Illinois Journal of Mathematics
Volume 47, Number 3, Fall 2003, Pages 831–845
S 0019-2082

EXTENSIONS, DILATIONS AND FUNCTIONAL MODELS OF DISCRETE DIRAC OPERATORS

B. P. ALLAHVERDIEV

ABSTRACT. A space of boundary values is constructed for minimal symmetric discrete Dirac operators in the limit-circle case. A description of all maximal dissipative, maximal accretive and self-adjoint extensions of such a symmetric operator is given in terms of boundary conditions at infinity. We construct a self-adjoint dilation of a maximal dissipative operator and its incoming and outgoing spectral representations, which make it possible to determine the scattering matrix of the dilation. We also construct a functional model of the dissipative operator and its characteristic function. Finally, we prove the completeness of the system of eigenvectors and associated vectors of dissipative operators.

1. Introduction

The theory of extensions of symmetric operators is one of the main branches of operator theory. The first fundamental results in this area were obtained by von Neumann [11], although the apparent origins can be found in the famous work of Weyl (see [13], [14]), and also in numerous papers on the classical problem of moments (see [1], [2], [3]). To describe the various classes of extensions of symmetric operators, theorems on the representation of linear relations have proved to be useful. The first result of this type is due to Rofe-Beketov [12]. Independently, in [4] and [7], the notion of a 'space of boundary values' was introduced and all maximal dissipative, accretive, self-adjoint, and other extensions of symmetric operators were described (see [6] and also the survey article [5]). However, regardless of the general scheme, the problem of the description of the maximal dissipative (accretive), self-adjoint and other extensions of a given symmetric operator via the boundary conditions is of considerable interest. This problem is particularly interesting in the case of singular operators, because at the singular ends of the interval under consideration the usual boundary conditions are in general meaningless.

Received August 16, 2002; received in final form September 30, 2002.
2000 *Mathematics Subject Classification.* Primary 47B39, 47B44. Secondary 39A70, 47A20, 47A40, 47A45, 47A75, 47B25, 47B37.

One general method of spectral analysis of dissipative operators is the method of contour integration of the resolvent. This method requires a sharp estimate of the resolvent on expanding contours separating the spectrum. The applicability of this method is restricted to weak perturbations of self-adjoint operators and operators with sparse discrete spectrum. Since for wide classes of singular systems there are no asymptotics of the solutions, the method cannot be applied in those cases.

It is known (see [8], [10]) that the theory of dilations with applications of operator models gives an adequate approach to the spectral theory of dissipative (contractive) operators. A central part in this theory is played by the characteristic function, which carries complete information on the spectral properties of the dissipative operator. Thus, in the incoming spectral representation of the dilation, the dissipative operator becomes the model. The problem of the completeness of the system of eigenvectors and associated vectors is solved in terms of the factorization of the characteristic function. The computation of the characteristic functions of dissipative operators requires the construction and investigation of a self-adjoint dilation and of the corresponding scattering problem, in which the characteristic function is realized as the scattering matrix.

In this paper we consider the minimal symmetric discrete Dirac operator in the space $l_A^2(\mathbb{N}; E)\,(\mathbb{N} := \{0, 1, 2, \ldots\}, E := \mathbb{C}^2)$ with defect index $(1, 1)$ (in Weyl's limit-circle case). We construct a space of boundary values of the minimal operator and describe all maximal dissipative, maximal accretive and self-adjoint extensions in terms of the boundary conditions at ∞. We also construct a self-adjoint dilation of the maximal dissipative operator and its incoming and outgoing spectral representations and determine the scattering matrix of the dilation, using the scheme of Lax and Phillips [9]. With the help of the incoming spectral representation we then construct a functional model of the dissipative operator and determine its characteristic function. Finally, using these results, we prove a theorem on the completeness of the system of eigenvectors and associated vectors of dissipative operators.

2. Extensions of symmetric discrete Dirac operators

For sequences $y^{(1)} = \{y_n^{(1)}\}$ and $y^{(2)} = \{y_n^{(2)}\}$ $(n \in \mathbb{N} := \{0, 1, 2, \ldots\})$ of complex numbers $y_n^{(1)}$ and $y_n^{(2)}$ we consider the discrete Dirac system

$$(l_1 y)_n := \begin{cases} -a_n y_{n+1}^{(2)} + b_n y_n^{(2)} + p_n y_n^{(1)} = \lambda c_n y_n^{(1)}, \\ b_n y_n^{(1)} - a_{n-1} y_{n-1}^{(1)} + q_n y_n^{(2)} = \lambda d_n y_n^{(2)}, \end{cases} \tag{2.1}$$

where λ is a complex spectral parameter, $a_{-1} \neq 0, a_n \neq 0, a_n, b_n, p_n, q_n \in \mathbb{R} := (-\infty, \infty)$ and $c_n > 0, d_n > 0$ $(n \in \mathbb{N})$.

System (2.1) is a discrete analog (for $a_{-1} = a_n = 1$, $n \in \mathbb{N}$) of the Dirac system given by

$$J\frac{dy(x)}{dx} + B(x)y(x) = \lambda A(x)y(x),\ x \in [0,\infty)\,, \tag{2.2}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$
$$B(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & r(x) \end{pmatrix}, \quad A(x) = \begin{pmatrix} c(x) & 0 \\ 0 & d(x) \end{pmatrix},$$

where $A(x) > 0$ for almost all $x \in [0,\infty)$, and the elements of the matrices $A(x)$ and $B(x)$ are real-valued, Lebesgue measurable and locally integrable functions on $[0,\infty)$. Equation (2.2) is the radial wave equation for a relativistic particle in a central field and is of interest in physics.

For two arbitrary vector-valued sequences

$$y := \{y_n\} := \begin{Bmatrix} y_n^{(1)} \\ y_n^{(2)} \end{Bmatrix} \quad \text{and} \quad z := \{z_n\} := \begin{Bmatrix} z_n^{(1)} \\ z_n^{(2)} \end{Bmatrix} \quad (n \in \{-1\} \cup \mathbb{N}),$$

denote by $[y,z]$ the sequence with components $[y,z]_n$ ($n \in \{-1\} \cup \mathbb{N}$) defined by

$$[y,z]_n = a_n \left(y_n^{(1)} \bar{z}_{n+1}^{(2)} - y_{n+1}^{(2)} \bar{z}_n^{(1)} \right). \tag{2.3}$$

It is easy to verify the Green's formula

$$\sum_{n=0}^{m} [((l_1 y)_n, z_n)_E - (y_n, (l_1 z)_n)_E] = [y,z]_m - [y,z]_{-1}, \quad m \in \mathbb{N}. \tag{2.4}$$

To pass from the system (2.1) to operators we introduce the Hilbert space $H := l_A^2(\mathbb{N}; E)$ consisting of all vector-valued sequences $y = \{y_n\}$ ($n \in \mathbb{N}$) such that

$$\sum_{n=0}^{\infty} (A_n y_n, y_n)_E = \sum_{n=0}^{\infty} \left(c_n \left| y_n^{(1)} \right|^2 + d_n \left| y_n^{(2)} \right|^2 \right) < \infty$$

with inner product $(y,z) = \sum_{n=0}^{\infty} (A_n y_n, z_n)_E$, where

$$A := \{A_n\}, \quad A_n := \begin{pmatrix} c_n & 0 \\ 0 & d_n \end{pmatrix} \quad (n \in \mathbb{N}).$$

Denote by $l_1(y)$ (resp. ly) the vector-valued sequences with components $(l_1 y)_n$ (resp. $(ly)_n := A_n^{-1}(l_1 y)_n$) ($n \in \mathbb{N}$). Next, denote by D the linear set of all vectors $y \in H$ such that $ly \in H$ and $y_{-1}^{(1)} = 0$. We define the *maximal operator* L on D by the equality $Ly = ly$.

It follows from (2.4) that for all $y, z \in D$ the limit $[y,z]_\infty = \lim_{m\to\infty} [y,z]_m$ exists and is finite. Therefore, passing to the limit as $m \to \infty$ in (2.4), we get that for arbitrary vectors y and z of D

$$(Ly, z) - (y, Lz) = [y,z]_\infty . \tag{2.5}$$

In H we consider the dense linear set D_0' consisting of finite vectors (i.e., vectors having only finitely many nonzero components). Denote by L_0' the restriction of the operator L to D_0'. It follows from (2.5) that L_0' is symmetric. Consequently, it admits closure. Its closure is denoted by L_0. The domain D_0 of L_0 consists of precisely those vectors $y \in D$ satisfying the condition

$$[y,z]_\infty = 0, \quad \forall z \in D. \tag{2.6}$$

The *minimal operator* L_0 is a symmetric operator with defect index $(0,0)$ or $(1,1)$ and satisfying $L = L_0^*$ (see [2], [3]). For defect index $(0,0)$ the operator L_0 is self-adjoint, that is, $L_0^* = L_0 = L$.

We assume that L_0 has defect index $(1,1)$, so that the *Weyl limit-circle case* holds for the discrete Dirac expression $l(y)$.

The *Wronskian* of two solutions $y = \{y_n\}$ and $z = \{z_n\}$ $(n \in \mathbb{N})$ of (2.1) is defined to be $W_n(y,z) := a_n(y_n^{(1)} z_{n+1}^{(2)} - y_{n+1}^{(2)} z_n^{(1)})$, so that $W_n(y,z) = [y,\bar{z}]_n$ $(n \in \mathbb{N})$. The Wronskian of two solutions of (2.1) does not depend on n and two solutions of this system are linearly independent if and only if their Wronskian is nonzero.

Denote by $P(\lambda) = \{P_n(\lambda)\}$ and $Q(\lambda) = \{Q_n(\lambda)\}$ $(n \in \mathbb{N})$ the solutions of (2.1) satisfying the initial conditions

$$P_{-1}^{(1)} = 1, \quad P_0^{(2)} = 0, \quad Q_{-1}^{(1)} = 0, \quad Q_0^{(2)} = 1/a_{-1}. \tag{2.7}$$

We have that $W_n[P(\lambda), Q(\lambda)] = 1$, $n \in \mathbb{N} \cup \{\infty\}$. Consequently $P(\lambda)$ and $Q(\lambda)$ form a fundamental system of solutions of (2.1). Since L_0 has defect index $(1,1)$, we have $P(\lambda), Q(\lambda) \in H$ for all $\lambda \in \mathbb{C}$.

Let $u = P(0)$ and $v = Q(0)$. Then we have:

LEMMA 2.1. *For arbitrary $\alpha, \beta \in \mathbb{C}$ there exists a vector $y \in D$ satisfying*

$$[y,u]_\infty = \alpha, \quad [y,v]_\infty = \beta. \tag{2.8}$$

Proof. Let f be an arbitrary vector in H satisfying

$$(f,u) = \alpha, \quad (f,v) = \beta. \tag{2.9}$$

Such a vector f exists, even among linear combinations of u and v. Indeed, if $f = c_1 u + c_2 v$, then the conditions (2.9) are a system of equations for the constants c_1 and c_2 whose determinant is the Gram determinant of the linearly independent vectors u and v, and hence is nonzero.

Denote by $y = \{y_n\}$ $(n \in \mathbb{N})$ the solution of $ly = f$ satisfying $y_{-1}^{(1)} = 0$. Then $y \in D$. We show that y is the desired vector. Indeed, using (2.4) as

$m \to \infty$, we get that

$$(2.10)\quad (f,u) = (ly,u) = [y,u]_\infty + (y,lu)\,, \quad (f,v) = (ly,v) = [y,v]_\infty + (y,v)\,.$$

But $lu = 0$, and thus $(y, lu) = 0$. Therefore, $(f,u) = \alpha = [y,u]_\infty$. Analogously, $[y,v]_\infty = \beta$. $\square$

LEMMA 2.2. *For arbitrary vectors $y, z \in D$, we have the equality ($n \in \mathbb{N} \cup \{\infty\}$)*

$$(2.11)\qquad [y,z]_n = [y,u]_n\,[\bar{z},v]_n - [y,v]_n\,[\bar{z},u]_n\,.$$

Proof. Since the sequences $u = \{u_n\}$ and $v = \{v_n\}$ are real and since $[u,v]_n = 1$ ($n \in \mathbb{N} \cup \{\infty\}$), we have

$$\begin{aligned}
&[y,u]_n\,[\bar{z},v]_n - [y,v]_n\,[\bar{z},u]_n \\
&\quad = a_n\left(y_n^{(1)}u_{n+1}^{(2)} - y_{n+1}^{(2)}u_n^{(1)}\right) a_n\left(\bar{z}_n^{(1)}v_{n+1}^{(2)} - \bar{z}_{n+1}^{(2)}v_n^{(1)}\right) \\
&\qquad - a_n\left(y_n^{(1)}v_{n+1}^{(2)} - y_{n+1}^{(2)}v_n^{(1)}\right) a_n\left(\bar{z}_n^{(1)}u_{n+1}^{(2)} - \bar{z}_{n+1}^{(2)}u_n^{(1)}\right) \\
&\quad = a_n^2\Big(y_n^{(1)}u_{n+1}^{(2)}\bar{z}_n^{(1)}v_{n+1}^{(2)} - y_n^{(1)}u_{n+1}^{(2)}\bar{z}_{n+1}^{(2)}v_n^{(1)} - y_{n+1}^{(2)}u_n^{(1)}\bar{z}_n^{(1)}v_{n+1}^{(2)} \\
&\qquad + y_{n+1}^{(2)}u_n^{(1)}\bar{z}_{n+1}^{(2)}v_n^{(1)} - y_n^{(1)}v_{n+1}^{(2)}\bar{z}_n^{(1)}u_{n+1}^{(2)} + y_n^{(1)}v_{n+1}^{(2)}\bar{z}_{n+1}^{(2)}u_n^{(1)} \\
&\qquad + y_{n+1}^{(2)}v_n^{(1)}\bar{z}_n^{(1)}u_{n+1}^{(2)} - y_{n+1}^{(2)}v_n^{(1)}\bar{z}_{n+1}^{(2)}u_n^{(1)}\Big) \\
&\quad = a_n\left(y_n^{(1)}\bar{z}_{n+1}^{(2)} - y_{n+1}^{(2)}\bar{z}_n^{(1)}\right) a_n\left(u_n^{(1)}v_{n+1}^{(2)} - u_{n+1}^{(2)}v_n^{(1)}\right) = [y,z]_n\,.
\end{aligned}$$

The lemma is proved. $\square$

THEOREM 2.3. *The domain D_0 of the operator L_0 consists of precisely those vectors $y \in D$ satisfying the boundary conditions*

$$(2.12)\qquad [y,u]_\infty = [y,v]_\infty = 0.$$

Proof. As noted above, the domain D_0 of L_0 coincides with the set of all vectors $y \in D$ satisfying (2.6). By virtue of Lemma 2.1, (2.6) is equivalent to

$$(2.13)\qquad [y,u]_\infty\,[\bar{z},v]_\infty - [y,v]_\infty\,[\bar{z},u]_\infty = 0.$$

Further, by Lemma 2.1, the numbers $[\bar{z},v]_\infty$ and $[\bar{z},u]_\infty$ ($z \in D$) can be arbitrary, and therefore (2.13) holds for all $z \in D$ if and if the conditions (2.12) hold. The theorem is proved. $\square$

The triple $(\mathcal{H}, \Gamma_1, \Gamma_2)$, where $\mathcal{H}$ is a Hilbert space and Γ_1 and Γ_2 are linear mappings of $D(\mathbf{A}^*)$ into $\mathcal{H}$, is called (see [4], [6], [7]) a *space of boundary values* of a closed symmetric operator $\mathbf{A}$ acting in a Hilbert space $\mathbf{H}$ with equal (finite or infinite) defect index if

(i) for any $f, g \in D(\mathbf{A}^*)$,

$$(\mathbf{A}^* f, g)_{\mathbf{H}} - (f, \mathbf{A}^* g)_{\mathbf{H}} = (\Gamma_1 f, \Gamma_2 g)_{\mathcal{H}} - (\Gamma_2 f, \Gamma_1 g)_{\mathcal{H}};$$

(ii) for any $F_1, F_2 \in \mathcal{H}$, there exists a vector $f \in D(\mathbf{A}^*)$ such that $\Gamma_1 f = F_1, \Gamma_2 f = F_2$.

In our case, we denote by Γ_1 and Γ_2 the linear mappings of D into $\mathbb{C}$ defined by

$$\Gamma_1 y = [y, u]_\infty, \quad \Gamma_2 f = [y, v]_\infty \, (y \in D). \tag{2.14}$$

Then we have:

THEOREM 2.4. *The triple* $(\mathbb{C}, \Gamma_1, \Gamma_2)$ *defined by* (2.14) *is the space of boundary values of the operator* L_0.

Proof. By Lemma 2.1, for arbitrary $y, z \in D$ we have

$$\begin{aligned}(Ly, z) - (y, Lz) = [y, z]_\infty &= [y, u]_\infty [\bar{z}, v]_\infty - [y, v]_\infty [\bar{z}, u]_\infty \\ &= (\Gamma_1 y, \Gamma_2 z) - (\Gamma_2 y, \Gamma_1 z),\end{aligned}$$

i.e., the first condition in the definition of the space of boundary values holds. The second condition holds by Lemma 2.1.

Recall that a linear operator T (with domain $D(T)$) on some Hilbert space H is called *dissipative (accretive)* if $\operatorname{Im}(Tf, f) \geq 0$ (respectively, $\operatorname{Im}(Tf, f) \leq 0$) for all $f \in D(T)$ and *maximal dissipative (accretive)* if it does not have a proper dissipative (accretive) extension. □

By [4] or [6], Theorem 2.4 implies the following result:

THEOREM 2.5. *Every maximal dissipative (accretive) extension* $\mathbf{L}_h$ *of* L_0 *is determined by the equality* $\mathbf{L}_h y = Ly$ *for the vectors* y *in* D *satisfying the boundary condition*

$$[y, u]_\infty - h [y, v]_\infty = 0, \tag{2.15}$$

where $\operatorname{Im} h \geq 0$ *or* $h = \infty$ *(resp.* $\operatorname{Im} h \leq 0$ *or* $h = \infty$*). Conversely, for an arbitrary* h *with* $\operatorname{Im} h \geq 0$ *or* $h = \infty$ *(resp.* $\operatorname{Im} h \leq 0$ *or* $h = \infty$*), the boundary condition* (2.15) *determines a maximal dissipative (accretive) extension on* L_0. *The self-adjoint extensions of* L_0 *are obtained precisely when* h *is a real number or infinity. For* $h = \infty$*, condition* (2.15) *should be replaced by* $[y, v]_\infty = 0$.

3. Completeness theorem, self-adjoint dilation and functional model of dissipative operators

In the sequel we shall study the maximal dissipative operators $\mathbf{L}_h$ (with $\operatorname{Im} h > 0$) generated by the expression ly and the boundary condition (2.15).

For the spectral analysis of the dissipative operators L_h we will apply the dilation and functional model theory of L_h as mentioned in introduction. We first develop the Lax-Phillips scattering theory for self-adjoint dilations of L_h and determine the scattering matrix of such a dilation via the characteristic function of L_h. The applications of these theories go far beyond the problem discussed here (see [8], [9], [10]).

Let us first state our main result for the operator L_h.

Theorem 3.1. *For all values of h with $\operatorname{Im} h > 0$, except possibly for a single value $h = h^0$, the characteristic function $S_h(\lambda)$ of the dissipative operator $\mathbf{L}_h$ is a Blaschke product and the spectrum of $\mathbf{L}_h$ is purely discrete and belongs to the open upper half plane. The operator $\mathbf{L}_h$ $(h \neq h^0)$ has a countable number of isolated eigenvalues with finite multiplicity and limit points at infinity, and the system of eigenvectors and associated vectors of this operator is complete in $l_A^2(\mathbb{N}; E)$.*

To prove the theorem, we first construct the self-adjoint dilation of the operator L_h.

Let us add to the space $H := l_A^2(\mathbb{N}; E)$ the 'incoming' and 'outgoing' channels $D_- := L^2(-\infty, 0)$ and $D_+ := L^2(0,\infty)$. We form the *main Hilbert space of the dilation* $\mathcal{H} = L^2(-\infty,0) \oplus H \oplus L^2(0,\infty)$, and in $\mathcal{H}$ we consider the operator $\mathcal{L}_h$ generated by the expression

$$\mathcal{L}\langle\varphi_-, y, \varphi_+\rangle = \left\langle i\frac{d\varphi_-}{d\xi}, l(y), i\frac{d\varphi_+}{d\varsigma}\right\rangle \tag{3.1}$$

on the set $D(\mathcal{L}_h)$ of vectors $\langle\varphi_-, y, \varphi_+\rangle$ satisfying $\varphi_- \in W_2^1(-\infty,0)$, $\varphi_+ \in W_2^1(0,\infty)$, $y \in D$, and

$$[y,u]_\infty - h\,[y,v]_\infty = \alpha\varphi_-(0), [y,u]_\infty - \bar{h}\,[y,v]_\infty = \alpha\varphi_+(0), \tag{3.2}$$

where $\alpha^2 := \operatorname{Im} h$, $\alpha > 0$, and W_2^1 is the Sobolev space. Then we have:

Theorem 3.2. *The operator $\mathcal{L}_h$ is self-adjoint in $\mathcal{H}$ and is a self-adjoint dilation of the operator $\mathbf{L}_h$.*

Proof. Suppose that $f, g \in D(\mathcal{L}_h)$, $f = \langle\varphi_-, y, \varphi_+\rangle$ and $g = \langle\psi_-, z, \psi_+\rangle$. Then, integrating by parts and using (2.14), we get that

$$\begin{aligned}(\mathcal{L}_h f, g)_{\mathcal{H}} &= \int_{-\infty}^0 i\varphi_-'\bar{\psi}_- d\xi + (ly,z)_H + \int_0^\infty \varphi_+'\bar{\psi}_+ d\xi \\ &= i\varphi_-(0)\,\bar{\psi}_-(0) - i\varphi_+(0)\,\bar{\psi}_+(0) + [y,z]_\infty + (f, \mathcal{L}_h)_{\mathcal{H}}.\end{aligned} \tag{3.3}$$

Next, using the boundary conditions (3.2) for the components of the vectors f, g and relation (2.13), we see by direct computation that $i\varphi_-(0)\,\bar{\psi}_-(0)$ $-i\varphi_+(0)\,\bar{\psi}_+(0) + [y,z]_\infty = 0$. Thus, $\mathcal{L}_h$ is symmetric. Therefore, to prove that $\mathcal{L}_h$ is self-adjoint, it suffices for us to show that $\mathcal{L}_h^* \subseteq \mathcal{L}_h$. Take $g = \langle\psi_-, z, \psi_+\rangle \in D(\mathcal{L}_h^*)$. Let $\mathcal{L}_h^* g = g^* = \langle\psi_-^*, z^*, \psi_+^*\rangle \in \mathcal{H}$, so that

$$(\mathcal{L}_h f, g)_{\mathcal{H}} = (f, g^*)_{\mathcal{H}}, \quad \forall f \in D(\mathcal{L}_h). \tag{3.4}$$

By choosing vectors with suitable components as the element $f \in D(\mathcal{L}_h)$ in (3.4), it is not difficult to show that $\psi_- \in W_2^1(-\infty, 0)$, $\psi_+ \in W_2^1(0, \infty)$, $z \in D$ and $g^* = \mathcal{L}g$, where the operation $\mathcal{L}$ is defined by (3.1). Consequently, (3.4) takes the form $(\mathcal{L}f, g)_{\mathcal{H}} = (f, \mathcal{L}g)_{\mathcal{H}}, \forall f \in D(\mathcal{L}_h)$. Therefore, the sum of the integral terms in the bilinear form $(\mathcal{L}f, g)_{\mathcal{H}}$ must be equal to zero, i.e.,

$$i\varphi_-(0)\bar{\psi}_-(0) - i\varphi_+(0)\bar{\psi}_+(0) + [y, z]_\infty = 0 \tag{3.5}$$

for all $f = \langle \varphi_-, y, \varphi_+ \rangle \in D(\mathcal{L}_h)$. Further, solving the boundary conditions (3.2) for $[y, u]_\infty$ and $[y, v]_\infty$, we find that $[y, v]_\infty = \frac{1}{i\alpha}(\varphi_+(0) - \varphi_-(0))$ and $[y, u]_\infty = \alpha\varphi_-(0) + \frac{h}{i\alpha}(\varphi_+(0) - \varphi_-(0))$. Therefore, using (2.11), we find that (3.5) is equivalent to the equation

$$\begin{aligned} i\varphi_-(0)\bar{\psi}_-(0) - i\varphi_+(0)\bar{\psi}_+(0) &= -[y, z]_\infty \\ &= \frac{1}{i\alpha}(\varphi_+(0) - \varphi_-(0))[\bar{z}, u]_\infty \\ &\quad - \left[\alpha\varphi_-(0) + \frac{h}{i\alpha}(\varphi_+(0) - \varphi_-(0))\right][\bar{z}, v]_\infty. \end{aligned}$$

Since the values $\varphi_\pm(0)$ can be arbitrary complex numbers, a comparison of the coefficients of $\varphi_\pm(0)$ on the left and right of this identity gives that the vector $g = \langle \psi_-, z, \psi_+ \rangle$ satisfies the boundary conditions $[z, u]_\infty - h[z, v]_\infty = \alpha\psi_-(0)$ and $[z, u]_\infty - \bar{h}[z, v]_\infty = \alpha\psi_+(0)$. Consequently, the inclusion $\mathcal{L}_h^* \subseteq \mathcal{L}_h$ is established, and hence $\mathcal{L}_h = \mathcal{L}_h^*$.

The self-adjoint operator $\mathcal{L}_h$ generates in $\mathcal{H}$ a unitary group $U_t = \exp[i\mathcal{L}_h t]$ $(t \in \mathbb{R})$. Denote by $P : \mathcal{H} \to H$ and $P_1 : H \to \mathcal{H}$ the mappings defined by $P : \langle \varphi_-, y, \varphi_+ \rangle \to y$ and $P_1 : y \to \langle 0, y, 0 \rangle$. Let $Z_t = PU_tP_1$ $(t \geq 0)$. The family $\{Z_t\}$ $(t \geq 0)$ of operators is a strongly continuous semigroup of completely nonunitary contractions on H. Denote by A_h the generator of this semigroup, i.e., $A_h y = \lim_{t \to +0}(it)^{-1}(Z_t y - y)$. The domain of A_h consists of all the vectors for which the limit exists. The operator A_h is a maximal dissipative operator. The operator $\mathcal{L}_h$ is called the self-adjoint dilation of A_h (see [8], [10]). We show that $A_h = \mathbf{L}_h$, and hence that $\mathcal{L}_h$ is a self-adjoint dilation of L_h. To do this, we first establish the relation (see [8], [10])

$$P(\mathcal{L}_h - \lambda I)^{-1} P_1 y = (\mathbf{L}_h - \lambda I)^{-1} y, \quad y \in H, \quad \operatorname{Im}\lambda < 0. \tag{3.6}$$

To this end, we set $(\mathcal{L}_h - \lambda I)^{-1} P_1 y = g = \langle \psi_-, z, \psi_+ \rangle$. Then $(\mathcal{L}_h - \lambda I) g = P_1 y$, and hence $Lz - \lambda z = y$, $\psi_-(\xi) = \psi_-(0) e^{-i\lambda\xi}$ and $\psi_+(\varsigma) = \psi_+(0) e^{-i\lambda\varsigma}$. Since $g \in D(\mathcal{L}_h)$ and hence $\psi_- \in L^2(-\infty, 0)$, it follows that $\psi_-(0) = 0$, and consequently, z satisfies the boundary condition $[z, u]_\infty - h[z, v]_\infty = 0$. Therefore, $z \in D(\mathcal{L}_h)$, and since a point λ with $\operatorname{Im}\lambda < 0$ cannot be an eigenvalue of dissipative operator, it follows that $z = (\mathbf{L}_h - \lambda I)^{-1} y$. We remark that $\psi_+(0)$ can be obtained from the formula $\psi_+(0) = \alpha^{-1}\left([z, u]_\infty - \bar{h}[z, v]_\infty\right)$.

Thus,

$$\left(\mathcal{L}_h - \lambda I\right)^{-1} P_1 y = \left\langle 0, (\mathbf{L}_h - \lambda I)^{-1} y, \alpha^{-1}\left([z,u]_\infty - \bar{h}\,[z,v]_\infty\right) e^{-i\lambda\varsigma}\right\rangle$$

for $y \in H$ and $\operatorname{Im}\lambda < 0$. On applying the mapping P, we obtain (3.6).

It is now easy to show that $A_h = \mathbf{L}_h$. Indeed, by (3.6),

$$\begin{aligned}(\mathbf{L}_h - \lambda I)^{-1} &= P\left(\mathcal{L}_h - \lambda I\right)^{-1} P_1 = -iP\int_0^\infty U_t e^{-i\lambda t} dt P_1 \\ &= -i\int_0^\infty Z_t e^{-i\lambda t} dt = \left(A_h - \lambda I\right)^{-1}, \quad \operatorname{Im}\lambda < 0,\end{aligned}$$

from which it is clear that $\mathbf{L}_h = A_h$. Theorem 3.2 is proved. □

The unitary group $U_t = \exp\left[i\mathcal{L}_h t\right]$ $(t \in \mathbb{R})$ has an important property which makes it possible to apply to it the Lax-Phillips scheme [9]. Namely, it has incoming and outgoing subspaces $D_- = \left\langle L^2\left(-\infty, 0\right), 0, 0\right\rangle$ and $D_+ = \left\langle 0, 0, L^2\left(0, \infty\right)\right\rangle$ possessing the following properties:

(1) $U_t D_- \subset D_-$, $t \leq 0$ and $U_t D_+ \subset D_+$, $t \geq 0$;
(2) $\bigcap_{t\leq 0} U_t D_- = \bigcap_{t\geq 0} U_t D_+ = \{0\}$;
(3) $\overline{\bigcup_{t\geq 0} U_t D_-} = \overline{\bigcup_{t\leq 0} U_t D_+} = \mathcal{H}$;
(4) $D_- \perp D_+$.

Property (4) is obvious. To prove property (1) for D_+ (the proof for D_- is similar), we set $R_\lambda = (\mathcal{L}_h - \lambda I)^{-1}$ for all λ with $\operatorname{Im}\lambda < 0$. Then, for any $f = \langle 0, 0, \varphi_+\rangle \in D_+$ we have

$$R_\lambda f = \left\langle 0, 0, -ie^{-i\lambda\xi}\int_0^\xi e^{-i\lambda s}\varphi_+(s)ds\right\rangle.$$

Hence, $R_\lambda f \in D_+$; therefore, if $g \perp D_+$, then

$$0 = \left(R_\lambda f, g\right)_{\mathcal{H}} = -i\int_0^\infty e^{-i\lambda t}\left(U_t f, g\right)_{\mathcal{H}} d\lambda, \quad \operatorname{Im}\lambda < 0.$$

From this it follows that $(U_t f, g)_{\mathcal{H}} = 0$ for all $t \geq 0$. Hence $U_t D_+ \subset D_+$ for $t \geq 0$, and property (1) has been proved.

To prove property (2), we denote by $P^+ : \mathcal{H} \to L^2\left(0,\infty\right)$ and $P_1^+ : L^2\left(0,\infty\right) \to D_+$ the mappings defined by $P^+ : \langle\varphi_-, u, \varphi_+\rangle \to \varphi_+$ and $P_1^+ : \varphi \to \langle 0, 0, \varphi\rangle$, respectively. We note that the semigroup of isometries $V_t = P^+ U_t P_1^+$, $t \geq 0$, is a one-sided shift in $L^2\left(0,\infty\right)$. Indeed, the generator of the semigroup of the one-sided shift V_t in $L^2\left(0,\infty\right)$ is the differential operator $i\left(d/d\xi\right)$ with boundary condition $\varphi\left(0\right) = 0$. On the other hand, the generator A of the semigroup of isometries U_t^+, $t \geq 0$, is the operator

$$A\varphi = P^+\mathcal{L}_h P_1^+ f = P^+\mathcal{L}_h\langle 0, 0, \varphi\rangle = P^+\left\langle 0, 0, i\frac{d\varphi}{d\xi}\right\rangle = i\frac{d\varphi}{d\xi},$$

where $\varphi \in W_2^1(0,\infty)$ and $\varphi(0) = 0$. But since a semigroup is uniquely determined by its generator, it follows that $U_t^+ = V_t$; hence,

$$\bigcap_{t\geq 0} U_t D_+ = \left\langle 0, 0, \bigcap_{t\geq 0} V_t L^2(0,\infty) \right\rangle = \{0\},$$

i.e., property (2) is proved.

In this scheme of the Lax-Phillips scattering theory, the scattering matrix is defined in terms of the theory of spectral representations. We now proceed to construct these representations. In the process, we also prove property (3) of the incoming and outgoing subspaces.

Lemma 3.3. *The operator $\mathbf{L}_h$ is totally nonself-adjoint (simple).*

Proof. Let $H' \subset H$ be a nontrivial subspace in which $\mathbf{L}_h$ induces a self-adjoint operator $\mathbf{L}'_h$ with domain $D(\mathbf{L}'_h) = H' \cap D(\mathbf{L}_h)$. If $y \in D(\mathbf{L}'_h)$, then $\operatorname{Im}(\mathbf{L}_h y, y) = 0$, and we get from $\operatorname{Im}(\mathbf{L}_h y, y) = (\operatorname{Im} h)\left|[y,v]_\infty\right|^2 = 0$ that $[y,v]_\infty = 0$. This and the boundary condition (2.17) also imply the equality $[y,u]_\infty = 0$. Thus,

$$[y,u]_\infty = [y,v]_\infty = 0,\ y \in D(\mathbf{L}'_h). \tag{3.7}$$

Denote by $\mathbf{L}_0$ and $\mathbf{L}_\infty$ the self-adjoint extensions of L_0 determined by the boundary conditions $[y,u]_\infty = 0$ and $[y,v]_\infty = 0$, respectively. By (3.7) $D(\mathbf{L}'_h)$ is contained in each of $D(\mathbf{L}_0)$ and $D(\mathbf{L}_\infty)$. Suppose that λ belongs to the spectrum of $\mathbf{L}'_h$. Then λ is real, and there exists a sequence of vectors $f_n \in D(\mathbf{L}'_h)$ such that $\|f_n\| = 1$ and $\|\mathbf{L}_h f_n - \lambda f_n\| \to 0$ as $n \to \infty$. This implies that λ belongs also to the spectra of the operators $\mathbf{L}_0$ and $\mathbf{L}_\infty$. Since the spectra of $\mathbf{L}_0$ and $\mathbf{L}_\infty$ are purely discrete, λ is an eigenvalue of these operators. The corresponding eigenvectors differ from $Q(\lambda)$ only by a scalar factor, because $Q(\lambda)$ is the only linearly independent solution of the system (2.1) with $y_{-1}^{(1)} = 0$. Consequently, $[Q(\lambda), u]_\infty = [Q(\lambda), v]_\infty = 0$. Further, from (2.11) with $y = P(\lambda)$ and $z = Q(\lambda)$ we have

$$[P(\lambda), Q(\lambda)]_\infty = [P(\lambda), u]_\infty [Q(\lambda), v]_\infty - [P(\lambda), v]_\infty [Q(\lambda), u]_\infty. \tag{3.8}$$

The right-hand side is equal to 0 in view of (3.8), while the left-hand side, as the value of the Wronskian of the solutions $P(\lambda)$ and $Q(\lambda)$ of (2.1), is equal to 1. This contradiction shows that $H' = \{0\}$. The lemma is proved. $\square$

We set $\mathcal{H}_- = \overline{\bigcup_{t\geq 0} U_t D_-}$, $\mathcal{H}_+ = \overline{\bigcup_{t\leq 0} U_t D_+}$

Lemma 3.4. $\mathcal{H}_- + \mathcal{H}_+ = \mathcal{H}$.

Proof. Using property (1) of the subspace D_+, it is easy to show that the subspace $\mathcal{H}' = \mathcal{H} \ominus (\mathcal{H}_- + \mathcal{H}_+)$ is invariant relative to the group $\{U_t\}$ and has the form $\mathcal{H}' = \langle 0, H', 0 \rangle$, where H' is a subspace in H. Therefore, if the subspace $\mathcal{H}'$ (and hence also H') were nontrivial, then the unitary group $\{U_t\}$, restricted to this subspace, would be a unitary part of the group $\{U_t\}$, and hence the restriction $\mathbf{L}_h'$ of $\mathbf{L}_h$ to H' would be a self-adjoint operator in H'. From the simplicity of the operator $\mathbf{L}_h$ it follows that $H' = \{0\}$, i.e., $\mathcal{H}' = \{0\}$. The lemma is proved. $\square$

Let us adopt the following notation:

$$n_h(\lambda) := [Q(\lambda), u]_\infty - h\,[Q(\lambda), v]_\infty\,,$$

$$n(\lambda) := \frac{[Q(\lambda), u]_\infty}{[Q(\lambda), v]_\infty}, \tag{3.9}$$

$$S_h(\lambda) := \frac{n_h(\lambda)}{n_{\bar{h}}(\lambda)} = \frac{n(\lambda) - h}{n(\lambda) - \bar{h}}. \tag{3.10}$$

From (3.9), it follows that $n(\lambda)$ is a meromorphic function on the complex plane $\mathbb{C}$ with a countable number of poles on the real axis. Further, it is possible to show that the function $n(\lambda)$ satisfies $\operatorname{Im}\lambda \operatorname{Im} n(\lambda) > 0$ for $\operatorname{Im}\lambda \neq 0$ and $n(\bar{\lambda}) = \overline{n(\lambda)}$ for $\lambda \in \mathbb{C}$ with the exception of the real poles of $n(\lambda)$.

Let

$$\mathcal{U}_\lambda^-(\xi,\zeta) = \left\langle e^{-i\lambda\xi}, \frac{\alpha}{n_h(\lambda)} Q(\lambda), \bar{S}_h(\lambda) e^{-i\lambda\zeta} \right\rangle.$$

We note that the vectors $U_\lambda^-(\xi,\zeta)$ for real λ do not belong to the space $\mathcal{H}$. However, these vectors satisfy the equation $\mathcal{L}U = \lambda U$ $(\lambda \in \mathbb{R})$ and the corresponding boundary conditions for the operator $\mathcal{L}_h$. With the help of the vectors $U_\lambda^-(\xi,\zeta)$, we define the transformation $\mathcal{F}_- : f \to \tilde{f}_-(\lambda)$ by $(\mathcal{F}_- f)(\lambda) := \tilde{f}_-(\lambda) := \frac{1}{\sqrt{2\pi}}(f, U_\lambda^-)_\mathcal{H}$ on the vectors $f = \langle \varphi_-, y, \varphi_+ \rangle$, where $\varphi_-(\xi)$ and $\varphi_+(\zeta)$ are compactly supported smooth functions, and $y = \{y_n\}$ $(n \in \mathbb{N})$ is a finite nonzero vector-valued sequence.

Lemma 3.5. *The transformation $\mathcal{F}_-$ maps $\mathcal{H}_-$ isometrically onto $L^2(\mathbb{R})$. For all vectors $f, g \in \mathcal{H}_-$ the Parseval equality and the inversion formula hold:*

$$(f,g)_\mathcal{H} = (\tilde{f}_-, \tilde{g}_-)_{L^2} = \int_{-\infty}^{\infty} \tilde{f}_-(\lambda)\overline{\tilde{g}_-(\lambda)}\,d\lambda, \quad f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}_-(\lambda) U_\lambda^-\,d\lambda,$$

where $\tilde{f}_-(\lambda) = (\mathcal{F}_- f)(\lambda)$ and $\tilde{g}_-(\lambda) = (\mathcal{F}_- g)(\lambda)$.

Proof. For $f, g \in D_-$, $f = \langle \varphi_-, 0, 0 \rangle$, $g = \langle \psi_-, 0, 0 \rangle$, we have that

$$\tilde{f}_-(\lambda) := \frac{1}{\sqrt{2\pi}}(f, U_\lambda^-)_\mathcal{H} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \varphi_-(\xi) e^{i\lambda\xi}\,d\xi \in H_-^2,$$

and in view of the usual Parseval equality for Fourier integrals

$$(f,g)_\mathcal{H} = \int_{-\infty}^{0} \varphi_-(\xi)\bar{\psi}_-(\xi)\,d\xi = \int_{-\infty}^{\infty} \tilde{f}_-(\lambda)\overline{\tilde{g}_-(\lambda)}\,d\lambda = (\mathcal{F}_- f, \mathcal{F}_- g)_{L^2}.$$

Here and below, $H^2_\pm$ denote the Hardy classes in $L^2(\mathbb{R})$ consisting of the functions analytically extendable to the upper and lower half-planes, respectively.

We now extend the Parseval equality to the whole space $\mathcal{H}_-$. To this end, we consider in $\mathcal{H}_-$ the dense set $\mathcal{H}'_-$ of vectors f obtained as follows from the smooth, compactly supported functions in D_- : $f \in \mathcal{H}'_-$ if $f = U_t f_0$, $f_0 = \langle \varphi_-, 0, 0\rangle$, $\varphi_- \in C_0^\infty(-\infty, 0)$, where $T = T_f$ is a non-negative number (depending on f). In this case, if $f, g \in \mathcal{H}'_-$, then for $T > T_f$ and $T > T_g$ we have that $U_{-T} f, U_{-T} g \in D_-$ and, moreover, the first components of these vectors belong to $C_0^\infty(-\infty, 0)$. Therefore, since the operators U_t $(t \in \mathbb{R})$ are unitary, by the equality $\mathcal{F}_- U_t f = (U_t f, U_\lambda^-)_\mathcal{H} = e^{i\lambda t}(f, U_\lambda^-)_\mathcal{H} = e^{i\lambda t}\mathcal{F}_- f$, we have

$$
\begin{aligned}
(3.11)\qquad (f, g)_\mathcal{H} &= (U_{-T} f, U_{-T} g)_\mathcal{H} = (\mathcal{F}_- U_{-T} f, \mathcal{F}_- U_{-T} g)_{L^2} \\
&= (e^{-i\lambda T}\mathcal{F}_- f, e^{-i\lambda T}\mathcal{F}_- g)_{L^2} = (\mathcal{F}_- f, \mathcal{F}_- g)_{L^2}.
\end{aligned}
$$

Taking the closure in (3.11), we obtain the Parseval equality for the whole space $\mathcal{H}_-$. The inversion formula follows from the Parseval equality if all integrals in it are understood as limits of integrals over finite intervals. Finally, we have

$$
\mathcal{F}_-\mathcal{H}_- = \overline{\bigcup_{t\ge 0} \mathcal{F}_- U_t D_-} = \overline{\bigcup_{t\ge 0} e^{-i\lambda t} H^2_-} = L^2(\mathbb{R}),
$$

i.e., $\mathcal{F}_-$ maps $\mathcal{H}_-$ onto the whole of $L^2(\mathbb{R})$. The lemma is proved. □

We set

$$
U_\lambda^+(\xi, \zeta) = \langle S_h(\lambda) e^{-i\lambda\xi}, \frac{\alpha}{n_{\bar{h}}(\lambda)} Q(\lambda), e^{-i\lambda\zeta}\rangle.
$$

We note that the vectors $U_\lambda^+(\xi, \zeta)$ for real λ do not belong to the space $\mathcal{H}$. However, these vectors satisfy the equation $\mathcal{L}U = \lambda U$ $(\lambda \in \mathbb{R})$ and the corresponding boundary conditions for the operator $\mathcal{L}_h$. With the help of the vectors $U_\lambda^+(\xi, \zeta)$, we define a transformation $\mathcal{F}_+ : f \to \tilde{f}_+(\lambda)$ on vectors $f = \langle \varphi_-, y, \varphi_+\rangle$, where $\varphi_-(\xi)$ and $\varphi_+(\zeta)$ are compactly supported smooth functions, and $y = \{y_n\}$ $(n \in \mathbb{N})$ is a finite vector-valued sequence, by setting $(\mathcal{F}_+ f)(\lambda) := \tilde{f}_+(\lambda) := 1/\sqrt{2\pi}(f, U_\lambda^+)_\mathcal{H}$. The proof of the next result is analogous to that of Lemma 3.5.

Lemma 3.6. *The transformation $\mathcal{F}_+$ maps $\mathcal{H}_+$ isometrically onto $L^2(\mathbb{R})$, and for all vectors $f, g \in \mathcal{H}_+$, the Parseval equality and the inversion formula hold:*

$$
(f, g)_\mathcal{H} = (\tilde{f}_+, \tilde{g}_+)_{L^2} = \int_{-\infty}^{\infty} \tilde{f}_+(\lambda)\overline{\tilde{g}_+(\lambda)} d\lambda, \;\; f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}_+(\lambda) U_\lambda^+ d\lambda,
$$

where $\tilde{f}_+(\lambda) = (\mathcal{F}_+ f)(\lambda)$ and $\tilde{g}_+(\lambda) = (\mathcal{F}_+ g)(\lambda)$.

According to (3.9), the function $S_h(\lambda)$ satisfies $|S_h(\lambda)| = 1$ for $\lambda \in \mathbb{R}$. Therefore, it follows from the explicit formula for the vectors U_λ^+ and U_λ^- that

$$U_\lambda^- = \bar{S}_h(\lambda) U_\lambda^+ \quad (\lambda \in \mathbb{R}). \tag{3.12}$$

By Lemmas 3.5 and 3.6 this implies $\mathcal{H}_- = \mathcal{H}_+$. Together with Lemma 3.4 this shows that $\mathcal{H} = \mathcal{H}_- = \mathcal{H}_+$. Hence property (3) for U_t above has been established for the incoming and outgoing subspace.

Thus, the transformation $\mathcal{F}_-$ maps $\mathcal{H}$ isometrically onto $L^2(\mathbb{R})$ with the subspace D_- mapped onto H_-^2 and the operators U_t mapped to operators of multiplication by $e^{i\lambda t}$. In other words, $\mathcal{F}_-$ is the incoming spectral representation for the group $\{U_t\}$. Similarly, $\mathcal{F}_+$ is the outgoing spectral representation for $\{U_t\}$. It follows from (3.12) that the passage from the $\mathcal{F}_+$-representation of an vector $f \in \mathcal{H}$ to its $\mathcal{F}_-$-representation is realized by multiplication by the function $S_h(\lambda) : \tilde{f}_-(\lambda) = S_h(\lambda)\tilde{f}_+(\lambda)$. According to [9], the scattering matrix (function) of the group $\{U_t\}$ with respect to the subspaces D_- and D_+ is the coefficient by which the $\mathcal{F}_-$-representation of a vector $f \in \mathcal{H}$ must be multiplied in order to get the corresponding $\mathcal{F}_+$ -representation: $\tilde{f}_+(\lambda) = \bar{S}_h(\lambda)\tilde{f}_-(\lambda)$. Thus, by [9], we have now proved the following result:

Theorem 3.7. *The function $\bar{S}_h(\lambda)$ is the scattering matrix of the group $\{U_t\}$ (of the self-adjoint operator $\mathcal{L}_h$).*

Let $S(\lambda)$ be an arbitrary inner function (see [10]) on the upper half-plane. Define $\mathcal{K} = H_+^2 \ominus SH_+^2$. Then $\mathcal{K} \neq \{0\}$ is a subspace of the Hilbert space H_+^2. We consider the semigroup of the operators Z_t $(t \geq 0)$ acting in $\mathcal{K}$ according to the formula $Z_t\varphi = P\left[e^{i\lambda t}\varphi\right]$, $\varphi := \varphi(\lambda) \in \mathcal{K}$, where P is the orthogonal projection from H_+^2 onto $\mathcal{K}$. The generator of the semigroup $\{Z_t\}$ is denoted by $T : T\varphi = \lim_{t\to+0}(it)^{-1}(Z_t\varphi - \varphi)$. T is a dissipative operator acting in $\mathcal{K}$ and its domain $D(T)$ consists of all functions $\varphi \in \mathcal{K}$ for which the above limit exists. The operator T is called a *model dissipative operator* (we remark that this model dissipative operator, which is associated with the names of Lax and Phillips [9], is a special case of a more general model dissipative operator constructed by Sz.-Nagy and Foiaş [10]). We claim that $S(\lambda)$ is the characteristic function of the operator T.

Let $K = \langle 0, H, 0\rangle$ so that $\mathcal{H} = D_- \oplus K \oplus D_+$. It follows from the explicit form of the unitary transformation $\mathcal{F}_-$ that under the mapping $\mathcal{F}_-$,

$$\begin{gathered}\mathcal{H} \to L^2(\mathbb{R}), \quad f \to \tilde{f}_-(\lambda) = (\mathcal{F}_- f)(\lambda), \quad D_- \to H_-^2, \quad D_+ \to S_h H_+^2, \\ K \to H_+^2 \ominus S_h H_+^2, \quad U_t f \to (\mathcal{F}_- U_t \mathcal{F}_-^{-1}\tilde{f}_-)(\lambda) = e^{i\lambda t}\tilde{f}_-(\lambda).\end{gathered} \tag{3.13}$$

The formulas (3.13) show that the operator $\mathbf{L}_h$ is a unitary equivalent to the model dissipative operator with characteristic function $S_h(\lambda)$. Since the characteristic functions of unitary equivalent dissipative operators coincide [10], we have proved the following result:

THEOREM 3.8. *The characteristic function of the dissipative operator* $\mathbf{L}_h$ *coincides with the function* $S_h(\lambda)$ *defined in* (3.10).

Proof of Theorem 3.1. It is known that the characteristic function $S_h(\lambda)$ of a dissipative operator $\mathbf{L}_h$ carries complete information about the spectral properties of this operator (see [8], [10]). For example, the absence of a singular factor $s(\lambda)$ of the characteristic function $S_h(\lambda)$ in the factorization $S_h(\lambda) = s(\lambda)\mathcal{B}(\lambda)$ (where $\mathcal{B}(\lambda)$ is a Blaschke product) guarantees the completeness of the system of eigenvectors and associated vectors of the dissipative operators $\mathbf{L}_h$.

It is clear from the explicit formula (3.10) that the function $S_h(\lambda)$ is an inner function in the upper half-plane and, moreover, is meromorphic in the whole λ-plane. Therefore, it can be factored in the form

$$S_h(\lambda) = e^{i\lambda c}\mathcal{B}_h(\lambda), \quad c = c(h) > 0, \tag{3.14}$$

where $\mathcal{B}_h(\lambda)$ is a Blaschke product. It follows from (3.14) that

$$|S_h(\lambda)| \leq e^{-c(h)\operatorname{Im}\lambda}, \quad \operatorname{Im}\lambda \geq 0. \tag{3.15}$$

Further, expressing $n(\lambda)$ in terms of $S_h(\lambda)$, we find from (3.10) that

$$n(\lambda) = \frac{\bar{h}S_h(\lambda) - h}{S_h(\lambda) - 1}. \tag{3.16}$$

If $c(h) > 0$ for some h ($\operatorname{Im} h > 0$), then (3.15) implies that $\lim_{t\to+\infty} S_h(it) = 0$. Therefore, by (3.16) it follows that $\lim_{t\to+\infty} n(it) = h$. Since $n(\lambda)$ does not depend on h, this implies that $c(h)$ can be nonzero at no more than a single point $h = h^0$ (given by $h^0 = \lim_{t\to+\infty} n(it)$). Hence, Theorem 3.1 is proved. □

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Department of Mathematics, Suleyman Demirel University, 32260 Isparta, Turkey

E-mail address: `bilender@fef.sdu.edu.tr`