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COMPLETE SPACELIKE HYPERSURFACES WITH CONSTANT MEAN CURVATURE IN THE DE SITTER SPACE: A GAP THEOREM

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ABSTRACT. Let M^n be a complete spacelike hypersurface with constant mean curvature H in the de Sitter space S_1^{n+1}. We use the operator $\phi = A - HI$, where A is the second fundamental form of M, and the roots $B_H^- \leq B_H^+$ of a certain second order polynomial, to prove that either $|\phi|^2 \equiv 0$ and M is totally umbilical, or $B_H^- \leq \sqrt{\sup|\phi|^2} \leq B_H^+$. For the case $H \geq 2\sqrt{n-1}/n$ we prove the following results: for every number B in the interval $[\max\{0, B_H^-\}, B_H^+]$ there is an example of a complete spacelike hypersurface such that $\sqrt{\sup|\phi|^2} = B$; if $\sqrt{\sup|\phi|^2} = B_H^-$ is attained at some point, then the corresponding M is a hyperbolic cylinder. We characterize the hyperbolic cylinders as the only complete spacelike hypersurfaces in S_1^{n+1} with constant mean curvature, non-negative Ricci curvature and having at least two ends. We also characterize all complete spacelike hypersurfaces of constant mean curvature with two distinct principal curvatures as rotation hypersurfaces or generalized hyperbolic cylinders.

1. Introduction and statement of results

Let $\mathbf{R}_1^{n+2}$ be the $(n+2)$-dimensional Euclidean space with the Lorentzian metric $\langle\ ,\ \rangle$ given by

$$\langle p, q\rangle = -p_0q_0 + p_1q_1 + \cdots + p_{n+1}q_{n+1}. \tag{1}$$

We define the de Sitter space by

$$S_1^{n+1} = \{p \in R_1^{n+2} \mid \langle p, p\rangle = 1\}.$$

Then S_1^{n+1} is a Lorentz manifold with constant sectional curvature 1. A hypersurface M^n immersed in S_1^{n+1} is said to be spacelike if the metric induced in M^n by the immersion in S_1^{n+1} is Riemannian. There are many interesting results in the study of spacelike hypersurfaces with constant mean curvature

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H. To begin with, complete hypersurfaces in the de Sitter space have been characterized by Q. M. Cheng [9] under the hypothesis that the mean curvature and the scalar curvature are linearly related. L. J. Alías, A. Romero and M. Sánchez [4] studied complete constant mean curvature spacelike hypersurfaces in connection with Bernstein-type problems. J. L. Barbosa and V. Oliker [6] proved the stability of a complete constant mean curvature spacelike hypersurface provided it is compact or satisfies $H^2 \geq 1$ or $H^2 < 4(n-1)/n^2$.

The study of this kind of hypersurface was inspired, in particular, by a conjecture posed by A. J. Goddard [12], stating that every complete spacelike hypersurface with constant mean curvature in S_1^{n+1} must be totally umbilical. The first result in this direction was obtained by J. Ramanathan [25] in 1987. He showed that if the constant mean curvature H of a complete spacelike hypersurface in S_1^3 satisfies $H^2 < 1$, then the surface is totally umbilical. Independently, and still in 1987, K. Akutagawa [2] proved Goddard's conjecture for the case $H^2 < 1$ if $n = 2$ and for the case $H^2 < 4(n-1)/n^2$ if $n > 2$. On the other hand, S. Montiel [15] proved the conjecture for the compact case. It turned out that the general conjecture was false, as shown by the existence of the so-called hyperbolic cylinders, which are defined and described at the end of Section 2.

We will restrict ourselves to complete spacelike hypersurfaces having non-zero constant mean curvature because the maximal hypersurfaces in the de Sitter space are totally geodesic (see [18]).

Given a spacelike hypersurface M^n with constant mean curvature H, for each $p \in M^n$ we define $\phi : T_pM \to T_pM$ by

$$\langle \phi X, Y\rangle = \langle AX, Y\rangle - H\langle X, Y\rangle,$$

where A is the operator associated to the second fundamental form of M. We observe that for non-zero constant mean curvature hypersurfaces the norm $|\phi|$ of the operator ϕ plays a role analogous to that of $|A|$ in the case of minimal hypersurfaces, as shown by Chern, do Carmo and Kobayashi [10].

The operator ϕ proved to be useful in the study of hypersurfaces with constant mean curvature in the Riemannian ambient; for example, H. Alencar and M. do Carmo [3] proved a gap theorem for compact hypersurfaces with constant mean curvature in spheres, characterizing the $H(r)$-torus by a pinching condition on $|\phi|^2$, as follows:

Theorem 1.1 (Alencar, do Carmo [3]). *Let M^n be a compact orientable hypersurface immersed in S^{n+1} with constant mean curvature $H > 0$. Suppose $|\phi|^2 \leq B_H$ for each $p \in M$, where B_H is the square of the positive root of the polynomial*

$$P_H(x) = x^2 + \frac{n(n-2)H}{\sqrt{n(n-1)}}x - n(1-H^2).$$

Then,

(i) $|\phi|^2 \equiv 0$ *and* M *is totally umbilical; or*

(ii) M *is the* $H(r)$*-torus* $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$*, where* $r^2 < (n-1)/n$.

(See also [26] for a generalization to higher codimension.)

Regarding the Lorentzian setting, U. H. Ki, H. J. Kim, and H. Nakagawa [13] used a Lorentzian version of the polynomial P_H to find a sharp upper bound for $|\phi|$. They showed that a complete spacelike hypersurface in S_1^{n+1} with constant mean curvature $H \geq 2\sqrt{n-1}/n$ such that $|\phi|$ is constant and equal to that upper bound must be a hyperbolic cylinder.

In this paper we extend some of the results given above by means of a detailed study of the polynomial P_H. In our first theorem, we use P_H to give a lower bound for $|\phi|$. For completeness, we include also the upper bounds proved in [2], [13] and [25].

THEOREM 1.2. *Let* M^n *be a complete spacelike hypersurface immersed in* S_1^{n+1}*,* $n \geq 3$*, with constant mean curvature* $H > 0$*. Then* $\sup|\phi|^2 < \infty$ *and*

(1) *either* $|\phi| \equiv 0$ *and* M *is totally umbilical; or*

(2) $B_H^- \leq \sqrt{\sup|\phi|^2} \leq B_H^+$*, where* $B_H^- \leq B_H^+$ *are the roots of the polynomial*

$$P_H(x) = x^2 - \frac{n(n-2)H}{\sqrt{n(n-1)}}x + n(1-H^2). \tag{2}$$

As we will show later in this paper, the polynomial P_H has real roots if and only if $H \geq 2\sqrt{n-1}/n$. As a consequence of Theorem 1.2, there are no complete spacelike hypersurfaces with constant mean curvature $H \geq 2\sqrt{n-1}/n$ such that $0 < \sqrt{\sup|\phi|^2} < B_H^-$, yielding a gap between the umbilical hypersurfaces ($|\phi| \equiv 0$) and the hypersurfaces with $\sqrt{\sup|\phi|^2} = B_H^-$.

We must mention the recent work [27] by Y. J. Suh, Y. S. Choi and H. Y. Yang, who proved independently from us the inequality $B_H^- \leq \sqrt{\sup|\phi|^2} \leq B_H^+$. (They studied a more general situation than ours and the notations differ significantly, but see equation (6.8) in the cited reference.) In that work, the authors used this inequality only to characterize the totally umbilical hypersurfaces. Apparently, the authors were not aware of the existence of results characterizing the hyperbolic cylinders which satisfy $\sqrt{\sup|\phi|^2} \equiv B_H^{\pm}$.

For the case $H \geq 2\sqrt{n-1}/n$ we also show that there is no gap between the roots of P_H, that is, we prove that for every number B in the interval $[\max\{0, B_H^-\}, B_H^+]$ there is a complete spacelike hypersurface with constant mean curvature H such that $\sqrt{\sup|\phi|^2} = B$. These examples constitute a class of new rotation hypersurfaces with constant mean curvature in S_1^{n+1}. In fact, we prove the following theorem.

Theorem 1.3. *Given an integer $n \geq 3$ and a number H such that $H \geq 2\sqrt{n-1}/n$, let $B_H^- \leq B_H^+$ be the roots of the polynomial P_H in equation* (2). *Then:*

(1) *For any value B in the interval $[\max\{0, B_H^-\}, B_H^+]$ there is a complete spacelike hypersurface in S_1^{n+1} with constant mean curvature H and $\sqrt{\sup|\phi|^2} = B$.*
(2) *If, in addition, $H \neq 2\sqrt{n-1}/n$, there exists a complete spacelike hypersurface in S_1^{n+1} with constant mean curvature H and $\sqrt{\sup|\phi|^2} = B_H^+$ which is not a hyperbolic cylinder.*

We also give some characterizations of the hyperbolic cylinders mentioned above. We generalize a result proved by Montiel [16] as follows.

Proposition 1.1. *Let M^n be a complete spacelike hypersurface immersed in S_1^{n+1}, $n \geq 3$, with constant mean curvature H such that $2\sqrt{n-1}/n \leq H < 1$, and*

$$\sqrt{\sup|\phi|^2} = B_H^-.$$

This supremum is attained if and only if M is isometric to the hyperbolic cylinder

$$H^1(\sinh r) \times S^{n-1}(\cosh r).$$

In the same paper, Montiel [16] characterized the hyperbolic cylinders as the only complete (non-compact) hypersurfaces in S_1^{n+1} with constant mean curvature $H = 2\sqrt{n-1}/n$ and having at least two ends. In the middle of the proof of this result he showed that the Ricci curvature of M is non-negative. In the following result we prove that among constant mean curvature hypersurfaces, these two properties, the non-negativity of the Ricci curvature and the existence of at least two ends, characterize all hyperbolic cylinders.

Theorem 1.4. *Let M be a complete spacelike hypersurface immersed in S_1^{n+1}, $n \geq 3$, with constant mean curvature. Then* $\mathrm{Ric} \geq 0$ *and M has at least two ends if and only if M is a hyperbolic cylinder.*

As our examples have two distinct principal curvatures at each point, to close this paper we prove some results on complete hypersurfaces with this property. In fact, we prove the following characterization of such hypersurfaces.

Proposition 1.2. *Let M be a complete spacelike hypersurface in S_1^{n+1} with constant mean curvature and two distinct principal curvatures with constant multiplicities k and $n-k$. Then the following statements are true:*

(1) *If $k = 1$ or $k = n-1$, then M is a rotation hypersurface.*

(2) *If* $1 < k < (n-1)$, *then* M *is isometric to*

$$H^k(\sinh r) \times S^{n-k}(\cosh r).$$

Our results may be interpreted as follows: We associate to each complete spacelike hypersurface M^n with constant mean curvature H the coordinate pair $(H, \sqrt{\sup|\phi|^2})$ in the first quadrant of a 2-plane, thus obtaining Figure 1. Results by Akutagawa [2], Ramanathan [25], Nishikawa [18], Ki, Kim and Nakagawa [13] and our Theorem 1.2 imply that there is no complete spacelike hypersurface such that the corresponding point $(H, \sqrt{\sup|\phi|^2})$ lies to the left

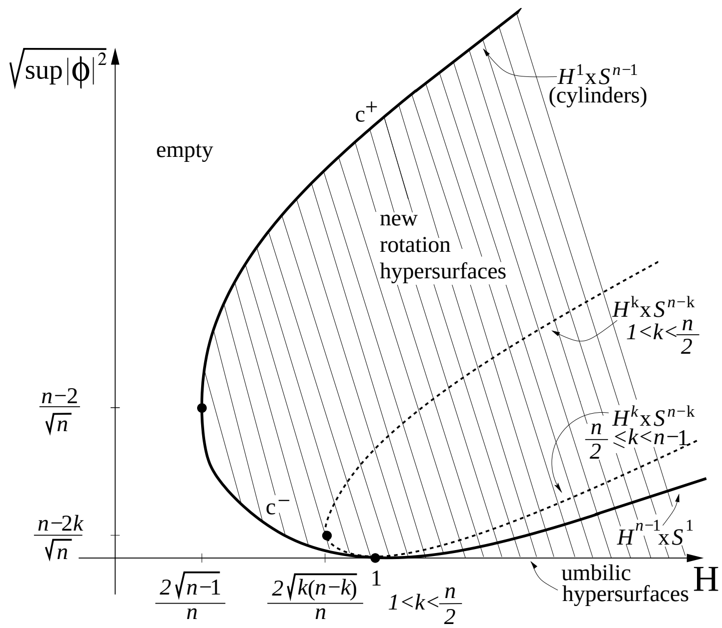

FIGURE 1. The plane $\left(H, \sqrt{\sup|\phi|^2}\right)$. The positive H-axis represents the totally umbilical hypersurfaces. Hyperbolic cylinders have their data represented on the solid curve $C = C^- \cup C^+$. Results in [2], [13], [18], [25] and our Theorem 1.2 together imply that the points of the region to the left of C (marked "empty") are not associated to any complete spacelike hypersurface with constant mean curvature. For the case $H \geq 2\sqrt{n-1}/n$, Theorem 1.3 gives a complete spacelike hypersurface corresponding to an arbitrary point to the right of C, and also a hypersurface corresponding to a point on C^+ which is not a hyperbolic cylinder. We also depict the curves corresponding to the hypersurfaces isometric to $H^k \times S^{n-k}$ characterized in Proposition 1.2.

of the curve C shown, thus yielding a gap. Theorem 1.3 gives, for each point $(H, \sqrt{\sup|\phi|^2})$ to the right of the curve C, a complete spacelike hypersurface corresponding to such a point. Theorem 1.4 characterizes the complete spacelike hypersurfaces with H constant, non negative Ricci curvature and having at least two ends as hyperbolic cylinders, so that the corresponding point $(H, \sqrt{\sup|\phi|^2})$ is on the curve C.

2. Preliminaries

Let M^n be an n-dimensional, complete manifold, immersed into the $(n+1)$-dimensional de Sitter space S_1^{n+1}. We say that M^n is a spacelike hypersurface if the semi-Riemannian metric of S_1^{n+1} induces a Riemannian metric on M^n.

If N is a timelike unit vector field everywhere normal to M, then we denote by A the operator associated to the second fundamental form corresponding to the choice of N; by k_i, $i = 1, \ldots, n$, the eigenvalues of A (or equivalently, the principal curvatures of M); and by $H = (1/n)\mathrm{tr}A$ the mean curvature of M. If R is the curvature tensor of M, we have the classical Gauss equation

$$R(u, v)w = \langle v, w\rangle u - \langle u, w\rangle v - \langle Av, w\rangle Au + \langle Au, w\rangle Av,$$

so that the Ricci curvature tensor Ric on M is given by

$$\mathrm{Ric}(u, v) = (n-1)\langle u, v\rangle - nH\langle Au, v\rangle + \langle A^2u, v\rangle. \tag{3}$$

We will use the following Simons type formula (for a proof, see [16]):

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + n\mathrm{tr}A^2 - n^2H^2 - nH\mathrm{tr}A^3 + (\mathrm{tr}A^2)^2. \tag{4}$$

Introducing the operator ϕ given by

$$\langle \phi X, Y\rangle = \langle AX, Y\rangle - H\langle X, Y\rangle, \tag{5}$$

it is easy to see that ϕ is traceless, that the eigenvalues of ϕ are of the form $\mu_i = \kappa_i - H$, and that

$$|\phi|^2 = \sum_i \mu_i^2 = \frac{1}{2n}\sum(\kappa_i - \kappa_j)^2.$$

Note that $|\phi|^2 \equiv 0$ if and only if M^n is totally umbilical.

Substituting ϕ in (4), we obtain the Lorentzian version of Simons' formula (also deduced in [16]):

$$\frac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 + (|\phi|^2)^2 - nH\mathrm{tr}\ \phi^3 + n(1-H^2)|\phi|^2. \tag{6}$$

We recall the standard examples of spacelike hypersurfaces with constant mean curvature in S_1^{n+1} (see [15] for details). First we have the totally umbilical hypersurfaces given by

$$M^n = \{p \in S_1^{n+1} \mid \langle p, a\rangle = \tau\},$$

where $a \in \mathbf{R}_1^{n+2}$, $|a|^2 = \rho = 1, 0, -1$ and $\tau^2 > \rho$. Their corresponding mean curvatures H satisfy

$$H^2 = \frac{\tau^2}{\tau^2 - \rho}.$$

M^n is isometric to a hyperbolic space, a Euclidean space or a sphere, if $\rho = 1, 0$ or -1, respectively. Also, we can prove that $|\phi|^2 \equiv 0$. Umbilicity implies that every direction is principal, with eigenvalue H, so that equation (3) gives the following expression for the Ricci curvature of M (we may take $|u| = 1$):

$$\begin{aligned}\operatorname{Ric}(u) &= (n-1)\langle u, u\rangle - nH\langle Au, u\rangle + \langle A^2u, u\rangle \\ &= (n-1)(1 - H^2).\end{aligned}$$

The other well-known complete spacelike hypersurfaces with constant mean curvature are given by

$$M^n = \{p \in S_1^{n+1} \mid p_{k+1}^2 + \cdots + p_{n+1}^2 = \cosh^2 r\},$$

with $r \in \mathbf{R}$ and $1 \le k \le n$. One can prove that M is isometric to the Riemannian product $H^k(\sinh r) \times S^{n-k}(\cosh r)$ of a k-dimensional hyperbolic space and an $(n-k)$-dimensional sphere of radii $\sinh r$ and $\cosh r$, respectively. M has k principal curvatures equal to $\coth r$ and $(n-k)$ principal curvatures equal to $\tanh r$, so the mean curvature is given by

$$nH = k \coth r + (n-k)\tanh r. \tag{7}$$

Multiplying by $\coth r$ and solving the resulting equation for $\coth r$, we have

$$\coth r = \frac{nH \pm \sqrt{n^2H^2 - 4k(n-k)}}{2k}. \tag{8}$$

In this case,

$$|\phi|^2 = \frac{k(n-k)}{n}(\coth r - \tanh r)^2.$$

Using (7) and (8), we may express $|\phi|^2$ as a function of H, namely,

$$|\phi|^2 = \frac{n}{4k(n-k)}\left((n-2k)H \pm \sqrt{n^2H^2 - 4k(n-k)}\right)^2. \tag{9}$$

To calculate the Ricci curvature of M, note that if u is an eigenvector of A with principal curvature $\coth r$, then (3) and (7) imply that

$$\begin{aligned}\operatorname{Ric}(u) &= (n-1)\langle u, u\rangle - nH\langle Au, u\rangle + \langle A^2u, u\rangle \\ &= (k-1)(1 - \coth^2 r).\end{aligned}$$

Note that $\operatorname{Ric}(u) = 0$ if and only if $k = 1$, and that otherwise $\operatorname{Ric}(u) < 0$.

On the other hand, if v is an eigenvector with principal curvature $\tanh r$, we use (7) again to get

$$\begin{aligned}\mathrm{Ric}(v) &= (n-1)\langle v,v\rangle - nH\langle Av,v\rangle + \langle A^2v,v\rangle \\ &= (n-k-1)(1-\tanh^2 r).\end{aligned}$$

Note that $\mathrm{Ric}(v)\geq 0$ if $n-k\geq 1$.

The case $k=1$ is the most important one for our purposes; in this case the hypersurface M is isometric to a Riemannian product $H^1(\sinh r)\times S^{n-1}(\cosh r)$ and is called a *hyperbolic cylinder.*

3. The gap theorem

In (2) we introduced the polynomial

$$P_H(x) = x^2 - \frac{n(n-2)H}{\sqrt{n(n-1)}}x + n(1-H^2).$$

By taking $x=|\phi|$, where ϕ is defined in (5), we will show that $P_H(|\phi|)=0$ if the hypersurface is the hyperbolic cylinder with constant mean curvature $H\geq 2\sqrt{n-1}/n$; this fact justifies the study of such a polynomial, which we carry out below.

First we recall the classic lemma due to M. Okumura [20], completed with the equality case proved in [3] by Alencar and do Carmo.

Lemma 3.1. *Let μ_i, $i=1,\dots,n$, be real numbers, with $\sum\mu_i=0$ and $\sum\mu_i^2=\beta^2\geq 0$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum\mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{10}$$

and equality holds if and only if either $(n-1)$ of the numbers μ_i are equal to $\beta/\sqrt{(n-1)/n}$ or $(n-1)$ of the numbers μ_i are equal to $-\beta/\sqrt{(n-1)/n}$.

The polynomial P_H arises analytically in the following lemma, whose proof uses Simons' formula (6).

Lemma 3.2. *Let M^n be a complete spacelike hypersurface immersed in S_1^{n+1}, $n\geq 3$, with constant mean curvature H. Then*

$$\frac{1}{2}\Delta|\phi|^2 \geq |\phi|^2 P_H(|\phi|), \tag{11}$$

where P_H is given in (2).

Proof. By Simons' formula (6),

$$\frac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 + (|\phi|^2)^2 - nH\mathrm{tr}\ \phi^3 + n(1-H^2)|\phi|^2.$$

Using the fact that $|\nabla\phi|^2 \geq 0$ and Lemma 3.1, we get

$$\begin{aligned}\frac{1}{2}\Delta|\phi|^2 &\geq (|\phi|^2)^2 - nH\mathrm{tr}\,\phi^3 + n(1-H^2)|\phi|^2\\ &\geq |\phi|^2\left(|\phi|^2 - \frac{n(n-2)H}{\sqrt{n(n-1)}}|\phi| + n(1-H^2)\right),\\ &= |\phi|^2 P_H(|\phi|),\end{aligned}$$

where equality holds in the first line if and only if $|\nabla\phi| = 0$. □

We state without proof some elementary properties of P_H.

Lemma 3.3. *If P_H is the polynomial defined in* (2)*, then:*

(1) *If $H^2 < 4(n-1)/n^2$, then $P_H(x) > 0$ for any $x \in \mathbf{R}$.*

(2) *If $H^2 = 4(n-1)/n^2$, we may write $H = 2\sqrt{n-1}/n$; the (double) root of P_H is*

$$B_H = \frac{n(n-2)}{2\sqrt{n(n-1)}}H = \frac{n-2}{\sqrt{n}},$$

so that $P_H(x) = (x-(n-2)/\sqrt{n})^2 \geq 0$ for all $x \in \mathbf{R}$.

(3) *If $H^2 > 4(n-1)/n^2$, then P_H has two real roots B_H^- and B_H^+ given by*

$$B_H^{\pm} = \sqrt{\frac{n}{4(n-1)}}\left((n-2)H \pm \sqrt{n^2H^2 - 4(n-1)}\right). \tag{12}$$

B_H^+ is always positive; on the other hand, $B_H^- > 0$ if and only if

$$(n-2)H - \sqrt{n^2H^2-4(n-1)}) > 0,$$

which holds if and only if $4(n-1)/n^2 \leq H^2 < 1$. Similarly, $B_H^- = 0$ if and only if $H^2 = 1$ and $B_H^- < 0$ if and only if $H^2 > 1$.

Now it is easy to conclude that $P_H(|\phi|) = 0$ for the constant value $|\phi| \equiv B_H^{\pm}$ (equation (9) for $k = 1$) corresponding to the hyperbolic cylinder with constant mean curvature $H \geq 2\sqrt{n-1}/n$.

For the proof of Theorem 1.2, we will also use the following principle due to H. Omori [21] and S. T. Yau [28].

Theorem 3.1. *Let M^n be an n-dimensional, complete Riemannian manifold whose Ricci curvature is bounded from below. Let f be a C^2 function, bounded from above on M. Then for each $\epsilon > 0$ there exists a point $p_\epsilon \in M$ such that*

$$\sup f - \epsilon < f(p_\epsilon) \leq \sup f, \quad |\nabla f(p_\epsilon)| < \epsilon, \quad \Delta f(p_\epsilon) < \epsilon.$$

Proof of Theorem 1.2. The Gauss equation implies

$$\mathrm{Ric} \geq (n-1) - \frac{n^2H^2}{4},$$

Thus, if $H < 2\sqrt{n-1}/n$ then $\mathrm{Ric} \geq \delta > 0$. By Bonnet-Myers' theorem M is compact. As H is constant, Montiel's theorem [15] implies that M is totally umbilical and so $|\phi|^2 \equiv 0$.

If $H \geq 2\sqrt{n-1}/n$, then we can use Theorem 1 in [13], which implies (stated in our terminology) that $|\phi|^2$ is bounded from above. As the Ricci curvature is still bounded from below, we may apply Theorem 3.1 to this function $|\phi|^2$, obtaining a sequence $\{p_k\}$ of points in M such that

$$\lim_{k\to\infty} |\phi|^2(p_k) = \sup |\phi|^2, \qquad |\nabla|\phi|^2(p_k)| < \frac{1}{k}, \qquad \Delta|\phi|^2(p_k) < \frac{1}{k}.$$

Substituting this in (11), we obtain

$$\frac{1}{2k} > \frac{1}{2}\Delta|\phi|^2(p_k) \geq |\phi|^2(p_k)P_H(|\phi|)(p_k),$$

Taking the limit as $k \to \infty$, it follows that

$$\sup|\phi|^2 \;\cdot\; P_H\left(\sqrt{\sup|\phi|^2}\right) \leq 0. \tag{13}$$

As $H \geq 2\sqrt{n-1}/n$, we know that P_H has two (not necessarily distinct) real roots. Suppose that the second conclusion in Theorem 1.2 does not hold, i.e., that

$$\sqrt{\sup|\phi|^2} < B_H^- \qquad \text{or} \qquad \sqrt{\sup|\phi|^2} > B_H^+.$$

Then $P_H(\sqrt{\sup|\phi|^2})$ is strictly positive, which implies that the left hand side of (13) is non-negative. This implies that $\sup|\phi|^2 = 0$, that is, $|\phi|^2 \equiv 0$, and thus that M^n is totally umbilical, proving the theorem. ☐

Remark 3.1. As noted before, the inequality $B_H^- \leq \sqrt{\sup|\phi|^2} \leq B_H^+$ was proved by Suh, Choi and Yang [27], independently from us.

4. Complete hypersurfaces with $B_H^- \leq \sqrt{\sup|\phi|^2} \leq B_H^+$

In this section we show that the bounds in Theorem 1.2 are sharp by exhibiting a family of complete hypersurfaces in S_1^{n+1} with constant mean curvature $H \geq 2\sqrt{n-1}/n$ and $\sqrt{\sup|\phi|^2} = B$, for any $B \in [\max\{0, B_H^-\}, B_H^+]$.

First we define a rotation hypersurface in S_1^{n+1}, following M. do Carmo and M. Dajczer [11], and H. Mori [17]. Recall that an *orthogonal transformation* on $\mathbf{R}_1^{n+2}$ is a linear map preserving the metric; these orthogonal transformations induce all isometries of S_1^{n+1}.

Let P^k be a k-dimensional vector subspace of $\mathbf{R}_1^{n+2}$. We say that P^k is Lorentzian (resp. Riemannian, degenerate) if the restriction of the metric to P^k is a Lorentzian (resp. Riemannian, degenerate) metric. We denote by

$O(P^k)$ the set of orthogonal transformations of $\mathbf{R}_1^{n+2}$ with positive determinant that leave P^k pointwise fixed.

Choose P^2, P^3 such that $P^2 \subset P^3$, and a regular, spacelike curve C in $S_1^{n+1} \cap (P^3 - P^2)$, parametrized by arc length. The orbit of C under $O(P^2)$ is called the spherical (resp. hyperbolic, parabolic) spacelike rotation hypersurface M in S_1^{n+1} generated by C, whenever P^2 is Lorentzian (resp. Riemannian, degenerate).

In fact, we only need the spherical case here. As the metric in $\mathbf{R}_1^{n+2}$ is given by (1), the canonical basis $e_0, \dots, e_n, e_{n+1}$ satisfies

$$\langle e_i, e_j \rangle = \epsilon_i \delta_{ij},$$

where $\epsilon_0 = -1$, and $\epsilon_i = 1$ otherwise. Take P^2 as the plane spanned by e_0 and e_1, P^3 as the plane spanned by e_0, e_1, e_2, and let $(y_0(s), y_1(s), y(s))$ be the arc length parametrization of the curve C. With these choices, a typical parametrization of a rotation hypersurface M is given by (see [17], for example)

$$(y_0(s), y_1(s), y(s)\Phi(u_1, \dots, u_{n-1})).$$

Here Φ is an orthogonal parametrization of the unit sphere in the vector subspace spanned by $e_2, \dots, e_{n+1}$ and $y_0(s)$ and $y_1(s)$ can be calculated in terms of $y(s)$ as

$$\begin{aligned} y_0 &= \sqrt{y^2 - 1}\cosh\varphi, \\ y_1 &= \sqrt{y^2 - 1}\sinh\varphi, \end{aligned}$$

as long as $y > 1$, where φ is given by

$$\varphi = \int_0^s \frac{\sqrt{y'^2 + y^2 - 1}}{y^2 - 1}\, ds.$$

The case $0 < y < 1$ can be treated similarly, but will not be needed here. We use the above parametrization to calculate the principal curvatures of M, namely,

$$\begin{aligned} \kappa_i &= \frac{\sqrt{y'^2 + y^2 - 1}}{y}, \quad i = 1, \dots, n-1, \\ \kappa_n &= \frac{y'' + y}{\sqrt{y'^2 + y^2 - 1}}, \end{aligned}$$

The mean curvature of M is given by

$$nH = \sum_{i=1,\dots,n} \kappa_i = (n-1)\frac{\sqrt{y'^2 + y^2 - 1}}{y} + \frac{y'' + y}{\sqrt{y'^2 + y^2 - 1}}. \tag{14}$$

This second order differential equation has a first integral (see [11], [14] and [23] for a detailed study), namely,

$$G(y, y') = y^{n-1} \left(\sqrt{y'^2 + y^2 - 1} - Hy \right). \tag{15}$$

By arguments similar to those given in the cited references it can be shown that the level curves of G are associated to spacelike rotation hypersurfaces. For our purposes, it suffices to analyze the level curves of G contained in the set

$$\{(y, y') \mid y > 0,\ y'^2 + y^2 - 1 \geq 0,\ G(y, y') \geq 0\}.$$

For example, observe that a level curve $G = 0$ ($y \neq 0$) is associated to an umbilical hypersurface, since

$$\kappa_i = \frac{\sqrt{y'^2 + y^2 - 1}}{y} = \frac{Hy}{y} = H,$$

where we have used that $\sqrt{y'^2 + y^2 - 1} - Hy = 0$. Thus,

$$\kappa_n = nH - (n-1)\kappa_i = H$$

and the hypersurface is umbilical. Recall that in this case we have $|\phi|^2 \equiv 0$ for the operator ϕ defined in (5).

Of particular importance are the critical points of G, given by the following lemma.

Lemma 4.1. *Let $H \geq 0$ and $G(y, y')$ the function defined in* (15). *Then:*

(1) *If $0 \leq H < 2\sqrt{n-1}/n$, G has no critical points.*
(2) *If $H = 2\sqrt{n-1}/n$, G has only one critical point of degenerate type.*
(3) *If $2\sqrt{n-1}/n < H < 1$, G has two distinct critical points.*
(4) *If $H \geq 1$, G has only one critical point.*

Proof. The critical points of G are located along the y'-axis and also at the y-axis if they satisfy the equation

$$y^2 - nHy\sqrt{y^2 - 1} + (n-1)(y^2 - 1) = 0.$$

To obtain explicit values of y, we make the substitution $y = \cosh r$ and divide the resulting expression by $\sinh^2 r$, obtaining

$$\coth^2 r - nH \coth r + (n-1) = 0.$$

Solving this equation for $\coth r$, we get

$$\coth r = \frac{nH \pm \sqrt{n^2H^2 - 4(n-1)}}{2}.$$

We recover y as

$$y = \cosh r = \frac{\coth r}{\sqrt{\coth^2 r - 1}}.$$

Using the above expression for $\coth r$, we obtain

$$y = \frac{nH \pm \sqrt{n^2H^2 - 4(n-1)}}{\sqrt{\left(nH \pm \sqrt{n^2H^2-4(n-1)}\right)^2 - 4}}. \tag{16}$$

The lemma follows easily from this last equation. $\square$

These critical points correspond exactly to the hyperbolic cylinders. If we take $y = \cosh r$ as before, it is easily seen that the principal curvatures of such a cylinder are

$$\kappa_i = \frac{\sqrt{y^2-1}}{y} = \frac{1}{\coth r} = \frac{2}{nH \pm \sqrt{n^2H^2-4(n-1)}}$$

for $i = 1, \ldots, n-1$, and

$$\kappa_n = \frac{y}{\sqrt{y^2-1}} = \coth r = \frac{nH \pm \sqrt{n^2H^2-4(n-1)}}{2}.$$

As calculated in (9) (at the end of Section 2), $|\phi|^2$ is given by

$$|\phi|^2 = \frac{n}{4(n-1)}\left((n-2)H \pm \sqrt{n^2H^2-4(n-1)}\right)^2,$$

which coincides with the expression for $(B_H^{\pm})^2$ given in (12).

The nature of the critical points may be determined by an analysis of the Hessian of G. A straightforward calculation shows that, if $H = 2\sqrt{n-1}/n$, the only critical point of G is of degenerate type. An equally straightforward (but long) calculation and subsequent analysis of the Hessian show also:

(1) When $2\sqrt{n-1}/n < H < 1$, the critical point with smallest y-coordinate is a saddle point, while the other is a center. The expression (16) for y shows that the center goes to infinity when $H \to 1^-$.
(2) If $H \geq 1$, we have only one critical point of saddle type.

The above remarks on the nature of the critical points confirm that the level curves of G in the case when $H \geq 2\sqrt{n-1}/n$ are given as shown in Figure 2 below.

Proof of Theorem 1.3. If $H = 2\sqrt{n-1}/n$, we have only one complete rotation hypersurface, namely the hyperbolic cylinder given by Montiel [16]. On the other hand, if $2\sqrt{n-1}/n < H < 1$, there is a family of closed curves surrounding the critical points of center type, as shown in Figure 2(b). This family represents a class of complete spacelike rotation hypersurfaces. We will study the behaviour of $|\phi|^2$ in this setting.

Fix a level curve of G, in order to study $|\phi|^2$ for a fixed hypersurface, say

$$G(y, y') = y^{n-1}\left(\sqrt{y'^2 + y^2 - 1} - Hy\right) \equiv K.$$

Then

$$
\begin{aligned}
|\phi|^2 &= \frac{1}{2n}\sum_{l,m}(\kappa_l-\kappa_m)^2 \\
&= \frac{n-1}{n}(\kappa_1-(nH-(n-1)\kappa_1))^2 \\
&= n(n-1)\left(\frac{\sqrt{y'^2+y^2-1}}{y}-H\right)^2 \\
&= n(n-1)\left(\frac{\sqrt{y'^2+y^2-1}-Hy}{y}\right)^2 \\
&= n(n-1)\left(\frac{K}{y^n}\right)^2.
\end{aligned}
$$

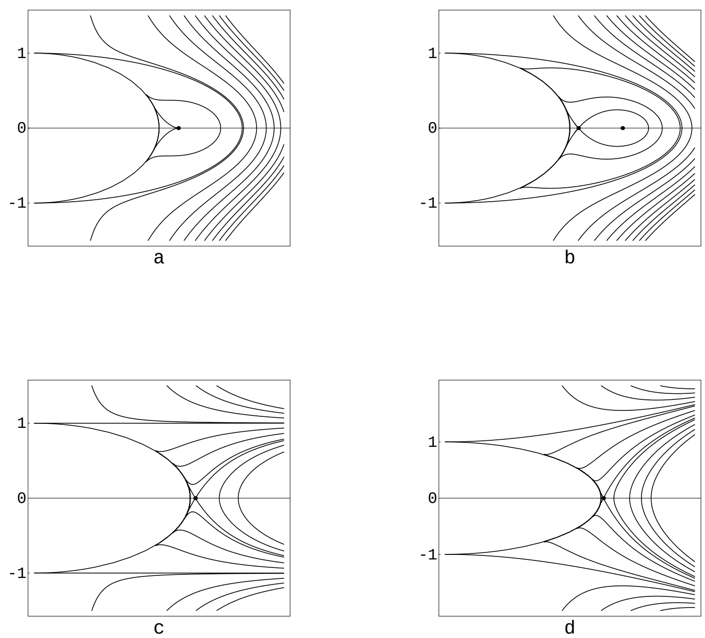

FIGURE 2. Level curves of G for different values of H. (a) $H = 2\sqrt{n-1}/n$, with a degenerate critical point; (b) $2\sqrt{n-1}/n < H < 1$, with two critical points, one saddle point and one center; (c) $H = 1$; and (d) $H > 1$. In each of the last two cases there is only one critical point of saddle type. Note also the level curves passing through the saddle points, which are used in the second part of Theorem 1.3.

This shows that, for a given level curve, $|\phi|^2$ is a decreasing function of y, so that $\sup|\phi|^2$ is attained at the left-most point of the chosen curve. We denote this point by $(y_0, 0)$. Note that for the family of closed curves we are interested in, y_0 can vary only between the y-coordinates of the critical points of G. Evaluating $|\phi|^2$ at these points (so that we now move from one level curve to another), we have

$$\begin{aligned}|\phi|^2(y_0,0) &= n(n-1)\left(\frac{\sqrt{y_0^2-1}-Hy_0}{y_0}\right)^2\\ &= n(n-1)\left(\sqrt{1-\frac{1}{y_0^2}}-H\right)^2.\end{aligned}$$

The continuity of $|\phi|^2$ guarantees that this function attains every number between its extreme values, namely, B_H^- and B_H^+. Consequently, for every H with $2\sqrt{n-1}/n \le H < 1$ and every $B \in [B_H^-, B_H^+]$ there is a complete rotation hypersurface of spherical type, constant mean curvature H and $\sqrt{\sup|\phi|^2} = B$, thus proving the first part of Theorem 1.3 for $2\sqrt{n-1}/n \le H < 1$.

In the case $H \ge 1$ the analysis is quite similar, except that in this case we analyze a family of open curves intersecting the y-axis to the right of the critical point (see (c) and (d) in Figure 2).

To prove the second part of the theorem, suppose first that $2\sqrt{n-1}/n < H < 1$; we will use a subset of the level curve passing through the saddle point $(y_1, 0)$, namely the subset of the level curve totally contained in the region $y > y_1$ (see (b) in Figure 2). We shall prove that this subset represents the complete hypersurface we are seeking by showing that the corresponding function $y = y(t)$ is defined on $(-\infty, \infty)$.

If $K := G(y_1, 0)$, then our curve is given by $G(y, y') = K$, $y > y_1$. Suppose first that $y' > 0$. Equation (15) implies

$$y'(t) = F(y(t)) = \frac{\sqrt{K^2 + 2KHy^n + (H^2-1)y^{2n} + y^{2(n-1)}}}{y^{n-1}}.$$

Integrating from t to a fixed value t_0, we have

$$t - t_0 = \int_{t_0}^{t} \frac{y'}{F(y(t))}dt = \int_{y_0}^{y} \frac{dy}{F(y)},$$

where $y(t_0) = y_0$. As F is continuous and $F(y_1) = 0$, the last integral diverges when $y \to y_1$. This means that t assumes every positive value. To prove that t assumes also every negative value we proceed similarly, analyzing the part of the curve below the y-axis, i.e., $y' < 0$. We obtain in this way a complete rotation hypersurface which satisfies $\sqrt{\sup|\phi|^2} = B_H^+$. As the principal curvatures are not constant, this hypersurface is different from the hyperbolic cylinder.

If $H \geq 1$ we use one of the two connected components of the subset of the level curve totally contained in the region $y > y_1$ (see (c) and (d) in Figure 2). As the analysis is similar to that above, we omit this part and conclude the proof of Theorem 1.3. $\square$

5. Characterization theorems

In this section we extend some results given in [16] for the case $H = 2\sqrt{n-1}/n$ to the case of any value of H satisfying $2\sqrt{n-1}/n \leq H < 1$.

Proof of Proposition 1.1. By Lemma 3.2,

$$\frac{1}{2}\Delta|\phi|^2 \geq |\phi|^2 P_H(|\phi|).$$

As $\sqrt{\sup|\phi|^2} = B_H^-$, we have $B_H^- \geq 0$ and hence $|\phi| \leq B_H^-$, so that each value $|\phi|$ lies to the left of the least root of P_H. Since the graph of P_H is a parabola opening upwards, we have $P_H(|\phi|) \geq 0$ for each $|\phi|$. This fact and the above inequality imply $\Delta|\phi|^2 \geq 0$, i.e, $|\phi|^2$ is subharmonic. By hypothesis, $\sup|\phi|^2$ is attained at some point of M, so we may apply the maximum principle (see, for example, [24, p. 53]) to show that $|\phi|^2$ is constant and hence $|\phi| \equiv B_H^-$. The proof now follows an argument similar to that given in [13] in the case $|\phi| \equiv B_H^+$, which we present here for completeness. Using again Lemma 3.2, we obtain

$$0 = \frac{1}{2}\Delta|\phi|^2 \geq |\phi|^2 P_H(|\phi|) = (B_H^-)^2 P_H(B_H^-) = 0,$$

so that equality holds in Lemma 3.1. By the equality case in that lemma, $(n-1)$ of the numbers $k_i - H$ are equal to

$$\frac{|\phi|}{\sqrt{n(n-1)}} = \frac{B_H^-}{\sqrt{n(n-1)}},$$

or equal to the negative of this last expression. This means that the hypersurface M has $(n-1)$ principal curvatures that are equal and constant. As $nH = \sum k_i$ and H is constant, the other principal curvature is constant as well, so M is isoparametric. The congruence theorem in [1] shows that M is isometric to a hyperbolic cylinder $H^1(\sinh r) \times S^{n-1}(\cosh r)$. $\square$

Before proceeding to prove Theorem 1.4 we state an auxiliary lemma proved in [5].

LEMMA 5.1. *Let V be a real vector space of dimension $n \geq 2$ with an inner product $\langle\ ,\ \rangle$, $A : V \to V$ a symmetric linear map, H given by $\mathrm{tr}A = nH$ and $|\phi|^2 = \mathrm{tr}A^2 - nH^2$. If $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$ are the eigenvalues of A, then for any unitary vector $v \in V$ we have*

$$\langle A^2 v, v\rangle \leq \frac{n-1}{n}|\phi|^2 + 2H\langle Av, v\rangle - H^2.$$

If $n \geq 3$ and equality holds for some unitary vector v, then $Av = \lambda_j v$, where j is such that $|\lambda_j - H| = \max|\lambda_i - H|$. Also, $\lambda_k = \lambda_l$ for all $k, l \neq j$, and $Aw = \lambda_k w$, $k \neq j$, for any w orthogonal to v.

Proof of Theorem 1.4. As $\mathrm{Ric} \geq 0$ and M has at least two ends, we may apply the Cheeger-Gromoll Splitting Theorem [8] to conclude that M is isometric to a Riemannian product $N \times \mathbf{R}$. This fact implies the existence of a direction determined by a unitary vector field u so that $\mathrm{Ric}(u) = 0$. As Ric attains its extrema at the principal directions, $Au = \lambda u$ and equation (3) imply

$$\begin{aligned} 0 = \mathrm{Ric}(u) &= (n-1)\langle u, u\rangle - nH\langle Au, u\rangle + \langle A^2u, u\rangle \\ &= (n-1) - nH\lambda + \lambda^2. \end{aligned}$$

This means that

$$\lambda = \lambda_{\pm} = \frac{nH \pm \sqrt{n^2H^2 - 4(n-1)}}{2}.$$

The continuity of u implies that λ is constant on M, and we must analyze two cases, $\lambda = \lambda_+$ and $\lambda = \lambda_-$.

Suppose first that $\lambda = \lambda_+$. We test the corresponding unitary eigenvector u in Lemma 5.1:

$$\begin{aligned} \lambda_+^2 = \langle A^2u, u\rangle &\leq \frac{n-1}{n}|\phi|^2 + 2H\langle Au, u\rangle - H^2 \\ &\leq \frac{n-1}{n}(B_H^+)^2 + 2H\lambda_+ - H^2 \\ &= \lambda_+^2, \end{aligned}$$

where the second inequality is a consequence of Theorem 1.2. We are now in a position to apply the equality case of Lemma 5.1 to conclude that M has $(n-1)$ principal curvatures equal to, say, μ. In fact, as $(n-1)\mu + \lambda_+ = nH$, μ is given by

$$\mu = \frac{nH + \sqrt{n^2H^2 - 4(n-1)}}{2(n-1)}.$$

Then M is isoparametric and its eigenvalues have multiplicities 1 and $n-1$. By the congruence theorem in [1], M is isometric to a hyperbolic cylinder $H^1(\sinh r) \times S^{n-1}(\cosh r)$.

Suppose now that $\lambda = \lambda_-$. We estimate $|\phi|^2$, expressed as

$$|\phi|^2 = \sum_{i=2}^{n} \kappa_i^2 + (\lambda_-^2 - nH^2),$$

subject to the constraint

$$\sum_{i=2}^{n} \kappa_i = nH - \lambda_- = \frac{nH + \sqrt{n^2H^2 - 4(n-1)}}{2}.$$

The Lagrange multiplier method assures us that the extrema of $|\phi|^2$ are attained when $\kappa_2 = \cdots = \kappa_n = \mu$; in that case,

$$\mu = \frac{nH - \lambda_-}{n-1} = \frac{nH + \sqrt{n^2H^2 - 4(n-1)}}{2(n-1)},$$

so that

$$\begin{aligned} |\phi|^2 \leq (n-1) & \left(\frac{nH + \sqrt{n^2H^2 - 4(n-1)}}{2(n-1)} \right)^2 \\ & + \left(\frac{nH - \sqrt{n^2H^2 - 4(n-1)}}{2} \right)^2 - nH^2 \\ = & \frac{n}{4(n-1)} \left(nH + \sqrt{(n-2)^2H^2 - 4(n-1)} \right)^2 \\ = & (B_H^-)^2. \end{aligned}$$

We now test the corresponding eigenvector v in Lemma 5.1. We have

$$\begin{aligned} \lambda_-^2 = \langle A^2 u, u \rangle & \leq \frac{n-1}{n} |\phi|^2 + 2H \langle Au, u \rangle - H^2 \\ & \leq \frac{n-1}{n} (B_H^-)^2 + 2H\lambda_- - H^2 \\ & = \lambda_-^2, \end{aligned}$$

and the conclusion follows as in the case $\lambda = \lambda_+$ by using the equality case of Lemma 5.1 and the congruence theorem in [1]. □

6. Hypersurfaces with two distinct principal curvatures

K. Nomizu [19] classified the isoparametric hypersurfaces in the de Sitter space. More precisely, he proved that these hypersurfaces are totally umbilical or have two distinct principal curvatures with multiplicities k and $n-k$. It is natural then to study the spacelike hypersurfaces with constant mean curvature $H \neq 0$ and two distinct principal curvatures. The first examples of this type were described by Montiel [15]. Now we characterize a hypersurface with constant mean curvature and two distinct principal curvatures in S_1^{n+1} (Proposition 1.2), adapting a result originally due to Otsuki [22].

Proposition 6.1. *Let M^n be a spacelike hypersurface in S_1^{n+1} such that the multiplicities of the principal curvatures are constant. Then the distribution D_λ of the space of principal vectors corresponding to each principal curvature λ is completely integrable. In particular, if the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of the principal vectors.*

The proof of this proposition uses an argument similar to that given in [7, p. 139], for Riemannian space forms and thus will be omitted.

We are now in a position to prove our characterization result.

Proof of Proposition 1.2. In the case $k = 1$, we have $\kappa_2 = \cdots = \kappa_n = \kappa$, with $\kappa = f(\kappa_1)$. By adapting the proof of Theorem 4.2 in [11, p. 701], we see that M^n is a rotation hypersurface. Suppose now that $1 < k < n-1$ and let λ and μ be the principal curvatures of multiplicities k and $n-k$, respectively. We may choose a local basis $E_1, \ldots, E_k, E_{k+1}, \ldots, E_n$ such that $E_i \in D_\lambda$, $1 \leq i \leq k$, and $E_j \in D_\mu$, $k+1 \leq j \leq n$. By Proposition 6.1 we have $E_i(\lambda) = 0$ for $1 \leq i \leq k$. As

$$k\lambda + (n-k)\mu = nH,$$

we have $(n-k)E_i(\mu) = 0$ for $1 \leq i \leq k$. Using Proposition 6.1 again, we get $E_j(\mu) = 0$ for $k+1 \leq j \leq n$ as well, so that μ is constant on M. Similarly, we obtain that λ is constant on M and then that M^n is isoparametric. By the congruence theorem in [1], it follows that M is isometric to $H^k(\sinh r) \times S^{n-k}(\cosh r)$. □

Added in proof. Professor Q. M. Cheng called our attention to his article *Complete spacelike hypersurfaces of a de Sitter space with constant mean curvature*, Tsukuba J. Math. **14** (1990), 353–370, where he proved part of our Proposition 1.2.

References

[1] N. Abe, N. Koike, and S. Yamaguchi, *Congruence theorems for proper semi-Riemannian hypersurfaces in a real space form*, Yokohama Math. J. **35** (1987), 123–136.

[2] K. Akutagawa, *On spacelike hypersurfaces with constant mean curvature in the de Sitter space*, Math. Z. **196** (1987), 13–19.

[3] H. Alencar and M. do Carmo, *Hypersurfaces with constant mean curvature in spheres*, Proc. Amer. Math. Soc. **120** (1994), 1223–1229.

[4] L. J. Alías, A. Romero, and M. Sánchez, *Spacelike hypersurfaces of constant mean curvature and Calabi-Bernstein type problems*, Tôhoku Math. J. **49** (1997), 337–345.

[5] A. C. Asperti and E. Costa, *Vanishing of homology groups, Ricci estimate for submanifolds and applications*, Kodai Math. J. **24** (2001), 313-328.

[6] J. L. Barbosa and V. Oliker, *Spacelike hypersurfaces with constant mean curvature in Lorentz space*, Mat. Contemporânea **4** (1993), 27–44.

[7] T. E. Cecil and P. J. Ryan, *Tight and taut immersions of manifolds*, Research Notes in Mathematics, vol. 107, Pitman, Boston, 1985

[8] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geom. **6** (1971), 119–128.

[9] Q. M. Cheng, *Complete spacelike hypersurfaces of a de Sitter space with $R = kH$*, Mem. Fac. Sci. Kyushu Univ. Ser. A **44** (1990), 67–77.

[10] S. S. Chern, M. do Carmo, and S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*, Functional Analysis and Related Fields (Proc. Conf. for M. Stone, Univ. Chicago, Chicago, Ill., 1968), Springer, New York, 1970, pp. 59–75.

[11] M. P. do Carmo and M. Dajczer, *Rotation hypersurfaces in spaces of constant curvature*, Trans. Amer. Math. Soc. **277** (1983), 685–709.

[12] A. J. Goddard, *Some remarks on the existence of spacelike hypersurfaces of constant mean curvature*, Math. Proc. Cambridge Phil. Soc. **82** (1977), 489–495.

[13] U. H. Ki, H. J. Kim, and H. Nakagawa, *On spacelike hypersurfaces with constant mean curvature of a Lorentz space form*, Tokyo J. Math. **14** (1991), 205–216.

[14] M. L. Leite, *Rotational hypersurfaces of space forms with constant scalar curvature*, Manuscripta Math. **67** (1990), 285–304.

[15] S. Montiel, *An integral inequality for compact spacelike hypersurfaces in the de Sitter space and applications to the case of constant mean curvature*, Indiana Univ. Math. J. **37** (1988), 909–917.

[16] ———, *A characterization of hyperbolic cylinders in the de Sitter space*, Tôhoku Math. J. **48** (1996), 23–31.

[17] H. Mori, *Rotational surfaces in a pseudo-Riemannian 3-sphere*, Tôhoku Math. J. **88** (1986), 29–36.

[18] S. Nishikawa, *On maximal spacelike hypersurfaces in a Lorentzian manifold*, Nagoya Math. J. **95** (1984), 117–124.

[19] K. Nomizu, *On isoparametric hypersurfaces in the Lorentzian space forms*, Japan. J. Math. (N.S.) **7** (1981), 217–226.

[20] M. Okumura, *Hypersurfaces and a pinching problem on the second fundamental tensor*, Amer. J. Math. **96** (1974), 207–213.

[21] H. Omori, *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan **19** (1967), 205–214.

[22] T. Otsuki, *Minimal hypersurfaces in a Riemannian manifold of constant curvature*, Amer. J. Math. **92** (1970), 145–173.

[23] O. Palmas, *Complete rotation hypersurfaces with H_k constant in space forms*, Bol. Soc. Bras. Mat. **30** (1999), 139–161.

[24] M. H. Protter and H. F. Weinberg, *Maximum principles in differential equations*, Springer-Verlag, New York, 1984.

[25] J. Ramanathan, *Complete spacelike hypersurfaces of constant mean curvature in the de Sitter space*, Indiana Univ. Math. J. **36** (1987), 349–359.

[26] W. Santos, *Submanifolds with parallel mean curvature vector in spheres*, Tôhoku Math. J. **46** (1994), 403–415.

[27] Y. J. Suh, Y. S. Choi and H. Y. Yang, *On spacelike hypersurfaces with constant mean curvature in a Lorentz manifold*, Houston J. Math. **28** (2002), 47–70.

[28] S. T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228.

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