Kan correspondence between the simplicial abelian groups and the nonnegative chain complexes of abelian groups, shifted cohomologically by $-t$, is called the motivic complex of Goodwillie and Lichtenbaum of *weight t*.

For each $(P, \theta_1, \ldots, \theta_t)$ in $K_0(R, \mathbb{G}_m^{\wedge t})$ there exists a projective module Q such that $P \oplus Q$ is free over R. Then $(P \oplus Q, \theta_1 \oplus 1_Q, \ldots, \theta_t \oplus 1_Q)$ represents the same element of $K_0(R, \mathbb{G}_m^{\wedge t})$ as $(P, \theta_1, \ldots, \theta_t)$. Thus $K_0(R\Delta^d, \mathbb{G}_m^{\wedge t})$ can be explicitly presented with generators and relations involving t-tuples of commuting matrices $(A_1, \ldots, A_t) = \big(A_1(T_1, \ldots, T_d), \ldots, A_t(T_1, \ldots, T_d)\big)$ in $GL_n(R\Delta^d)$, $n \geq 1$.

Throughout this paper, for a regular local ring R the motivic cohomology $H^q_{\mathcal{M}}\big(\operatorname{Spec} R, \mathbb{Z}(t)\big)$ will be the $(2t - q)$-th homology group of the Goodwillie-Lichtenbaum complex of weight t. In particular,

$$H^q_{\mathcal{M}}\big(\operatorname{Spec}\mathbb{C}, \mathbb{Z}(t)\big) = \pi_{2t-q}\Omega^{-t}|d \mapsto K_0(\mathbb{C}\Delta^d, \mathbb{G}_m^{\wedge t})|$$
$$= \pi_{t-q}|d \mapsto K_0(\mathbb{C}\Delta^d, \mathbb{G}_m^{\wedge t})|.$$

Walker showed in [17] that this cohomology group agrees with the definition of motivic cohomology given by Voevodsky in [16] and thus various other definitions of motivic cohomology for smooth schemes over a field.

Our description relies on the construction of an analogue of the n-th polylogarithm as a certain alternating multilinear function from the set of n-tuples of holomorphic functions on an open domain $U \subset \mathbb{C}^{n-1}$ to the group of closed $(n-1)$-forms. We show that it gives rise to a homomorphism from $H^1_{\mathcal{M}}\big(X, \mathbb{Z}(n)\big)$ into $\mathbb{R}$. An argument using simultaneous triangularization of n-tuples of commuting matrices to obtain n-tuples of local holomorphic functions leads to the local construction of an $(n-1)$-form on an open set in $\mathbb{C}^{n-1}$ whose complement is thin. Using the Analytic Desingularization Theorem [1], we obtain an appropriate proper mapping such that the pullbacks of the eigenvalues are well-defined globally. Using this mapping, the form is extended smoothly to all of $\mathbb{C}^{n-1}$. The homomorphism is obtained by integrating the form over the standard $(n-1)$-simplex in $\mathbb{C}^{n-1}$. The classical monodromy computation for the polylogarithm is replaced by finding a suitable proper mapping.

2. Constructing closed forms

To define a homomorphism from $H^1_{\mathcal{M}}\big(\operatorname{Spec}\mathbb{C},\, \mathbb{Z}(n)\big)$, we will want to construct a differential form with certain properties on $\mathbb{C}^s$, where $s = n - 1$. However, for technical reasons we allow s to be different from $n - 1$ in this section. Let $u(z_1, \ldots, z_s) = u(x_1 + iy_1, \ldots, x_s + iy_s)$ be a smooth real-valued function on a domain $U \subset \mathbb{C}^s (\cong \mathbb{R}^{2s})$. Then we have a 1-form

$$du = \sum_{i=1}^{s} \left(\frac{\partial u}{\partial x_i} dx_i + \frac{\partial u}{\partial y_i} dy_i \right).$$

For a 1-form

$$\omega = \sum_{i=1}^{s} (f_i dx_i + g_i dy_i)$$

we define its conjugate form as the 1-form

$$\star\omega = \sum_{i=1}^{s} \left(-g_i dx_i + f_i dy_i \right).$$

In particular, for a smooth real-valued function u on U we have

$$\star du = \sum_{i=1}^{s} \left(-\frac{\partial u}{\partial y_i} dx_i + \frac{\partial u}{\partial x_i} dy_i \right).$$

This agrees with the notation of [2] when $s = 1$. We also notice that if u is the real part of a holomorphic function f on U, then

$$\star du = \sum_{i=1}^{s} \left(\mathrm{Im} \left(\frac{\partial f}{\partial z_i} \right) dx_i + \mathrm{Re} \left(\frac{\partial f}{\partial z_i} \right) dy_i \right).$$

Then $\star$ is a $C^{\infty}(U)$-linear operator on the group $A^1(U, \mathbb{R})$ of smooth 1-forms on U such that $\star\star = -1$. We may also define a linear operator $\star : A^p(U, \mathbb{R}) \to A^p(U, \mathbb{R})$ such that

$$\star(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_p) = \star\omega_1 \wedge \star\omega_2 \wedge \cdots \wedge \star\omega_p,$$

but we will not need it.

REMARK 2.1.   If $\omega_1, \omega_2, \ldots, \omega_{2s}$ are 1-forms on $U \subset \mathbb{C}^s$, then

$$\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_{2s} = \star\omega_1 \wedge \star\omega_2 \wedge \cdots \wedge \star\omega_{2s}.$$

Proof. Locally, we may write, for $i = 1, 2, \ldots, 2s$,

$$\omega_i = \sum_{j=1}^{n-1} \left(a_{ij} dx_j + b_{ij} dy_j \right),$$

where a_{ij} and b_{ij} are smooth real-valued functions on a local neighborhood in U. Then we are reduced to proving that the determinant of the matrix

$$\begin{pmatrix} a_{11} & b_{11} & a_{12} & b_{12} & \ldots & a_{1\,s} & b_{1\,s} \\ a_{21} & b_{21} & a_{22} & b_{22} & \ldots & a_{2\,s} & b_{2\,s} \\ . & . & . & . & & . & . \\ . & . & . & . & & . & . \\ a_{2s\,1} & b_{2s\,1} & a_{2s\,2} & b_{2s\,2} & \ldots & a_{2s\,s} & b_{2s\,s} \end{pmatrix}$$

is equal to the determinant of the matrix

$$\begin{pmatrix} -b_{11} & a_{11} & -b_{12} & a_{12} & \ldots & -b_{1\,s} & a_{1\,s} \\ -b_{21} & a_{21} & -b_{22} & a_{22} & \ldots & -b_{2\,s} & a_{2\,s} \\ . & . & . & . & & . & . \\ . & . & . & . & & . & . \\ -b_{2s\,1} & a_{2s\,1} & -b_{2s\,2} & a_{2s\,2} & \ldots & -b_{2s\,s} & a_{2s\,s} \end{pmatrix}.$$

But this can be shown with basic properties of determinants.   □

The following lemma is straightforward.

LEMMA 2.2.   *If u is pluriharmonic (i.e., locally the real part of a holomorphic function) on $U \subset \mathbb{C}^s$, then*

$$d \star du = 0.$$

Proof. We just compute the differential $d \star du$:

$$d \star du = \sum_{i,j} \left(\frac{\partial^2 u}{\partial y_j \partial y_i} + \frac{\partial^2 u}{\partial x_i \partial x_j} \right) dx_i \wedge dy_j$$

$$+ \sum_{i \neq j} \left(\frac{\partial^2 u}{\partial x_j \partial y_i} - \frac{\partial^2 u}{\partial x_i \partial y_j} \right) dx_i \wedge dx_j$$

$$+ \sum_{i \neq j} \left(\frac{\partial^2 u}{\partial y_i \partial x_j} - \frac{\partial^2 u}{\partial y_j \partial x_i} \right) dy_i \wedge dy_j.$$

All terms on the right-hand side are zero, by the Cauchy-Riemann equations.

Actually, if v is the imaginary part of the holomorphic function, then we have $\star du = dv$ locally. Thus the lemma is nothing more than the usual equality $d^2 v = 0$. $\qquad\square$

LEMMA 2.3. *If $\omega_1, \omega_2, \ldots, \omega_n$ are smooth 1-forms on $U \subset \mathbb{C}^s$, then the n-form*

$$\varphi \left(= \varphi(\omega_1, \omega_2, \ldots, \omega_n) \right) := \sum_{k=0}^{[(n-1)/2]} (-1)^k \sum_{\substack{A \subset \{1,2,\ldots,n\} \\ |A|=n-(2k+1)}} \varphi_A(\omega_1, \omega_2, \ldots, \omega_n)$$

on U vanishes if $s \leq n - 1$. Here, $\varphi_A(\omega_1, \omega_2, \ldots, \omega_n)$ denotes the n-form $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$, where

$$\varphi_i = \begin{cases} \star\omega_i & \text{if } i \in A, \\ \omega_i & \text{otherwise.} \end{cases}$$

Proof. It is sufficient for us to prove that the above n-form φ vanishes in the group $A^n(U, \mathbb{R}) \otimes \mathbb{C}$. Observe first that $dz_1, d\bar{z}_1, \ldots, dz_s, d\bar{z}_s$ form a $C^\infty(U) \otimes \mathbb{C}$-basis of the module $A^1(U, \mathbb{R}) \otimes \mathbb{C}$, where $dz_j = dx_j + idy_j$ and $d\bar{z}_j = dx_j - idy_j$ for each $j = 1, \ldots, s$. We then have $\star dz_j = idz_j$ and $\star d\bar{z}_j = -id\bar{z}_j$ for any $j = 1, 2, \ldots, s$. Now, by linearity, we need to prove the vanishing of the above n-form only when $\omega_1, \ldots, \omega_n$ are n distinct elements from the basis. Under this assumption, for $j = 1, \ldots, n$, we have $\star\omega_j = \lambda_j \omega_j$, where λ_j is either i or $-i$. Then

$$\sum_{\substack{A \subset \{1,2,\ldots,n\} \\ |A|=n-(2k+1)}} \varphi_A(\omega_1, \omega_2, \ldots, \omega_n) = S_{n-(2k+1)}(\lambda_1, \lambda_2, \ldots, \lambda_n)\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n,$$

where $S_{n-(2k+1)}(X_1, X_2, \ldots, X_n)$ is the elementary symmetric polynomial of degree $n - (2k + 1)$ in $X_1, X_2, \ldots, X_n$. Hence we need only to show that

$$S_{n-1}(\lambda_1, \lambda_2, \ldots, \lambda_n) - S_{n-3}(\lambda_1, \lambda_2, \ldots, \lambda_n) + \cdots = 0.$$

We obtain this equality by expanding and dividing by $2i$ the equality

$$(\lambda_1 + i)(\lambda_2 + i)\cdots(\lambda_n + i) - (\lambda_1 - i)(\lambda_2 - i)\cdots(\lambda_n - i) = 0.$$

Both terms on the left-hand side vanish because at least one of the λ_j's is i (respectively, $-i$) since $\omega_1, \ldots, \omega_n$ are n *distinct* elements from the basis and so one of them is $d\bar{z}_j$ (respectively, dz_j) for some j. $\qquad\square$

Now, given functions $u_1, \ldots, u_n$ which are locally the real parts of holomorphic functions on $U \subset \mathbb{C}^s$, we want to construct an $(n-1)$-form $\omega_P(u_1, u_2, \ldots, u_n)$ such that its differential is equal to the n-form given in Lemma 2.3 when $\omega_i = du_i$ for each $i = 1, \ldots, n$.

DEFINITION 2.4. Given pluriharmonic functions $u_1, \ldots, u_n$ on $U \subset \mathbb{C}^s$, we define $w_P(u_1, \ldots, u_n)$ to be the $(n-1)$-form

$$\sum_{k=0}^{[(n-1)/2]} (-1)^k w_k(u_1, \ldots, u_n).$$

Here $w_k(u_1, \ldots, u_n)$ denotes the $(n-1)$-form

$$\frac{1}{\alpha!(n-\alpha)!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, u_{\sigma(1)} \wedge \varphi_{\sigma(2)} \wedge \cdots \wedge \varphi_{\sigma(n)},$$

where $\alpha = n - (2k+1)$ and

$$\varphi_i = \begin{cases} \star du_i & \text{if } i \leq \alpha, \\ du_i & \text{otherwise.} \end{cases}$$

The following proposition states the desired properties of $w_P(u_1, \ldots, u_n)$. Note that the form $w_k(u_1, \ldots, u_n)$ in Definition 2.4 is chosen so that the assignment $(u_1, \ldots, u_n) \mapsto w_k(u_1, \ldots, u_n)$ is alternating, i.e., $w_k(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) = \mathrm{sgn}(\sigma)w_k(u_1, \ldots, u_n)$, whenever σ is a permutation in S_n.

LEMMA 2.5. *The form $w_P(u_1, \ldots, u_n)$ defined in Definition 2.4 is a closed $(n-1)$-form if $s \leq n-1$, and $w_P(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) = \mathrm{sgn}(\sigma)w_P(u_1, \ldots, u_n)$, whenever σ is a permutation of the set $\{1, 2, \ldots, n\}$.*

Proof. By Lemma 2.2,

$$dw_k(u_1, \ldots, u_n) = \frac{1}{\alpha!(n-\alpha)!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, \varphi_{\sigma(1)} \wedge \varphi_{\sigma(2)} \wedge \cdots \wedge \varphi_{\sigma(n)},$$

which, in turn, is equal to

$$\sum_{\substack{A \subset \{1,2,\ldots,n\} \\ |A| = n-(2k+1)}} \varphi_A(\omega_1, \omega_2, \ldots, \omega_n).$$

By Lemma 2.3, $dw_P(u_1, \ldots, u_n) = 0$. Thus $w_P(u_1, \ldots, u_n)$ is a closed $(n-1)$-form whenever $s \leq n - 1$.

The second statement is clear from our construction. $\qquad\square$

In particular, we will pay attention to the case where $f_1, \ldots, f_n$ are holomorphic functions on $U \subset \mathbb{C}^s$ and $u_1(z) = \log|f_1(z)|, \ldots, u_n(z) = \log|f_n(z)|$. We record the following straightforward computation as a lemma.

LEMMA 2.6. *Suppose that $f_1, \ldots, f_n$ are non-vanishing holomorphic functions on $U \subset \mathbb{C}^s$, and let*

$$u_1(z) = \log|f_1(z)|, \ldots, u_n(z) = \log|f_n(z)|.$$

Then we have, for each $j = 1, \ldots, n$,

$$du_j = \sum_{k=1}^{s} \left(\mathrm{Re}\left(\frac{1}{f_j} \frac{\partial f_j}{\partial z_k} \right) dx_k - \mathrm{Im}\left(\frac{1}{f_j} \frac{\partial f_j}{\partial z_k} \right) dy_k \right),$$

$$\star du_j = \sum_{k=1}^{s} \left(\mathrm{Im}\left(\frac{1}{f_j} \frac{\partial f_j}{\partial z_k} \right) dx_k + \mathrm{Re}\left(\frac{1}{f_j} \frac{\partial f_j}{\partial z_k} \right) dy_k \right).$$

Proof. See the computation before Remark 2.1. $\qquad\square$

EXAMPLE 2.7. For $n = 2$ and $s = 1$, $w_P(u_1, u_2)$ is the 1-form

$$u_1 \wedge \star du_2 - \star du_1 \wedge u_2 = \log|f_1(z)| \left(\mathrm{Im}\left(\frac{f_2'(z)}{f_2(z)} \right) dx + \mathrm{Re}\left(\frac{f_2'(z)}{f_2(z)} \right) dy \right)$$

$$- \log|f_2(z)| \left(\mathrm{Im}\left(\frac{f_1'(z)}{f_1(z)} \right) dx + \mathrm{Re}\left(\frac{f_1'(z)}{f_1(z)} \right) dy \right).$$

If we integrate this form over a singular 1-simplex (i.e., a path) γ in $\mathbb{C} - 0$, then we get

$$\int_0^1 \log|f_1(\gamma(t))| \left(\mathrm{Im}\left(\frac{f_2'(\gamma(t))}{f_2(\gamma(t))} \right) \mathrm{Re}(\gamma'(t)) dt \right.$$

$$\left. + \mathrm{Re}\left(\frac{f_2'(\gamma(t))}{f_2(\gamma(t))} \right) \mathrm{Im}(\gamma'(t)) dt \right)$$

$$- \int_0^1 \log|f_2(\gamma(t))| \left(\mathrm{Im}\left(\frac{f_1'(\gamma(t))}{f_1(\gamma(t))} \right) \mathrm{Re}(\gamma'(t)) dt \right.$$

$$\left. + \mathrm{Re}\left(\frac{f_1'(\gamma(t))}{f_1(\gamma(t))} \right) \mathrm{Im}(\gamma'(t)) dt \right)$$

$$= \int_0^1 \log|f_1(\gamma(t))| \mathrm{Im}\left(\frac{f_2'(\gamma(t))}{f_2(\gamma(t))} \gamma'(t) \right) dt$$

$$- \int_0^1 \log|f_2(\gamma(t))| \mathrm{Im}\left(\frac{f_1'(\gamma(t))}{f_1(\gamma(t))} \gamma'(t) \right) dt.$$

In particular, with the path $\gamma(t) = t$ the integral is equal to

$$\int_0^1 \log|f_1(t)|\text{Im}\left(\frac{f_2'(t))}{f_2(t)}\right)dt - \int_0^1 \log|f_2(t)|\text{Im}\left(\frac{f_1'(t))}{f_1(t)}\right)dt.$$

This is a bilinear form of dilogarithm considered in [14].

3. Constructing a homomorphism from $H^1_{\mathcal{M}}\big(\text{Spec}\,\mathbb{C},\,\mathbb{Z}(n)\big)$ into $\mathbb{R}$

Recall that

$$H^q_{\mathcal{M}}\big(\text{Spec}\,\mathbb{C},\,\mathbb{Z}(n)\big) = \pi_{n-q}|d \mapsto K_0(\mathbb{C}\Delta^d,\,\mathbb{G}_m^{\wedge n})|,$$

where the group $K_0(\mathbb{C}\Delta^s,\,\mathbb{G}_m^{\wedge n})$ is generated by n-tuples $(A_1,\ldots,A_n) = \big(A_1(T_1,\ldots,T_s),\ldots,A_n(T_1,\ldots,T_s)\big)$ of commuting invertible $r \times r$ matrices for various $r \geq 0$. This motivic complex is discussed in the introduction of this paper.

For each $x \in \mathbb{C}^s$, we denote by $\mathcal{O}_x$ the local ring of germs of holomorphic functions at x.

LEMMA 3.1 (Simultaneous triangularization). *For commuting matrices $A_1,\ldots,A_n$ in $GL_r(\mathbb{C}[T_1,\ldots,T_s])$, let $x \in \mathbb{C}^s$ be such that, for each $i = 1,\ldots,n$, the characteristic polynomial P_{A_i} of A_i is factored as*

$$P_{A_i}(\lambda) = (\lambda - a_{i,1})(\lambda - a_{i,2})\cdots(\lambda - a_{i,r})$$

for some analytic functions $a_{i,1}, a_{i,2},\ldots,a_{i,r}$ on an open neighborhood $U \subset \mathbb{C}^s$ of x. Then there exists $S \in GL_n(K)$ such that $S^{-1}A_iS$ is an upper triangular matrix in $GL_r(K)$ for every index i, where K is the fraction field of $\mathcal{O}_x$.

Proof. The existence of S can be proved by induction on the size r of the matrices. If $r = 1$, there is nothing to prove. We may assume that at least one of $A_1,\ldots,A_n$, say A_1, is not a diagonal matrix. Take an eigenvalue, say $a \in K$, of A_1. Then the eigenspace $E = \{v \in K^r \mid A_1v = av\}$ is neither 0, nor K^r. We also have $A_iE \subset E$ for every i since $a(A_iv) = A_i(av) = A_iA_1v = A_1(A_iv)$. Choose a basis $v_1, v_2,\ldots,v_s$ of E and extend it to a basis $v_1, v_2,\ldots,v_r$ of K^r. Let $P \in GL_r(K)$ be the matrix whose column vectors are these basis vectors $v_1, v_2,\ldots,v_r$. Then each $P^{-1}A_1P$ is of the form

$$\begin{pmatrix} A_{i,11} & A_{i,12} \\ 0 & A_{i,22} \end{pmatrix},$$

where the block matrices $A_{i,11}$ and $A_{i,22}$ are $s \times s$ and $(r-s) \times (r-s)$-matrices, respectively. By the induction hypothesis, we have $S_1 \in GL_s(K)$ and $S_2 \in GL_{r-s}(K)$ such that $S_1^{-1}A_{i,11}S_1$ and $S_2^{-1}A_{i,22}S_2$ are upper triangular matrices for every $i = 1,\ldots,n$. Then

$$\begin{pmatrix} S_1^{-1} & 0 \\ 0 & S_2^{-1} \end{pmatrix} P^{-1}A_iP \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$$

is upper triangular for every $i = 1, \ldots, n$. Therefore we have obtained a matrix $S \in GL_r(K)$ such that $S^{-1}A_iS$ is upper triangular in $GL_r(K)$ for every $i = 1, \ldots, n$. □

COROLLARY 3.2. *With the same assumptions as in Lemma 3.1, let* $(a_{i,1}, a_{i,2}, \ldots, a_{i,r})$ *be the ordered r-tuple of diagonal entries of $S^{-1}A_iS$ for each $i = 1, \ldots, n$ Then the set of n-tuples*

$$\{(a_{1,1}, a_{2,1}, \ldots, a_{n,1}), \ldots, (a_{1,r}, a_{2,r}, \ldots, a_{n,r})\}$$

of elements of $\mathcal{O}_x$ is determined only by $(A_1, \ldots, A_n)$ and $x \in \mathbb{C}^s$ and is independent of the choice of S.

Proof. Let K be the quotient field of $\mathcal{O}_x$. Let $i \in \{1, \ldots, n\}$ be any index. Then, by the theory of Jordan canonical forms, there exists $P \in GL_r(K)$ be such that $P^{-1}A_iP$ is a block-diagonal matrix $A_{i,1} \oplus A_{i,2} \oplus \cdots \oplus A_{i,l}$ in $GL_r(K)$, where each block matrix $A_{i,k}$ has a unique eigenvalue $\lambda_i \in \mathcal{O}_x$ and the eigenvalues of any two diagonal blocks $A_{i,k}$ and $A_{i,k'}$ are different whenever $k \neq k'$. Now let B denote any of the matrices $A_{i'}$, $i' \neq i$. Write

$$P^{-1}BP = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1l} \\ B_{21} & B_{22} & \cdots & B_{2l} \\ . & & & . \\ B_{l1} & B_{l2} & \cdots & B_{ll} \end{pmatrix},$$

where B_{kk} is a matrix of the same size as $A_{i,k}$, for each $k = 1, \ldots, l$.

We may regard $M = K^r$ as a left $K[X]$-module by declaring $Xv = A_iv$ for every $v \in K^r$. Then M is isomorphic to $M_1 \oplus \cdots \oplus M_l$ as a $K[X]$-module, where X acts on each M_k via $A_{i,k}$ and $\mathrm{Ann}(M_k) = (X - \lambda_k)^{e_k}$ for a positive integer e_k. Since $A_{i,k}B_{km} = B_{km}A_{i,m}$, the K-linear map $B_{km} : M_m \to M_k$ can be considered as a $K[X]$-linear map. But $\mathrm{Ann}(M_m) = (X - \lambda_m)^{e_m}$, $\mathrm{Ann}(M_k) = (X - \lambda_k)^{e_k}$, and these two ideals are relatively prime in the ring $K[X]$ if $k \neq m$. Therefore, any element in the image of map B_{km} is killed by the whole ring $K[X]$. Hence $B_{km} = 0$ whenever $k \neq m$, and for each k the restriction of the linear automorphism $B : M \to M$ to M_k is just B_{kk}.

Therefore the eigenvalues of B_{kk} are exactly those of $B = A_{i'}$ that correspond to λ_i. Since the modules $M_k = \{v \in M \mid (A_i - \lambda_iI)^\nu v = 0 \text{ for some } \nu > 0\}$, $k = 1, \ldots, l$, do not depend on the choice of P, the proof is complete. □

Let S_A be the set of points in $\mathbb{C}^s$ which are roots of an irreducible factor of the discriminant of one of the characteristic polynomials $P_{A_1}(\lambda), \ldots, P_{A_n}(\lambda)$ of the matrices $A_1, \ldots, A_n$. The set S_A is a proper closed algebraic subset of the affine space $\mathbb{C}^s$. Note that S_A is the zero set of a single polynomial $f \in \mathbb{C}[z_1, \ldots, z_s]$. We also set $U_A = \mathbb{C}^s - S_A$ so that U_A is an open domain in $\mathbb{C}^s$.

Now, let $u_{i,j}(z_1, \ldots, z_n) = \log|a_{i,j}(z_1, \ldots, z_n)|$ for $i = 1, \ldots, n$ and $j = 1, \ldots, r$ on each open neighborhood U of $x \in U_A$. We then define the $(n-1)$-form $\omega_P(A_1, \ldots, A_n)$ on U_A as follows.

DEFINITION 3.3. For the n-tuple of commuting $r \times r$ matrices $(A_1, \ldots, A_n)$, the closed $(n-1)$-form $\omega_P(A_1, \ldots, A_n)$ on U_A is defined as the sum

$$\sum_{j=1}^{r} \omega_P(u_{1,j}, u_{2,j}, \ldots, u_{n,j}),$$

where $u_{i,j}$, for $i = 1, \ldots, n$ and $j = 1, \ldots, r$, are as above.

Using Lemma 2.6 as an intermediate definition, we see that the coefficients of the possible exterior product of dx_j or dy_j which appear in $\omega_P(A_1, \ldots, A_n)$ are well-defined and smooth on $\mathbb{C}^s$ except on the set S_A. Since being closed is a local property for a differential form, we see that $\omega_P(A_1, \ldots, A_n)$ is a closed $(n-1)$-form on $U_A \subset \mathbb{C}^s$. Our next task is to extend $\omega_P(A_1, \ldots, A_n)$ to a smooth form on $\mathbb{C}^s$. We will need several known results to do that.

LEMMA 3.4 (Push-forward of holomorphic forms for proper mappings). *Suppose that $\psi : W \to V$ is a proper, surjective holomorphic mapping, where both V and W are irreducible analytic varieties of the same dimension. Then there exists the push-forward mapping $\psi_* : \Omega^q(W) \to \Omega^q(V)$, which commutes with the differential d, where Ω^q is the sheaf of holomorphic q-forms. Whenever U is a small open set in V such that the inverse image $\psi^{-1}(U) = U_1 \cup \cdots \cup U_m$ decomposes into m disjoint open sets U_ν and $\psi : U_\nu \to U$ is an isomorphism with inverse s_ν, we have $\psi_*(f) = s_1^*(f) + \cdots + s_m^*(f)$.*

Proof. See [10] for a proof. Note that f_* is not necessarily an algebra homomorphism between the exterior algebras of holomorphic forms. □

COROLLARY 3.5 (Push-forward of real analytic forms for proper mappings). *With the same assumption as in Lemma 3.4, there exists a real analytic form $\psi_*(f)$ for any real analytic q-form f on W such that, whenever U is a small open set in V such that the inverse image $\psi^{-1}(U) = U_1 \cup \cdots \cup U_m$ decomposes into m disjoint open sets U_ν and $\psi : U_\nu \to U$ is an isomorphism with inverse s_ν, we have $\psi_*(f) = s_1^*(f) + \cdots + s_m^*(f)$.*

Proof. Let $\hat{\psi} : \hat{W} \to \hat{V}$ be the complexification (cf. [13]) of ψ. Then the complexification $\hat{f}$ is a holomorphic form on an open set of $\hat{W}$ containing W. Note that we have the following commutative diagram.

$$
\begin{array}{ccc}
W & \xrightarrow{\psi} & V \\
\downarrow & & \downarrow \\
\hat{W} & \xrightarrow{\hat{\psi}} & \hat{V}
\end{array}
$$

Now let $\bar{W}$ and $\bar{V}$ be copies of W and V, respectively. We may choose the map $W \to \hat{W} = W \times \bar{W}$ given by $z \mapsto (z, \bar{z})$ as a complexification of W, and similarly for V. Then $\hat{\psi}(z_1, z_2) = (\psi(z_1), \psi(z_2))$ and it is clear that $\hat{\psi} : \hat{W} \to \hat{V}$ is a proper surjective holomorphic mapping. By Lemma 3.4, we have a push-forward $\hat{\psi}_*(\hat{f})$, which is holomorphic on $\hat{V}$. Its restriction to V gives a real analytic q-form which is an integer multiple of $\psi_*(f)$. $\qquad\square$

LEMMA 3.6 (Analytic desingularization theorem). *Let V be biholomorphic to $\mathbb{C}^s$ and let f be a nonzero holomorphic function on V. Then we can construct an iterated analytic monoidal transform W of V such that the hypersurface defined by the pullback of f in W has a normal crossing at every point Q of W.*

Proof. See the appendix of [1] for a proof of the theorem. Actually, this result should be regarded as a step toward resolution of singularities problem in characteristic 0 (cf. [12]). That the hypersurface $f = 0$ has a normal crossing at a point Q means that, for some basis $y_1, \ldots, y_s$ of the maximal ideal m_Q of the local ring of Q, we have $f = \delta y_1^{b_1} \cdots y_s^{b_s}$, where $b_1, \ldots, b_s$ are nonnegative integers and δ is a unit in the local ring of Q. $\qquad\square$

PROPOSITION 3.7. *The closed $(n-1)$-form $\omega_P(A_1, \ldots, A_n)$ on U_A in Definition 3.3 can be extended to a closed smooth $(n-1)$-form on all of $\mathbb{C}^s$.*

Proof. Let P be any point in $S_A = \{z \mid f(z) = 0\}$. Let V be a connected open neighborhood of P and let γ be a closed path with endpoint P_0 in $V - S_A$. With the same notation as in Corollary 3.2, let us consider the analytic continuation $a_{i,j}{}^\gamma$ of the holomorphic functions $a_{i,j} \in \mathcal{O}_{P_0}$ along the path γ, for $i = 1, \ldots, n$, $j = 1, \ldots, r$ and $k = 1, \ldots, l$. First of all, $a_{i,j}{}^\gamma$ is a root of the characteristic polynomial P_{A_i} of A_i, so there exists an index j' such that $a_{i,j}{}^{\gamma_k} = a_{i,j'}$. Furthermore, via the operation $a_{i,j} \mapsto a_{i,j}{}^\gamma$, the group $G = \pi_1(V - S_A, P_0)$ acts on the finite set of n-tuples $\{(a_{1,1}, a_{2,1}, \ldots, a_{n,1}), \ldots, (a_{1,r}, a_{2,r}, \ldots, a_{n,r})\}$, with the notation of Corollary 3.2. In particular, there exists a positive integer m such that for every $g \in G$, g^m fixes the whole set of n-tuples. For example, we may take $m = r!$. Now, by Lemma 3.6, there exists a proper surjective holomorphic mapping $\psi' : W \to V$ such that the hypersurface f has a normal crossing at every point of W. Let Q be a preimage of P under ψ'. Changing the basis for the maximal ideal m_Q of the local ring of Q, we may assume that there exists a polydisk $U \simeq \{(z_1, \ldots, z_s) \mid |z_1| < r_1, \ldots, |z_s| < r_s\} \subset W$ containing Q such that $\psi'(V - S_A) \cap U$ is a punctured polydisk $U' \simeq \{(z_1, \ldots, z_s) \mid |z_1| < r_1, \ldots, |z_e| < r_e, 0 < |z_{e+1}| < r_{e+1}, \ldots 0 < |z_s| < r_s\}$. We have a finite surjective mapping $\psi'' : U \to U$ which sends U' onto U' such that

$\pi_1(\psi'')(\pi_1(U')) \subseteq \pi_1(U')^m$. For example, we may take

$$\psi''(z_1, \ldots, z_s) = \left(z_1, \ldots, z_e, \frac{z_{e+1}{}^m}{r_{e+1}{}^{m-1}}, \ldots, \frac{z_s{}^m}{r_s{}^{m-1}} \right).$$

Now, by shrinking V if necessary, we may assume that $W - U$ is a thin subset of W. By Riemann's extension theorem, we may extend $\psi' \circ \psi''$ to a proper holomorphic mapping $\psi : W \to V$. By our choice of m, $a_{i,j} \circ \psi$ has an analytic continuation to all of U for every $i = 1, \ldots, n$ and every $j = 1, \ldots, r$. Furthermore, $a_{i,j}$ is bounded on U since it is a root of a monic polynomial P_{A_i} whose coefficients are bounded on U. Therefore, by Riemann's extension theorem, $a_{i,j} \circ \psi$ extends to a holomorphic function on W. We define a smooth form ω' on W by

$$\omega' = \sum_{j=1}^{r} \omega_P(u_{1,j}, u_{2,j}, \ldots, u_{n,j}),$$

where $u_{i,j}$ is the pluriharmonic function $u_{i,j} = \log|a_{i,j}|$ for $i = 1, \ldots, n$ and $j = 1, \ldots, r$. The form $\omega_P(u_{1,j}, u_{2,j}, \ldots, u_{n,j})$ was introduced in Definition 2.4. In particular, ω' is real analytic. By Corollary 3.5, we have a smooth closed $(n-1)$-form $\psi_*(\omega')$ which is $m' \omega_P(A_1, \ldots, A_n)$ for some positive integer m'. Thus we have extended $\omega_P(A_1, \ldots, A_n)$ to V. The extension is unique since two real analytic functions on a connected open set which agree on an open subset are identical. Since P was an arbitrary point of S_A, the proof is complete. $\qquad\square$

PROPOSITION 3.8.

(i) *(Skew-Symmetry) For n commuting matrices $A_1, A_2, \ldots, A_n$ in $GL_r(\mathbb{C}[T_1, \ldots, T_s])$ and a permutation $\sigma \in S_n$, we have*

$$\omega_P(A_{\sigma(1)}, A_{\sigma(1)}, \ldots, A_{\sigma(n)}) = \mathrm{sgn}(\sigma)\, \omega_P(A_1, A_2, \ldots, A_n).$$

(ii) *(Bilinearity) For commuting matrices $A, B, A_2, A_3, \ldots, A_n$ in $GL_r(\mathbb{C}[T_1, \ldots, T_s])$, we have*

$$\omega_P(AB, A_2, A_3, \ldots, A_n) = \omega_P(A, A_2, A_3, \ldots, A_n) \\ + \omega_P(B, A_2, A_3, \ldots, A_n).$$

Proof. Part (i) follows directly from Lemma 2.5 and (ii) is basically a consequence of the fact that $\log|fg| = \log|f| + \log|g|$. $\qquad\square$

THEOREM 3.9 (Polylogarithm).   *The map P_n from $H^1_{\mathcal{M}}\big(\operatorname{Spec}\mathbb{C},\,\mathbb{Z}(n)\big)$ into $\mathbb{R}$ defined by*

$$P_n(A_1,\ldots,A_n) = \int_{\Delta^{n-1}} \omega_P(A_1,\ldots,A_n)$$

is a well-defined homomorphism.

Proof. Note that the elements of the form $(A_1,\ldots,A_n)$ generate $K_0(\mathbb{C}\Delta^{n-1},\,\mathbb{G}_m^{\wedge n})$ and the map P_n extends $\mathbb{Z}$-linearly to a homomorphism from $K_0(\mathbb{C}\Delta^{n-1},\,\mathbb{G}_m^{\wedge n})$ to $\mathbb{R}$ since

$$\omega_P((A_1,\ldots,A_n) + (A'_1,\ldots,A'_n)) = \omega_P((A_1 \oplus A'_1,\ldots,A_n \oplus A'_n)$$
$$= \omega_P((A_1,\ldots,A_n) + \omega_P((A'_1,\ldots,A'_n))$$

by the way we constructed ω_P in Definition 3.3.

We now prove that the map P_n is a homomorphism from $K_0(\mathbb{C}\Delta^{n-1},\,\mathbb{G}_m^{\wedge n})/\partial K_0(\mathbb{C}\Delta^n,\,\mathbb{G}_m^{\wedge n})$ to $\mathbb{R}$. To prove the vanishing of P_n on the boundary elements in $\partial K_0(\mathbb{C}\Delta^n,\,\mathbb{G}_m^{\wedge n})$, we need to prove that

$$\int_{\Delta^{n-1}} \omega_P(\partial(B_1,\ldots,B_n)) = 0$$

for each $(B_1,\ldots,B_n) \in K_0(\mathbb{C}\Delta^n,\,\mathbb{G}_m^{\wedge n})$.

Let $\phi : \mathbb{R}^n \to \mathbb{C}^n$ be defined by $(x_1,\ldots,x_n) \mapsto (x_1,0,x_2,0,\ldots,x_n,0)$, i.e, ϕ is the inclusion of $\mathbb{R}^n$ as the real part of $\mathbb{C}^n$. Then we have

$$\int_{\Delta^{n-1}} \omega_P(\partial(B_1,\ldots,B_n)) = \int_{\partial\Delta^n} \phi^*\omega_P(B_1,\ldots,B_n)$$
$$= \int_{\Delta^n} \phi^* d\omega_P(B_1,\ldots,B_n).$$

The first equality follows from the definition of the boundary map ∂ on the motivic complex, and the second equality follows from Stokes' theorem. We will show this integral is 0. Note that $d\omega_P(B_1,\ldots,B_n)$ is locally equal to

$$\sum_{j=1}^{r} \varphi(du_{1,j}, du_{2,j}, \ldots, du_{n,j})$$

$$= \sum_{j=1}^{r} \sum_{k=0}^{[(n-1)/2]} (-1)^k \sum_{\substack{B\subset\{1,2,\ldots,n\} \\ |B|=n-(2k+1)}} \varphi_B(du_{1,j}, du_{2,j}, \ldots, du_{n,j}),$$

where the notations are as in Lemma 2.3 and Definition 3.3, except that we are using $(B_1,\ldots,B_n)$ instead of $(A_1,\ldots,A_n)$.

Letting $u_i = \log|f_i|$ for each i, by Lemma 2.6, $\phi^* d\varphi(du_1, du_2, \ldots, du_n)$ is equal to the following sum of $n \times n$ determinants:

$$
\begin{vmatrix}
\mathrm{Re}\left(\frac{\partial f_1}{\partial z_1}/f_1\right)dx_1 & \ldots & \mathrm{Re}\left(\frac{\partial f_1}{\partial z_n}/f_1\right)dx_n \\
\mathrm{Im}\left(\frac{\partial f_2}{\partial z_1}/f_2\right)dx_1 & \ldots & \mathrm{Im}\left(\frac{\partial f_2}{\partial z_n}/f_2\right)dx_n \\
\mathrm{Im}\left(\frac{\partial f_3}{\partial z_1}/f_3\right)dx_1 & \ldots & \mathrm{Im}\left(\frac{\partial f_3}{\partial z_n}/f_3\right)dx_n \\
. & \ldots & . \\
. & \ldots & . \\
. & \ldots & . \\
\mathrm{Im}\left(\frac{\partial f_n}{\partial z_1}/f_n\right)dx_1 & \ldots & \mathrm{Im}\left(\frac{\partial f_n}{\partial z_n}/f_n\right)dx_n
\end{vmatrix}
+
\begin{vmatrix}
\mathrm{Im}\left(\frac{\partial f_1}{\partial z_1}/f_1\right)dx_1 & \ldots & \mathrm{Im}\left(\frac{\partial f_1}{\partial z_n}/f_1\right)dx_n \\
\mathrm{Re}\left(\frac{\partial f_2}{\partial z_1}/f_2\right)dx_1 & \ldots & \mathrm{Re}\left(\frac{\partial f_2}{\partial z_n}/f_2\right)dx_n \\
\mathrm{Im}\left(\frac{\partial f_3}{\partial z_1}/f_3\right)dx_1 & \ldots & \mathrm{Im}\left(\frac{\partial f_3}{\partial z_n}/f_3\right)dx_n \\
. & \ldots & . \\
. & \ldots & . \\
. & \ldots & . \\
\mathrm{Im}\left(\frac{\partial f_n}{\partial z_1}/f_n\right)dx_1 & \ldots & \mathrm{Im}\left(\frac{\partial f_n}{\partial z_n}/f_n\right)dx_n
\end{vmatrix}
$$

$$
+\ \ldots\ +
\begin{vmatrix}
\mathrm{Im}\left(\frac{\partial f_1}{\partial z_1}/f_1\right)dx_1 & \ldots & \mathrm{Im}\left(\frac{\partial f_1}{\partial z_n}/f_1\right)dx_n \\
\mathrm{Im}\left(\frac{\partial f_2}{\partial z_1}/f_2\right)dx_1 & \ldots & \mathrm{Im}\left(\frac{\partial f_2}{\partial z_n}/f_2\right)dx_n \\
\mathrm{Im}\left(\frac{\partial f_3}{\partial z_1}/f_3\right)dx_1 & \ldots & \mathrm{Im}\left(\frac{\partial f_3}{\partial z_n}/f_3\right)dx_n \\
. & \ldots & . \\
. & \ldots & . \\
. & \ldots & . \\
\mathrm{Re}\left(\frac{\partial f_n}{\partial z_1}/f_n\right)dx_1 & \ldots & \mathrm{Re}\left(\frac{\partial f_n}{\partial z_n}/f_n\right)dx_n
\end{vmatrix}
$$

$$
-
\begin{vmatrix}
\mathrm{Re}\left(\frac{\partial f_1}{\partial z_1}/f_1\right)dx_1 & \ldots & \mathrm{Re}\left(\frac{\partial f_1}{\partial z_n}/f_1\right)dx_n \\
\mathrm{Re}\left(\frac{\partial f_2}{\partial z_1}/f_2\right)dx_1 & \ldots & \mathrm{Re}\left(\frac{\partial f_2}{\partial z_n}/f_2\right)dx_n \\
\mathrm{Re}\left(\frac{\partial f_3}{\partial z_1}/f_3\right)dx_1 & \ldots & \mathrm{Re}\left(\frac{\partial f_3}{\partial z_n}/f_3\right)dx_n \\
\mathrm{Im}\left(\frac{\partial f_4}{\partial z_1}/f_4\right)dx_1 & \ldots & \mathrm{Im}\left(\frac{\partial f_4}{\partial z_n}/f_4\right)dx_n \\
. & \ldots & . \\
. & \ldots & . \\
\mathrm{Im}\left(\frac{\partial f_n}{\partial z_1}/f_n\right)dx_1 & \ldots & \mathrm{Im}\left(\frac{\partial f_n}{\partial z_n}/f_n\right)dx_n
\end{vmatrix}
-
\begin{vmatrix}
\mathrm{Re}\left(\frac{\partial f_1}{\partial z_1}/f_1\right)dx_1 & \ldots & \mathrm{Re}\left(\frac{\partial f_1}{\partial z_n}/f_1\right)dx_n \\
\mathrm{Re}\left(\frac{\partial f_2}{\partial z_1}/f_2\right)dx_1 & \ldots & \mathrm{Re}\left(\frac{\partial f_2}{\partial z_n}/f_2\right)dx_n \\
\mathrm{Im}\left(\frac{\partial f_3}{\partial z_1}/f_3\right)dx_1 & \ldots & \mathrm{Im}\left(\frac{\partial f_3}{\partial z_n}/f_3\right)dx_n \\
\mathrm{Re}\left(\frac{\partial f_4}{\partial z_1}/f_4\right)dx_1 & \ldots & \mathrm{Re}\left(\frac{\partial f_4}{\partial z_n}/f_4\right)dx_n \\
. & \ldots & . \\
. & \ldots & . \\
\mathrm{Im}\left(\frac{\partial f_n}{\partial z_1}/f_n\right)dx_1 & \ldots & \mathrm{Im}\left(\frac{\partial f_n}{\partial z_n}/f_n\right)dx_n
\end{vmatrix}
$$

$$
-\ \ldots\ +
\begin{vmatrix}
\mathrm{Re}\left(\frac{\partial f_1}{\partial z_1}/f_1\right)dx_1 & \ldots & \mathrm{Re}\left(\frac{\partial f_1}{\partial z_n}/f_1\right)dx_n \\
. & \ldots & . \\
. & \ldots & . \\
\mathrm{Re}\left(\frac{\partial f_5}{\partial z_1}/f_5\right)dx_1 & \ldots & \mathrm{Re}\left(\frac{\partial f_5}{\partial z_n}/f_5\right)dx_n \\
\mathrm{Im}\left(\frac{\partial f_6}{\partial z_1}/f_6\right)dx_1 & \ldots & \mathrm{Im}\left(\frac{\partial f_6}{\partial z_n}/f_6\right)dx_n \\
. & \ldots & . \\
. & \ldots & . \\
\mathrm{Im}\left(\frac{\partial f_n}{\partial z_1}/f_n\right)dx_1 & \ldots & \mathrm{Im}\left(\frac{\partial f_n}{\partial z_n}/f_n\right)dx_n
\end{vmatrix}
+\ \ldots
$$

But this is equal to $\pm\mathrm{Re}J(f_1,\ldots,f_n)$ if n is odd, and $\pm\mathrm{Im}J(f_1,\ldots,f_n)$ if n is even, where

$$
J(f_1,\ldots,f_n)=
\begin{vmatrix}
\frac{\partial f_1}{\partial z_1}/f_1 & \cdots & \frac{\partial f_1}{\partial z_n}/f_1 \\
\frac{\partial f_2}{\partial z_1}/f_2 & \cdots & \frac{\partial f_2}{\partial z_n}/f_2 \\
. & \cdots & . \\
. & \cdots & . \\
\frac{\partial f_n}{\partial z_1}/f_n & \cdots & \frac{\partial f_n}{\partial z_n}/f_n
\end{vmatrix}
dx_1 dx_2 \ldots dx_n.
$$

We remark that if $f_1, \ldots, f_n$ are analytic on $U \subset \mathbb{C}^n$, then $J(f_1, \ldots, f_n)$ is the pullback of the volume form $\frac{dw_1}{w_1} \wedge \cdots \wedge \frac{dw_n}{w_n}$ on $(\mathbb{C}^*)^n$ under the map $\mathbb{R}^n \cap U \hookrightarrow U \to (\mathbb{C}^*)^n$, where the first map is the inclusion of $\mathbb{R}^n \cap U$ as the real part of $\mathbb{C}^n$ and the second map is defined by $(f_1, \ldots, f_n)$.

Let us define

$$J(B_1, \ldots, B_n) := \sum_{j=1}^r J(f_{1,j}, f_{2,j}, \ldots, f_{n,j}) \quad \text{on } \mathbb{C}^n - S_B,$$

where $f_{1,j}, f_{2,j}, \ldots, f_{n,j}$ are locally n-tuples of eigenvalues of $(B_1, \ldots, B_n)$ as in Corollary 3.2. Then $\phi^* d\omega_P(B_1, \ldots, B_n) = \pm\text{Im} \ (i^n J(B_1, \ldots, B_n))$ on $\mathbb{C}^n - S_B$. Hence the vanishing of the map P_n on a boundary element $\partial(B_1, \ldots, B_n)$ will follow if we have $J(B_1, \ldots, B_n) = 0$, since then $\phi^* d\omega_P(B_1, \ldots, B_n) = 0$.

Now let the symmetric group S_n operate on $K_0(\mathbb{C}\Delta^n, \mathbb{G}_m^{\wedge n})$ by permuting the indeterminates $T_1, \ldots T_n$. For a transposition $\sigma \in S_n$ we have a decomposition $K_0(\mathbb{C}\Delta^n, \mathbb{G}_m^{\wedge n}) \otimes \mathbb{Q} = V^+ \oplus V^-$, where $V^+ = \{v \mid \sigma(v) = v\}$ and $V^- = \{v \mid \sigma(v) = -v\}$. For example, when $n = 2$, the decomposition is given by the formula

$(B_1(T_1, T_2), B_2(T_1, T_2))$

$$= \frac{1}{2} \left((B_1(T_1, T_2), B_2(T_1, T_2)) + (B_1(T_2, T_1), B_2(T_2, T_1))\right)$$

$$+ \frac{1}{2} \left((B_1(T_1, T_2), B_2(T_1, T_2)) - (B_1(T_2, T_1), B_2(T_2, T_1))\right).$$

But $J(B_1, \ldots, B_n) = 0$ for any $(B_1, \ldots, B_n) \in V^-$ since

$$J\left(\sigma(B_1, \ldots, B_n)\right) = J(B_1, \ldots, B_n) = 0 \text{ for every } \sigma \in S_n.$$

Therefore it only remains to prove the vanishing of the map on the set of elements $v \in K_0(\mathbb{C}\Delta^n, \mathbb{G}_m^{\wedge n})$, satisfying $v = \sigma(v)$ for any transposition $\sigma \in S_n$. If $v = \sum_k (B_{1k}, \ldots, B_{nk})$ is in this set, then we have

$$\partial v = \partial_0 v$$

$$= \sum_k (B_{1k}(1 - T_1 - \cdots - T_{n-1}, T_1, \ldots, T_{n-1}),$$

$$\ldots, B_{nk}(1 - T_1 - \cdots - T_{n-1}, T_1, \ldots, T_{n-1}))$$

when n is even, and

$$\partial v = \partial_0 v - \partial_1 v$$

$$= \sum_k (B_{1k}(1 - T_1 - \cdots - T_{n-1}, T_1, \ldots, T_{n-1}),$$

$$\ldots, B_{nk}(1 - T_1 - \cdots - T_{n-1}, T_1, \ldots, T_{n-1}))$$

$$- \sum_k (B_{1k}(0, T_1, \ldots, T_{n-1}), \ldots, B_{nk}(0, T_1, \ldots, T_{n-1}))$$

when n is odd. In either case, the vanishing of P_n follows directly from the symmetry since if we interchange the two variables T_i and T_j, then the orientation of the integral $\int_{\Delta^{n-1}} \omega_P(v)$ changes, but the form $\omega_P(v)$ remains the same, so that

$$\int_{\Delta^{n-1}} \omega_P\left(\sum_k (B_{1k}, \ldots, B_{nk})\right) = \int_{\Delta^{n-1}} \omega_P\left(\sigma\left(\sum_k (B_{1k}, \ldots, B_{nk})\right)\right)$$

$$= -\int_{\Delta^{n-1}} \omega_P\left(\sum_k (B_{1k}, \ldots, B_{nk})\right).$$

In particular, if $n = 2$, we have

$$\omega_P(\partial B)(T) = \sum_k \omega_P(B_{1k}(1-T,T), B_{2k}(1-T,T))$$

$$= \sum_k \omega_P(B_{1k}(T,1-T), B_{2k}(T,1-T))$$

$$= \omega_P(\partial B)(1-T)$$

whenever $B = \sum_k (B_{1k}, B_{2k})$ is in $V^+ \subset K_0(\mathbb{C}\Delta^2, \mathbb{G}_m^{\wedge 2}) \otimes \mathbb{Q}$. Hence we have $\int_0^1 \omega_P(\partial B)(T) = \int_1^0 \omega_P(\partial B)(T)$, and so $\int_0^1 \omega_P(\partial B)(T) = 0$. $\qquad\square$

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