

SIMPLICITY OF THE REDUCED C^* -ALGEBRAS OF CERTAIN COXETER GROUPS

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ABSTRACT. Let (G, S) be a finitely generated Coxeter group such that the Coxeter system is indecomposable and the canonical bilinear form is indefinite but non-degenerate. We show that the reduced C^* -algebra of G is simple with unique normalised trace.

For an arbitrary finitely generated Coxeter group we prove the validity of a Haagerup inequality: There exist constants $C > 0$ and $\Lambda \in \mathbb{N}$ such that, for any function $f \in l^2(G)$ supported on elements of length n with respect to the generating set S and for all $h \in l^2(G)$, $\|f * h\| \leq C(n+1)^{\frac{3}{2}\Lambda} \|f\|$.

1. Introduction

For a discrete group G we denote by $l^2(G)$ the Hilbert space of all square summable complex functions on G and by $B(l^2(G))$ the von Neumann algebra of all bounded operators on $l^2(G)$.

The group G acts on $l^2(G)$ by the left regular representation:

$$\lambda(g)f(h) = f(g^{-1}h), \quad g, h \in G, f \in l^2(G).$$

The reduced (or (left) regular) C^* -algebra $C_r^*(G)$ of G is the operator norm closure of the linear span of the set of operators $\{\lambda(g) : g \in G\}$. We often think of its elements as certain l^2 functions on G . The natural normalised trace on this algebra is then just the evaluation of a function at the group identity.

When G is a non-abelian free group on two generators, Powers [23] showed that $C_r^*(G)$ is a simple C^* algebra, that is, it contains no non-trivial twosided ideals. This result has since been generalised by several authors to various groups and extended to (reduced) cross products; see [1], [2], [3], [6], [14], [17], [22].

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On the other hand, when G is an amenable group (or contains amenable normal subgroups), then the kernel of the trivial representation (or the representation induced from the trivial representation of an amenable normal subgroup) is a non-trivial twosided ideal in $C_r^*(G)$. These facts suggest that the question of the simplicity of $C_r^*(G)$ is related to the Tits alternative for linear groups (i.e., that a linear group either is amenable or contains non-abelian free subgroups).

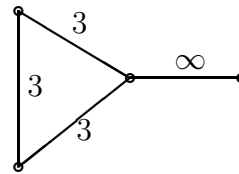
In this note we consider finitely generated Coxeter groups, for which de la Harpe [15] gave an elaboration on the Tits alternative. We show that a finitely generated infinite Coxeter group either contains a normal solvable (even nilpotent) subgroup or has a simple reduced C^* -algebra.

For the geometric representation σ of a Coxeter group (G, S) , with $\#S < \infty$, on $E = \mathbb{R}^S$ we adopt the usual notation of [9]. Assume that (G, S) is indecomposable. The canonical $\sigma(G)$ -invariant bilinear form B can be strictly positive definite, positive semidefinite, or strictly indefinite. In these three cases, the group is, respectively, finite, an affine Coxeter group and hence amenable, or non-amenable [15]. In the last case B may be degenerate. The orthogonal E^0 of E for B then is fixed pointwise by all $\sigma(g)$, $g \in G$, and the kernel of the representation $\hat{\sigma} : G \rightarrow \text{Gl}(\hat{E})$ induced on the quotient $\hat{E} = E/E^0$ is a non-trivial nilpotent normal subgroup. (By choosing an appropriate basis for E it is easily seen to be mapped by σ into a group of unipotent matrices.)

If (G, S) is a decomposable Coxeter system, then G can be written as a direct product of indecomposable Coxeter groups. The reduced C^* -algebra of G is the spatial tensor product of the reduced C^* -algebras of the factors. This spatial tensor product is known to be simple if and only if each factor is a simple C^* -algebra [27].

Hence we shall always assume that B is indecomposable and strictly indefinite but non-degenerate. In the course of our arguments we shall see that under these conditions a Coxeter group is an icc-group (i.e., conjugacy classes of elements different from the identity are infinite) and that the normalised trace on $C_r^*(G)$ is unique. (For a decomposable Coxeter group an argument similar to the one given in the last paragraph shows that the trace is unique if and only if the trace is unique for each indecomposable factor; see [4].)

One might think that Coxeter groups of the above kind are Gromov hyperbolic and arguments like those in [16] combined with [4] would allow us to prove the simplicity of the reduced C^* -algebra, but the group with the Coxeter graph shown in the figure does not contain a finite index Gromov hyperbolic subgroup.



Gromov hyperbolic Coxeter groups have been characterized by Moussong [21] as those Coxeter groups (G, S) which do not contain two infinite commuting parabolic subgroups and further have the property that no subset $T \subset S$

generates a parabolic subgroup (G_T, T) which is an affine Coxeter group of rank at least 3.

When B has signature $(n - 1, 1)$ it can happen that G is a hyperbolic Coxeter group (in the classical sense). Therefore it is a lattice in the real Lie group $O(n - 1, 1)$, and hence Zariski dense in it. A theorem of Bekka, Cowling and de la Harpe [3] then applies. We do not know whether a Coxeter group of the kind considered here is always Zariski dense in some simple real Lie group¹.

To prove the simplicity of the reduced C^* -algebra we have to deal with the combinatorics in G . As a byproduct we obtain a Haagerup inequality, valid for all finitely generated Coxeter groups:

There exist constants $C > 0$ and $\Lambda \in \mathbb{N}$ such that any function f supported on elements of word-length n with respect to the generating set S satisfies

$$\|\lambda(f)\| \leq C(n + 1)^{\frac{3}{2}\Lambda} \|f\|_2.$$

The constant Λ in this inequality can be obtained in terms of the geometric representation of (G, S) . Examples show that it is not best possible. We conjecture that the optimal constant is just the virtual cohomological dimension of G and refer the reader to [5] for motivation.

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2. Trees

In this section we define certain trees on which a finite index torsion free normal subgroup Γ of the Coxeter group acts by simplicial automorphisms of the trees. For trees we use the standard notation of [26]; for the existence of a finite index torsion free normal subgroup see [9, Chap. V, § 4, Ex. 9]. We shall show that the action of Γ on the product of those trees is free. Our construction is similar to that of Januszkiewicz [19]. For the convenience of the reader we shall work with the classical Tits cone U and the transposed geometric action σ^* of G on it. Let us introduce some notation and recall some facts.

The word-length of an element $g \in G$ with respect to the generating set S is defined as $\mathbf{l}(g) = \inf\{n : g = s_1 \cdots s_n, s_1, \dots, s_n \in S\}$. We denote by $T = \{g^{-1}sg : g \in G, s \in S\}$ the set of reflections of G . For $g \in G$ let $N_g = \{t \in T : \mathbf{l}(tg) < \mathbf{l}(g)\}$. With these notations we have for $g, h \in G$

$$\mathbf{l}(g) = \#\{t \in T : \mathbf{l}(tg) < \mathbf{l}(g)\}$$

and (see [10])

$$\mathbf{l}(g^{-1}h) = \#N_g \triangle N_h.$$

¹**Added in proof:** This has recently been shown by Y. Benoist and P. de la Harpe (Adh erence de Zariski de groupes de Coxeter, to appear in *Compositio Mathematica*).

We decompose the set of reflections $T \subset G$ into disjoint Γ -orbits with respect to conjugation:

$$(1) \quad T = T_1 \dot{\cup} T_2 \dot{\cup} \dots \dot{\cup} T_\Lambda.$$

Given $t \in T$, we denote by M_t the hyperplane in E^* fixed by $\sigma^*(t)$ and call it the mirror of t . For $i \in \{1, \dots, \Lambda\}$ we define a graph \mathcal{T}_i as follows: The vertices are the connected components of $U \setminus (\bigcup_{t \in T_i} M_t)$ and two such vertices are connected by an edge if, as connected components, they are separated by just one mirror.

LEMMA 1. *The above defined graph is a tree.*

Proof. We have to show that a closed path in \mathcal{T}_i contains backtracking.

Let $C_0, C_1, \dots, C_n = C_0, n \geq 1$, be the sequence of vertices of a non-trivial closed path. Choose points $c_i \in C_i$, where we may assume $c_0 = c_n$, and elements e_1, \dots, e_n of E , considered as functionals on E^* , with $e_i(c_0) < 0$ for all $i \in \{1, \dots, n\}$, in such a way that e_i vanishes on the hyperplane which defines the edge $\{C_{i-1}, C_i\}$. We may assume that $e_i = e_j$ if the defining edges $\{C_{i-1}, C_i\}$ and $\{C_{j-1}, C_j\}$ are equal.

Consider the function $f : \{0, \dots, n\} \mapsto \mathbb{Z}$ defined by

$$f(i) = \sum_{j=1}^n \text{sign } e_j(c_i).$$

It satisfies $f(0) = f(n) = -n$, and $f(1) = f(n-1) = -n+2$. Since $f(i+1) \in \{f(i)+2, f(i)-2\}$, there exists i_0 such that $f(i_0) > f(i_0+1) = f(i_0-1)$. Hence $e_{i_0}(c_{i_0}) > 0, e_{i_0+1}(c_{i_0}) > 0, e_{i_0}(c_{i_0-1}) < 0, e_{i_0+1}(c_{i_0+1}) < 0$, and, of course, $e_{i_0}(c_0) < 0, e_{i_0+1}(c_0) < 0$. Since U is convex, the hyperplanes $e_{i_0} = 0$ and $e_{i_0+1} = 0$ must intersect inside U and we conclude from [13, Lemma 3] that they coincide. Thus we have found a backtracking. \square

The Coxeter group G acts via σ^* on the chamber system \mathcal{C} defined from the mirrors on U ; see [9] and also [25]. Moreover, two points x, y are separated by a mirror M_t if and only if, for $g \in G, \sigma^*(g)x$ and $\sigma^*(g)y$ are separated by $M_{gtg^{-1}}$. Since we defined the trees \mathcal{T}_i with respect to a Γ -orbit in T , we have:

LEMMA 2. *The contragradient representation σ^* induces an action of Γ on \mathcal{T}_i by automorphisms of the tree.*

Proof. First we show that the action of Γ is well defined. Let $\gamma \in \Gamma$ and a component C be given. For $c_0, c_1 \in C$ we have to show that $\sigma^*(\gamma)c_0$ and $\sigma^*(\gamma)c_1$ are not separated by a mirror of a reflection in \mathcal{T}_i . Indeed, if $\sigma^*(\gamma)c_0$ and $\sigma^*(\gamma)c_1$ were separated by M_t , then by the remark preceding the lemma $M_{\gamma^{-1}t\gamma}$ would separate c_0 and c_1 .

Let $\gamma \in \Gamma$ be given. If C_0 and C_1 are connected by an edge, then there exist exactly one $t \in T_i$ such that $\overline{C_0} \cap M_t \neq \emptyset$ and $\overline{C_1} \cap M_t \neq \emptyset$. Clearly this

is the case if and only if $\overline{\sigma^*(\gamma)C_0} \cap M_{\gamma t \gamma^{-1}} \neq \emptyset$ and $\overline{\sigma^*(\gamma)C_1} \cap M_{\gamma t \gamma^{-1}} \neq \emptyset$. Since $\gamma t \gamma^{-1} \in T_i$ if and only if $t \in T_i$, we are done. \square

We consider the product

$$(2) \quad \mathcal{G} = \mathcal{T}_1 \times \cdots \times \mathcal{T}_\Lambda$$

as a product of chamber systems (see [25, p. 2]). On the vertices $V_{\mathcal{G}}$ of \mathcal{G} we use the metric

$$d^1(x, y) = \sum_{i=1}^{\Lambda} d_i(x_i, y_i),$$

where $x = (x_1, \dots, x_\Lambda), y = (y_1, \dots, y_\Lambda) \in V_{\mathcal{G}}$. The action of Γ on $V_{\mathcal{G}}$ is isometric with respect to this metric.

LEMMA 3. Γ acts freely on the vertices of \mathcal{G} without bounded orbit. Moreover, no non-trivial subgroup of Γ has a bounded orbit.

Proof. Denoting by C_0 the fundamental chamber in U , we have the injection $g \mapsto \sigma^*(g)C_0$ of G onto the chambers of \mathcal{C} . If $[C]_i$ denotes the connected component of the chamber C in $U \setminus (\bigcup_{t \in T_i} M_t)$, we obtain a map of the chambers of \mathcal{C} into the set of vertices of the product \mathcal{G} : $[\cdot] : C \mapsto ([C]_1, \dots, [C]_\Lambda)$. This map is an injection, since two chambers C, C' in \mathcal{C} are different if they are separated by a mirror, say M_t , where $t \in T$, for then there is $j_0 \in \{1, \dots, \Lambda\}$ with $t \in T_{j_0}$, whence $[C]_{j_0} \neq [C']_{j_0}$.

The composition of these two maps defines an embedding of G into the vertices of \mathcal{G} and the action of Γ on this subset is free, since it is just the transferred left multiplication in the group G . Moreover, no non-trivial subgroup of Γ has a bounded orbit. To see this, we note first that the injection $g \mapsto [\sigma^*(g)C_0]$ is an isometry from G endowed with the left invariant distance coming from the word-length with respect to the generating set S into the vertices of \mathcal{G} endowed with the metric d^1 . Indeed, for $g, h \in G$ we have

$$\begin{aligned} \mathbf{l}(g^{-1}h) &= \#N_g \Delta N_h \\ &= \#\{M_t : M_t \text{ separates } \sigma^*(g)C_0 \text{ from } \sigma^*(h)C_0\} \\ &= d^1([\sigma^*(g)C_0], [\sigma^*(h)C_0]). \end{aligned}$$

So, if $[\sigma^*(\gamma^n)C_0], n \in \mathbb{Z}$, were bounded in \mathcal{G} , then the set of mirrors which separate C_0 from a chamber in $\bigcup_{n \in \mathbb{Z}} \sigma^*(\gamma^n)C_0$ would have finite cardinality, and hence $\sup_{n \in \mathbb{Z}} \mathbf{l}(\gamma^n) < \infty$. Therefore the set $\{\gamma^n : n \in \mathbb{Z}\}$ would be finite, contradicting the fact that Γ is torsion free.

Now, if $x \in \mathcal{G}$ has a stabiliser $\Gamma_x \subset \Gamma$, then a vertex $w = [\sigma^*(g)C_0]$ in the image of G in \mathcal{G} would have a bounded Γ_x -orbit since Γ acts by isometries. This follows from

$$d^1(\gamma w, w) \leq d^1(\gamma w, \gamma x) + d^1(\gamma x, x) + d^1(x, w) \leq 2d^1(x, w), \quad \forall \gamma \in \Gamma_x.$$

Hence $\Gamma_x = \{\mathbf{e}\}$. □

3. The action on the trees

In this section we collect some auxiliary results for later use.

LEMMA 4. *If $t_1, t_2 \in T$ are reflections such that the corresponding edges are distinct but in the same tree, then $t_1 t_2$ acts as a translation on this tree.*

Proof. First note that t_1 and t_2 are Γ conjugate, $\gamma^{-1} t_1 \gamma = t_2$, say, since their edges belong to the same tree, \mathcal{T}_i , say. Therefore, $t_1 t_2 = t_1 \gamma^{-1} t_1 \gamma \in \Gamma$.

An oriented line segment in the Tits-cone from a point $v \in M_{t_1}$ to its image $\sigma^*(t_2)v$ is just reversed by $\sigma^*(t_2)$. Since the edges are distinct, this implies that this segment is non-trivial. Since $\sigma^*(t_1)$ maps this line segment to a segment that is adjacent (since both segments contain v), but differently oriented, we conclude that the composition $\sigma^*(t_1)\sigma^*(t_2)$ maps the original segment to a coherently oriented one.

Its image, under $\sigma^*(t_1 t_2)$, and the line segment itself can be connected to a coherently oriented broken line in the cone. The mirrors crossed by the line segment and those crossed by its $\sigma^*(t_1 t_2)$ -image are separated by M_{t_1} . Hence, in \mathcal{T}_i , this broken line defines a coherently oriented geodesic. □

The edges of one of the trees \mathcal{T}_i (identified with the set of reflections T_i), as a Γ -orbit of a reflection, generate a subgroup in G . By a theorem independently proved by Deodhar [11] and Dyer [12] this subgroup is itself a Coxeter group. Clearly this subgroup is normalised by Γ , but in general we cannot expect that all its reflections are contained in T_i .

For subgroups generated by a G -conjugation invariant set of reflections we can say more:

LEMMA 5. *Let $T' \subset T$ be a set of reflections of G , invariant under conjugation. Let W' denote the subgroup generated by T' in G . The subgroup W' is, with respect to a subset $S' \subset T'$, a Coxeter group, normal in G , and its set of reflections coincides with T' .*

Proof. From the theorem of [11], or rather from Step 1 of its proof (see also Theorem 3.4 and Corollary 3.11 in [12]) it is clear that (W', S') is a Coxeter system for some set $S' \subset T'$. A reflection in W' is conjugate, by an element of W' , to a reflection in S' . Since T' is G -conjugation invariant, any reflection of W' is in T' . The other assertions of the lemma are immediate. □

Now we view the set of edges of the product of trees (2) as a fiber bundle $p : \text{edges}(\mathcal{T}_1 \times \cdots \times \mathcal{T}_\Lambda) \rightarrow \{1, \dots, \Lambda\}$ with base space $\{1, \dots, \Lambda\}$. Indeed, two vertices $x = (x_1, \dots, x_\Lambda)$, $y = (y_1, \dots, y_\Lambda)$ are connected by an edge, say $e(x, y)$, if for some $j \in \{1, \dots, \Lambda\}$ the vertices x_j and y_j are connected by an edge in \mathcal{T}_j and for all $i \neq j$ we have $x_i = y_i$. We define $p(e(x, y)) = j$.

Since Γ leaves the fibers invariant we obtain an action of G/Γ by permutations of $\{1, \dots, \Lambda\}$, which we denote by $\pi : G/\Gamma \rightarrow \text{Sym}_\Lambda$. If $\mathcal{O} \subset \{1, \dots, \Lambda\}$ is a $\pi(G/\Gamma)$ -orbit, then, by Lemma 5, the edges of $p^{-1}(\mathcal{O})$ are the reflections of a Coxeter group $W_{\mathcal{O}} \triangleleft G$.

LEMMA 6. For $i \in \{1, \dots, \Lambda\}$ and $g \in G$, $\sigma^*(g)$ induces a morphism of trees:

$$g : \mathcal{T}_i \rightarrow \mathcal{T}_{\pi(\dot{g})(i)}.$$

Here $g \mapsto \dot{g}$ denotes the quotient morphism $G \rightarrow G/\Gamma$.

Proof. An edge of \mathcal{T}_i is the mirror M_t of some reflection $t \in T_i$. Its image under $\sigma^*(g)$ is the mirror of $gtg^{-1} \in T_{\pi(\dot{g})(i)}$. Hence it defines an edge in $\mathcal{T}_{\pi(\dot{g})(i)}$.

If C is a component of $U \setminus \{M_t : t \in T_i\}$, then some $c_0, c_1 \in C$ would have images $\sigma^*(g)c_0, \sigma^*(g)c_1$ in different components of $U \setminus \{M_t : t \in T_{\pi(\dot{g})(i)}\}$ if there is a mirror of some $t' \in T_{\pi(\dot{g})(i)}$ separating the images. But then $g^{-1}t'g \in T_i$ would have a mirror separating c_0 and c_1 . This contradiction shows that $\sigma^*(g)$ defines a map from the vertices of \mathcal{T}_i to those of $\mathcal{T}_{\pi(\dot{g})(i)}$.

If C_0 and C_1 are connected by an edge in \mathcal{T}_i , then there exists exactly one $t \in T_i$ such that $\overline{C_0} \cap M_t \neq \emptyset$ and $\overline{C_1} \cap M_t \neq \emptyset$. Clearly this is the case if and only if $\overline{\sigma^*(g)C_0} \cap M_{gtg^{-1}} \neq \emptyset$ and $\overline{\sigma^*(g)C_1} \cap M_{gtg^{-1}} \neq \emptyset$. Since $gtg^{-1} \in T_{\pi(\dot{g})(i)}$ if and only if $t \in T_i$, we are done. \square

4. A Haagerup inequality

Following Rammage, Robertson and Steger [24] we first prove a Haagerup inequality for the torsion free subgroup Γ of G . We then apply a theorem of Jolissaint [20] to the group extension $0 \rightarrow \Gamma \rightarrow G \rightarrow G/\Gamma \rightarrow 0$.

We consider the product of trees \mathcal{G} as a building of type $\tilde{A}_1 \times \dots \times \tilde{A}_1$. Its apartments are Λ -dimensional Euclidian spaces tessellated by unit cubes. We have a shape defined on pairs of vertices

$$\sigma : V_{\mathcal{G}} \times V_{\mathcal{G}} \rightarrow \mathbb{Z}_+ \times \dots \times \mathbb{Z}_+$$

by

$$\sigma(u, w) = (d_1(u_1, w_1), \dots, d_\Lambda(u_\Lambda, w_\Lambda)).$$

It is clear from Lemma 2 that the action of Γ is shape preserving and we define a shape on Γ by fixing a vertex $v_0 \in V_{\mathcal{G}}$: $\sigma(\gamma) = \sigma(v_0, \gamma v_0)$. Let $p(n_1, \dots, n_\Lambda) = \prod_{i=1}^\Lambda (n_i + 1)$. The following theorem can be proved almost verbatim as the $\tilde{A}_1 \times \tilde{A}_1$ case of [24, Theorem 1.1]:

THEOREM 1. If $h \in l^2(\Gamma)$ is supported on elements of shape (n_1, \dots, n_Λ) , then for all $f \in l^2(\Gamma)$,

$$\| f * h \|_2 \leq p(n_1, \dots, n_\Lambda) \| f \|_2 \| h \|_2.$$

COROLLARY 1. *Let (G, S) be a Coxeter group. There exist constants $C > 0$ and $\Lambda \in \mathbb{N}$ such that for any function $h \in l^2(G)$ supported on elements of length n and for all $f \in l^2(G)$,*

$$\| f * h \|_2 \leq C(n + 1)^{\frac{3}{2}\Lambda} \| f \|_2 \| h \|_2.$$

Proof. Let Γ be a torsion free normal subgroup of finite index in G and denote by Λ the cardinality of distinct conjugation orbits of Γ on the set of reflections of G . Since the length of an element of Γ is just the sum of the components of its shape, the set of elements of length n decomposes into less than $k = (n + 1)^\Lambda$ sets of elements of different shapes. Obviously $p(\sigma(\gamma)) \leq (\mathbf{l}(\gamma) + 1)^\Lambda$. Hence, for any $h \in l^2(\Gamma)$ with support on elements of length n we have

$$\begin{aligned} \| f * h \|_2 &= \left\| \sum_{j=1}^k f * h_j \right\|_2 \\ &\leq (n + 1)^\Lambda \sum_{j=1}^k \| f \|_2 \| h_j \|_2 \\ &\leq (n + 1)^\Lambda \sqrt{k} \| f \|_2 \| h \|_2, \end{aligned}$$

where $h = \sum_{j=1}^k h_j$ is the orthogonal decomposition of h into functions h_j supported on elements of the same shape.

Since Γ is of finite index in G , we may apply [20, Lemma 2.1.2]. □

5. Free subgroups

As before let Γ be a torsion free subgroup of finite index in the Coxeter group G and $\mathcal{T}_1, \dots, \mathcal{T}_\Lambda$ the associated trees.

It is obvious from the definition of the trees that for each such tree the action of Γ on the set of its edges is transitive. Hence each $\Gamma \backslash \mathcal{T}_i$ is either a simple loop or a single edge with two endpoints, depending on whether Γ has one or two orbits on the set of vertices.

LEMMA 7. *For $\gamma \in \Gamma$ there exists a tree, among $\mathcal{T}_1, \dots, \mathcal{T}_\Lambda$, on which γ acts as a translation.*

Proof. Denote by $v_i(\mathbf{e})$ the vertex in \mathcal{T}_i defined by the equivalence class of the group identity. Since $\mathbf{l}(\gamma^n) \rightarrow \infty$ as $n \rightarrow \infty$, the formula

$$\mathbf{l}(\gamma^n) = \sum_1^\Lambda d_i(\gamma^n v_i(\mathbf{e}), v_i(\mathbf{e}))$$

shows that for at least one i the sequence $d_i(\gamma^n v_i(\mathbf{e}), v_i(\mathbf{e}))$ must be unbounded. Since γ acts as an isometry, for any other vertex $v \in \mathcal{T}_i$ we have

$$\begin{aligned} d_i(\gamma^n v_i(\mathbf{e}), v_i(\mathbf{e})) &\leq d_i(\gamma^n v, v) + d_i(\gamma^n v, \gamma^n v_i(\mathbf{e})) + d_i(v, v_i(\mathbf{e})) \\ &\leq d_i(\gamma^n v, v) + 2d_i(v, v_i(\mathbf{e})). \end{aligned}$$

We infer that γ does not stabilise any finite set of vertices of \mathcal{T}_i . In particular, γ acts without inversion. Now [26, Proposition 25] implies the assertion. \square

REMARK 1. If $m(s, t) < \infty$ for all $s, t \in S$, then the Coxeter group itself has property FA of Serre. (Concerning property FA, see [26, Ex. 3, p. 66].)

Denote by I_1, I_2 and I_3 the sets of indices $i \in \{1, \dots, \Lambda\}$ such that the corresponding trees have, respectively, only one edge, only vertices of valencies at most two, and at least one vertex of valency at least three. Lemma 4 shows that the existence of one vertex of valency two implies that the tree contains an infinite axis of a translation of amplitude two.

Clearly, if G is finite then $I_2 = I_3 = \emptyset$. On the other hand, denote by $H_i \triangleleft \Gamma$ the intersection of the kernels of the homomorphisms $\pi_j : \gamma \rightarrow \text{Aut}(\mathcal{T}_j), j \notin I_i$. Then, whenever G (or, equivalently, Γ) is infinite we have that $H_1 = \{\mathbf{e}\}$ is trivial, H_2 is a solvable, normal subgroup of Γ , and H_3 , if not trivial, contains non-abelian free subgroups.

That H_2 is solvable follows since the group of automorphisms of a tree of degree two is just $\mathbb{Z}_2 \times \mathbb{Z}$ and the set $\pi_j, j \in I_2$, separates the points of H_2 . Indeed, H_2 embeds as a subgroup in a direct sum of solvable groups.

PROPOSITION 1. *Let \mathcal{T} be a tree with at least one vertex of valency at least three. Assume that every pair of adjacent edges $e_1 = \{y, x\}, e_2 = \{x, z\}$ defines a translation $u = u(e_1, e_2)$ on \mathcal{T} with $uy = z$. If h_1, \dots, h_l are non-trivial translations of \mathcal{T} , then there is a pair of adjacent edges, defining a translation v , such that for each $j \in \{1, \dots, l\}$ the group generated by h_j and v in $\text{Aut}(\mathcal{T})$ is isomorphic to the free product $\mathbb{Z} * \mathbb{Z}$.*

Proof. We shall use Klein’s table tennis criterion, of which a suitable formulation for our needs can be found in [3, Lemma 4.1].

Since a group is acting on the tree there can at most be two valencies of vertices. More precisely, each vertex is either of valency at least three or has a neighbour of this kind. It is known that the boundary of the tree is infinite.

By assumption, each h_j has an attracting boundary point b_j^+ and a repulsing boundary point b_j^- , which are connected by the axis a_j of the corresponding translation. We take a boundary point not contained in $\{b_1^+, \dots, b_l^+\} \cup \{b_1^-, \dots, b_l^-\}$ and choose on a straight path towards this point a vertex x of valency at least three not belonging to one of the axes a_1, \dots, a_l . The translations $\{h_1, \dots, h_l\}$ do not fix this vertex and our proof is completed by the following well-known argument:

Let e be an edge adjacent to x , but not belonging to one of the geodesics from x to $h_i x$ or from $h_i^{-1} x$ to x , $i = 1, \dots, l$. We split the tree into two disjoint trees cutting this edge. Let V denote the part not containing the above geodesics and U the part containing them. By assumption there exists a translation u moving x into V . The vertex ux is adjacent to two different edges which lie inside V , since as an image of x it has valency greater than two. Again by our assumption there exists a translation v whose axis lies entirely in V and contains ux .

Now, given $h \in \{h_1, \dots, h_l\}$ and $j \in \mathbb{Z} \setminus \{0\}$ it is clear that $h^j V \subset U$, while, on the other hand, $v^j U \subset V$. Since v is of infinite order, Klein's criterion implies that the group $\langle h, v \rangle$ generated by v and h in the group of automorphisms of \mathcal{T} is the free product $\langle h \rangle * \mathbb{Z} = \mathbb{Z} * \mathbb{Z}$. \square

6. Factoriality

We consider again the geometric representation $\sigma : G \rightarrow Gl(E)$. Associated to G and S there is the bilinear form $B : E \times E \rightarrow \mathbb{R}$ whose matrix, with respect to the standard unit vectors of $E = \bigoplus_{s \in S} \mathbb{R}e_s$, has entries

$$B(e_s, e_t) = \begin{cases} 1 & \text{if } s = t, \\ -\cos(\pi/m(s, t)) & \text{if } m(s, t) < \infty, \\ -1 & \text{if } m(s, t) = \infty. \end{cases}$$

We call (G, S) decomposable if there exist non-empty subsets $S_1, S_2 \subset S$, such that $s, t \in S$ commute whenever $s \in S_1$ and $t \in S_2$, or equivalently $B(e_s, e_t) = 0$, and indecomposable otherwise.

It is well known that the (left) regular representation of a discrete group is factorial if and only if the group is icc, that is, the conjugation class of any group element different from the identity is infinite. Our proof of this for a certain class of Coxeter groups relies very much on an irreducibility lemma of de la Harpe for finite index subgroups of Coxeter groups ([15, Lemma 1]). It is not immediately clear that the complexified representations remain irreducible, and we shall provide a proof of this fact.

PROPOSITION 2. *Let (G, S) be an indecomposable Coxeter system, with G infinite. If the associated bilinear form B is indefinite and non-degenerate, then G is an icc-group.*

Proof. For $w \in G$ denote by $C(w)$ its centraliser. The conjugation class of w is finite if and only if the index of $C(w)$ in G is finite. By [15, Lemma 1] the image $\sigma(C(w))$ acts irreducibly on E in this case. By Schur's lemma the commutant $\sigma(C(w))'$ is a division algebra over \mathbb{R} . As it is finite dimensional, it is isomorphic to \mathbb{R}, \mathbb{C} , or \mathbb{H} . We claim that for any $u \in G$ with $\sigma(u) \in \sigma(C(w))'$ the operator $\sigma(u)$ is a real multiple of the identity.

The claim implies the proposition because it implies that $\sigma(w)$, which is obviously an element of $\sigma(C(w))'$, commutes with all elements from $\sigma(G)$.

Since σ is a faithful representation of G we conclude that w is in the centre of the Coxeter group. (This is trivial; see [18, Section 6.3].)

To establish the claim it suffices to show that any such $\sigma(u)$ has real spectrum. If $\xi + i\eta \in \text{Sp}(u)$, with $\xi, \eta \in \mathbb{R}$, then by [7, Chap. 1, Theorem 8] $(\xi - \sigma(u))^2 + \eta^2$ is singular and hence equals 0.

If $\xi = 0$, then $\sigma(u^2) = \sigma(u)^2 = -\eta^2$. Since $\det \sigma(u^2) \in \{+1, -1\}$, we would have $\eta^2 = 1$ and $\sigma(u^2) = -1$, which is impossible in an infinite Coxeter group.

Now from $(\xi - \sigma(u))^2 + \eta^2 = 0$ we see that $2\xi = \sigma(u^{-1})(\sigma(u^2) + \xi^2 + \eta^2)$. Here the adjoint to the right hand side leaves the Tits cone U invariant. Hence ξ must be strictly positive.

We conclude that any u with $\sigma(u) \in \sigma(C(w))'$ has its spectrum in the right half plane $\{z : \Re z > 0\}$. If some $z \in \text{Sp}(u)$ has non-vanishing imaginary part, then, on the one hand, for some $k \in \mathbb{N}$, z^k has negative real part, but on the other hand, by the spectral mapping theorem, z^k is an element of the spectrum of $\sigma(u)^k = \sigma(u^k) \in \sigma(C(w))'$. □

COROLLARY 2. *A Coxeter group as in the above proposition is not a finite extension of an abelian group.*

Let $\sigma_{\mathbb{C}} : G \rightarrow \text{Gl}(E \otimes_{\mathbb{R}} \mathbb{C})$ be the complexification of the geometric representation, i.e., $\sigma_{\mathbb{C}}(g) = \sigma(g) \otimes_{\mathbb{R}} \text{Id}_{\mathbb{C}}$ for all $g \in G$, and extend B canonically to a bilinear form $B_{\mathbb{C}}$, which clearly remains non-degenerate, if B is non-degenerate.

LEMMA 8. *Suppose that G is infinite and B non-degenerate. Every subgroup of finite index in G acts, by $\sigma_{\mathbb{C}}$, irreducibly on $E \otimes_{\mathbb{R}} \mathbb{C}$.*

Proof. We follow the first part of the arguments of de la Harpe. We may assume that Γ is normal in G . Assuming $L_1 \neq E \otimes_{\mathbb{R}} \mathbb{C}$ to be a non-trivial $\sigma_{\mathbb{C}}(\Gamma)$ -invariant subspace, we find a generator $s \in S$ such that $L_1 \cap \sigma_{\mathbb{C}}(s)L_1 = \{0\}$.

The complex codimension one subspace $H_s = \bigoplus_{s' \in S \setminus \{s\}} \mathbb{C}e_{s'}$ is stabilised by $\sigma_{\mathbb{C}}(s)$ and has non-trivial intersection with $L_1 \oplus \sigma_{\mathbb{C}}(s)L_1$, because the dimension of the latter subspace is at least two. On the other hand, L_1 intersects trivially with H_s since $\sigma_{\mathbb{C}}(s)$ does not fix any of its non-zero elements. We conclude that L_1 complements H_s and is one-dimensional. In particular, we have $e_s = v + h$ for some $v \in L_1$ and some $h \in H_s$. Now

$$-e_s = \sigma_{\mathbb{C}}(s)e_s = \sigma_{\mathbb{C}}(s)v + h.$$

Subtracting, we see that

$$e_s = \frac{1}{2}(v - \sigma_{\mathbb{C}}(s)v) \in L_1 \oplus \sigma_{\mathbb{C}}(s)L_1.$$

The G -orbit $\mathcal{L} = \{L_1, \dots, L_N\}$ of L_1 is finite since Γ is of finite index in G , and all of those complex lines are Γ -invariant, by the normality of Γ . As

$\dim L_j = 1$ there exist homomorphisms $\lambda_j : \Gamma \rightarrow \mathbb{C}^*$ by which Γ acts. Because G acts irreducibly (notice that the extension to complex scalars is included in the Corollaire of Chap. V, §4, Sec. 7 of [9]) on $E \otimes_{\mathbb{R}} \mathbb{C}$, the G -invariant sum $\bigoplus_{j=1}^N L_j$ equals the whole space. Since $\sigma_{\mathbb{C}}$ is a faithful representation we conclude that Γ is abelian, in contradiction to the above corollary. \square

REMARK 2. As in the proof of Proposition 2 one sees that under the conditions of the above lemma a finite index subgroup of G has a trivial centraliser.

PROPOSITION 3. *If G is infinite and B non-degenerate, then every torsion free normal subgroup Γ of finite index in G contains no non-trivial solvable normal subgroup.*

Proof. Let $H \triangleleft \Gamma$ be a solvable normal subgroup of Γ as in the statement of the proposition. We denote by \overline{H}^Z and $\overline{\Gamma}^Z$ the Zariski closures of $\sigma_{\mathbb{C}}(H)$, respectively $\sigma_{\mathbb{C}}(\Gamma)$, in $\text{Gl}(E \otimes_{\mathbb{R}} \mathbb{C})$. Clearly, \overline{H}^Z is a normal divisor of $\overline{\Gamma}^Z$. Moreover, the connected component \overline{H}^0 (in the Zariski topology) of the identity in \overline{H}^Z is, on the one hand, still normal in $\overline{\Gamma}^Z$ and, on the other hand, of finite index in \overline{H}^Z . We claim that it reduces to the identity. This claim proves the proposition, since it implies that \overline{H}^Z and hence H are finite groups. Because Γ is torsion-free this is possible only if $H = \{e\}$.

The solvable Zariski connected group \overline{H}^0 has a common eigenvector $v \in E \otimes_{\mathbb{R}} \mathbb{C}$, as follows from the Lie-Kolchin Theorem; see, e.g., [8, Corollary 10.5]. Therefore, there exists a character (a continuous multiplicative function) $\alpha_v : \overline{H}^0 \rightarrow \mathbb{C}^*$ from \overline{H}^0 to the multiplicative group of \mathbb{C} such that $h v = \alpha_v(h) v$. Since \overline{H}^0 is normal in $\overline{\Gamma}^Z$, any vector γv , with $\gamma \in \overline{\Gamma}^Z$, is also a common eigenvector, and the corresponding character is $\alpha_{\gamma v}(\cdot) = \alpha_v(\gamma^{-1} \cdot \gamma)$.

Let

$$\mathcal{V} = \{u \in E \otimes_{\mathbb{R}} \mathbb{C} : h u = \alpha_u(h)u \text{ for some } \alpha_u \text{ as above } \}.$$

This set is $\overline{\Gamma}^Z$ invariant and spans an $\overline{\Gamma}^Z$ -invariant subspace. We have seen that it is non-trivial, and by the irreducibility of $\sigma_{\mathbb{C}}(\Gamma)$ it must equal $E \otimes_{\mathbb{R}} \mathbb{C}$.

Now the trace of $B_{\mathbb{C}}$ is positive and \mathcal{V} contains a basis. Hence for some $u \in \mathcal{V}$ we have $B_{\mathbb{C}}(u, u) \neq 0$. From

$$\begin{aligned} \alpha_u(h)B_{\mathbb{C}}(u, u) &= B_{\mathbb{C}}(hu, u) = B_{\mathbb{C}}(u, h^{-1}u) \\ &= \alpha_w(h^{-1})B_{\mathbb{C}}(u, u) \quad \forall h \in \overline{H}^0 \end{aligned}$$

we infer that $\alpha_u \in \{+1, -1\}$.

As above we conclude that the set

$$\mathcal{V}_1 = \{u \in E \otimes_{\mathbb{R}} \mathbb{C} : h u = \alpha_u(h)u, \alpha_u : \overline{H}^0 \rightarrow \{+1, -1\} \text{ is a character } \}$$

contains a basis. With respect to one such basis the elements of \overline{H}^0 consist of diagonal matrices with entries from $\{+1, -1\}$. Since \overline{H}^0 is Zariski connected, it must be trivial. \square

7. Simplicity of the regular C^* -algebra

THEOREM 2. *If (G, S) is an indecomposable Coxeter system, with G infinite, such that the associated bilinear form B is indefinite and non-degenerate, then its (left) regular C^* -algebra is simple with unique trace.*

Before proving the theorem we establish a lemma.

LEMMA 9. *If (G, S) is as in the theorem, then all trees $\mathcal{T}_1, \dots, \mathcal{T}_\Lambda$ have vertices of valency at least three.*

Proof. Assume that \mathcal{T}_i has only vertices of valency two. From Lemma 6 we see that for each j in the orbit $\mathcal{O} = \{\pi(\dot{g})i : \dot{g} \in G/\Gamma\}$ the tree \mathcal{T}_j is isomorphic to \mathcal{T}_i , and hence has only vertices of valency two.

The group W generated by the reflections $\cup_{j \in \mathcal{O}} \mathcal{T}_j$ contains $\Gamma' := W \cap \Gamma$ as a normal torsion-free subgroup of finite index, which is also normal in G . The set of homomorphisms $\pi_j|_{\Gamma'} : \Gamma' \rightarrow \text{Aut}(\mathcal{T}_j)$, $j \in \mathcal{O}$, is faithful, since the assumption that $\pi_j(\gamma) = \text{Id}$ for all $j \in \mathcal{O}$ implies that γ acts, by conjugation, trivially on all reflections in $\cup_{j \in \mathcal{O}} \mathcal{T}_j$, i.e., on all reflections in W . From this and the fact that $\gamma \in W$ it follows that γ is in the centre of W . But the centre is trivial, since W is infinite. We infer that Γ' is solvable. By Proposition 3 it is trivial and W finite. This is a contradiction. \square

Proof of the theorem: Let Γ be a torsion free normal subgroup of finite index in G . Since G is an icc-group, the results of Bekka and de la Harpe [4] show that it suffices to prove the assertions for Γ . We shall use the concept of weak Powers groups in the sense of Boca and Nitica [6].

Let $F \subset C_h$ be a finite subset of the Γ -conjugation-class of some $h \in \Gamma$. Lemma 7 shows that there is a tree \mathcal{T} on which h , and hence all elements of its conjugation-class, act as translations. This tree has a vertex of valency at least three. We can now apply Proposition 1 to obtain $v \in \Gamma$ such that for any $k \in F$ the subgroup $\langle k, v \rangle$ generated by k and v is isomorphic to $\mathbb{Z} * \mathbb{Z}$.

The proof of [3, Lemma 2.2] shows that there exists a constant $C > 0$ and $v \in \Gamma$ such that for all $k \in F$

$$(3) \quad \left\| \sum_{j=1}^{\infty} a_j \lambda_{\Gamma}(v^{-j} k v^j) \right\| \leq C \|a\|_2 \quad \forall a \in l^2(\mathbb{Z}^+).$$

Armed with this, the computations in the proof of [6, Lemma 2.2] prove the following fact which we state as a lemma.

LEMMA 10. *Given a finite linear combination $x = \sum_{k \in F} a_k k \in \mathbb{C}\Gamma$ with $e \notin F$ and $\epsilon > 0$ there exist $n \in \mathbb{N}$ and $v_1, \dots, v_n \in \Gamma$ such that in $C_\lambda^*(\Gamma)$,*

$$(4) \quad \left\| \frac{1}{n} \sum_{j=1}^n \lambda_\Gamma(v_j) \lambda_\Gamma(x) \lambda_\Gamma(v_j)^* \right\| \leq \epsilon.$$

The arguments in the proof of [3, Lemma 2.1] show that $C_\lambda^*(\Gamma)$ is simple. Finally, the uniqueness of the trace is an immediate consequence of inequality (4). \square

REMARK 3. It is not hard to see that Γ is a weak Powers group. By Remark 2, the centraliser of Γ in G is trivial. Hence the Coxeter group itself is an ultra-weak Powers group in the sense of Bédos [2].

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