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ANOTHER APPROACH TO BITING CONVERGENCE OF JACOBIANS

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Abstract. We give new proof of the theorem of K. Zhang [Z] on biting convergence of Jacobian determinants for mappings of Sobolev class $\mathscr{W}^{1,n}(\Omega,\mathbb{R}^n)$. The novelty of our approach is in using $\mathscr{W}^{1,p}$-estimates with the exponents $1 \leqslant p < n$, as developed in [IS1], [IL], [I1]. These rather strong estimates compensate for the lack of equi-integrability. The remaining arguments are fairly elementary. In particular, we are able to dispense with both the Chacon biting lemma and the Dunford-Pettis criterion for weak convergence in $\mathscr{L}^1(\Omega)$. We extend the result to the so-called Grand Sobolev setting.

Biting convergence of Jacobians for mappings whose cofactor matrices are bounded in $\mathscr{L}^{\frac{n}{n-1}}(\mathbb{R}^n)$ is also obtained. Possible generalizations to the wedge products of differential forms are discussed.

1. Introduction and overview

Throughout this paper Ω will be a bounded open subset of $\mathbb{R}^n$, $n \geqslant 2$. We shall study vector functions $f = (f^1, f^2, \dots, f^n)\colon \Omega \to \mathbb{R}^n$ called Sobolev mappings, whose coordinates f^i, $i = 1, 2, \dots, n$ belong to certain Sobolev spaces $\mathscr{W}^{1,p_i}(\Omega)$ with $1 < p_i < \infty$. The linear differential map $\mathfrak{D}f(x)\colon \mathbb{R}^n \to \mathbb{R}^n$ is defined at almost every point $x \in \Omega$ and is represented by the Jacobi matrix, also denoted by $\mathfrak{D}f(x)$. Thus

$$\mathfrak{D}f(x) = \left[\frac{\partial f^i(x)}{\partial x_j}\right]_{i,j=1,\dots,n} \in \mathbb{R}^{n\times n}, \tag{1.1}$$

where $\mathbb{R}^{n\times n}$ denotes the space of $n \times n$ matrices. This space is equipped with the inner product $\langle A, B\rangle = \mathrm{Trace}(A^*B)$ and the induced Hilbert-Schmidt norm $\|A\| = \langle A, A\rangle^{1/2}$. Our main object of study is the Jacobian determinant

$$J(x, f) = \det \mathfrak{D}f(x)\,. \tag{1.2}$$

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Let us make a few brief comments on exterior algebra. First note that the Jacobian determinant gives rise to a wedge product of n exact 1-forms

$$J(x, f)\, dx = df^1 \wedge df^2 \wedge \cdots \wedge df^n \,. \tag{1.3}$$

Arranging these 1-forms into disjoint groups yields many more decompositions of the Jacobian, namely

$$J(x, f)\, dx = \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^m \,, \tag{1.4}$$

where the θ's are closed forms of different degree, such as

$$\theta^i = df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_l}, \qquad 1 \leqslant i_1 < \cdots < i_l \leqslant n \,. \tag{1.5}$$

Now to every pair of ordered l-tuples, $\boldsymbol{i} = (i_1, i_2, \dots, i_l)$ and $\boldsymbol{j} = (j_1, j_2, \dots, j_l)$ with $1 \leqslant i_1 < \cdots < i_l \leqslant n$ and $1 \leqslant j_1 < \cdots < j_l \leqslant n$, there corresponds a $l \times l$ sub-determinant of the differential matrix, denoted by

$$\frac{\partial f^{\boldsymbol{i}}}{\partial x_{\boldsymbol{j}}} = \frac{\partial(f^{i_1}, \dots, f^{i_l})}{\partial(x^{j_1}, \dots, x^{j_l})} \,. \tag{1.6}$$

When $l = 1$ in the above definition, we get all the entries $\partial f^i / \partial x_j$, $i, j = 1, \dots, n$, and when $\boldsymbol{i} = \boldsymbol{j} = (1, 2, \dots, n)$ we get the Jacobian determinant. Note the formula

$$df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} = \sum_{\boldsymbol{j}} \frac{\partial f^{\boldsymbol{i}}}{\partial x_{\boldsymbol{j}}} dx_{\boldsymbol{j}}, \tag{1.7}$$

where $dx_{\boldsymbol{j}} = dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_l}$, for $\boldsymbol{j} = (j_1, \dots, j_l)$.

Numerous results on Jacobian determinants which we shall discuss here remain valid for wedge products of arbitrary closed differential forms [GIM], [I1], [IL], [RRT].

There are several natural assumptions on the mapping $f = (f^1, f^2, \dots, f^n)$ under which the $\mathscr{L}^1$-theory of Jacobians can be developed. For instance, suppose the coordinate functions f^i belong to the Sobolev space $\mathscr{W}^{1,p_i}(\Omega)$, $i = 1, 2, \dots, n$, where the exponents satisfy Hölder's relation

$$1/p_1 + \cdots + 1/p_n = 1 \,.$$

In this case, $J(x, f)$ is integrable. Mappings in the Sobolev space $\mathscr{W}^{1,n}(\Omega, \mathbb{R}^n)$ may be treated as a special case of the situation just mentioned, with $p_1 = \cdots = p_n = n$. Reasoning as above, one can work with other natural settings. If, for instance, $f \in \mathscr{W}^{1,l}(\Omega, \mathbb{R}^n)$ for some $1 \leqslant l \leqslant n$, then it suffices to assume that the $l \times l$ minors $\partial f^{\boldsymbol{i}} / \partial x_{\boldsymbol{j}}$ belong to $\mathscr{L}^{n/l}(\Omega)$. Indeed, we have the pointwise estimate

$$|J(x, f)| \leqslant C(n) \sum_{\boldsymbol{i}, \boldsymbol{j}} \left| \frac{\partial f^{\boldsymbol{i}}}{\partial x_{\boldsymbol{j}}} \right|^{n/l} \in \mathscr{L}^1(\Omega) \,. \tag{1.8}$$

The $\mathscr{L}^1$-integrability properties of Jacobians are studied under much less restrictive hypotheses in [GIOV], [G2], [H], [IO], [IV], [KZ], [L], [M3], [MTS].

One important concept in this theory is that of a *weak Jacobian* [B], [G], [I2], [M4], [M5], [M6], [M7], [M8]. The weak (or distributional) Jacobian of a mapping f is a Schwartz distribution, denoted by $\mathfrak{J}_f$, which operates on the test functions $\Phi \in \mathscr{C}_0^\infty(\Omega)$ by the rule

$$\mathfrak{J}_f[\Phi] = -\int_\Omega f^1 d\Phi \wedge df^2 \wedge \cdots \wedge df^n\,. \tag{1.9}$$

The reader may guess that this definition has resulted from formal integration by parts of $\int \Phi(x)\, J(x,f)\, dx$. This integration is legitimate for $f^i \in \mathscr{W}^{1,p_i}_{\mathrm{loc}}(\Omega)$ with Hölder conjugate exponents p_i, whereas using the Sobolev inequality we see that the integral defining $\mathfrak{J}_f$ also converges if $f^i \in \mathscr{W}^{1,s_i}_{\mathrm{loc}}(\Omega)$, where the Sobolev exponents $1 \leqslant s_1, \dots, s_n < \infty$ need only satisfy the so-called Sobolev relation

$$1/s_1 + \cdots + 1/s_n = 1 + 1/n \qquad (1 \leqslant s_1 < n)\,.$$

However, in the natural setting we can use Hölder's inequality to obtain:

Proposition 1.1. *For mappings $f, g \in \mathscr{W}^{1,n}(\Omega, \mathbb{R}^n)$ and $\Phi \in \mathscr{C}_0^\infty(\Omega)$ we have*

$$\begin{aligned}\big|\mathfrak{J}_f[\Phi] - \mathfrak{J}_g[\Phi]\big| &= \left|\int_\Omega \Phi(x)[J(x,f) - J(x,g)]\, dx\right| \\ &\leqslant C(n)\, \|\nabla\Phi\|_\infty \|f-g\|_n (\|\mathfrak{D}f\|_n + \|\mathfrak{D}g\|_n)^{n-1}\,.\end{aligned}$$

This is a warm-up to an even more general estimate in Lemma 3.1. Now, as a consequence of the compactness of the embedding $\mathscr{W}^{1,n}(\Omega) \hookrightarrow \mathscr{L}^n(\Omega)$, we see that the distributional Jacobian $\mathfrak{J}\colon \mathscr{W}^{1,n}(\Omega, \mathbb{R}^n) \to \mathscr{D}'(\Omega)$ is a continuous operator with respect to the weak topology in $\mathscr{W}^{1,n}(\Omega, \mathbb{R}^n)$. Precisely, we have:

Theorem 1.2 (Weak Continuity). *If $f_k \rightharpoonup f$ weakly in $\mathscr{W}^{1,n}(\Omega, \mathbb{R}^n)$, then we have $\mathfrak{J}_{f_k} \to \mathfrak{J}_f$ in the sense of Schwartz distributions. By definition, this means that*

$$\lim_{k\to\infty} \int_\Omega \Phi(x) J(x, f_k)\, dx = \int_\Omega \Phi(x) J(x,f)\, dx \tag{1.10}$$

for every test function $\Phi \in C_0^\infty(\Omega)$.

This elegant result should be credited to R. Caccioppoli [C1] and C. B. Morrey [M1], [M2]. The utility of weak convergence of Jacobians was clearly recognized in quasiconformal geometry [GI], [IM], [R1], [R2], variational calculus [AF], [M2], [BM1] and nonlinear elasticity [A], [B], [C2]. It is very easy to see, by an approximation argument, that (1.10) remains valid for all continuous test functions vanishing on $\partial\Omega$, that is, in the space $\mathscr{C}_0(\overline{\Omega})$. We recall that $\mathscr{C}_0(\overline{\Omega})$ is dual to the space of Radon signed measures. What Theorem 1.2 tells us is that the sequence of measures $d\mu_k = J(x, f_k)\,dx$ converges weakly to $d\mu = J(x, f)\,dx$.

More generally, it may be shown that (1.10) remains valid if $\Phi \in \mathscr{VMO}(\Omega)$, the completion of $\mathscr{C}_0^\infty(\Omega)$ in the space $\mathscr{BMO}(\Omega)$ of functions of bounded mean oscillation. This is why the Jacobians actually converge in a biting sense. We only mention that the space $\mathscr{BMO}(\Omega)$ is defined by means of the norm

$$\|\Phi\|_{\mathscr{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |\Phi - \Phi_Q|\,dx\,,$$

where Q is a cube contained in Ω and Φ_Q denotes the integral mean $|Q|^{-1}\int_Q \Phi$. For details, we refer the reader to the seminal work [CLMS] and the references therein. It is natural to ask what really is the class of test functions for which (1.10) holds.

The reader may wish to note that (1.10) fails for compactly supported bounded test functions. Even finding a subsequence of $\{f_k\}$ which satisfies (1.10) may be impossible. To see this, consider $f_k(x) = h(2^k x)$ in the unit ball $\Omega = \{x : |x| < 1\}$, where h can be any smooth mapping supported in the annulus $1 < |x| < 2$ such that $J(x, h) \not\equiv 0$. This sequence converges to zero weakly in $\mathscr{W}^{1,n}(\Omega, \mathbb{R}^n)$, since for all $k \in \mathbb{N}$ we have $\|f_k\|_n = 2^{-k}\|h\|_n$ and $\|\mathfrak{D} f_k\|_n = \|\mathfrak{D} h\|_n$. On the other hand, if we take the test function $\Phi = \sum_{j \geqslant 1} J(2^j x, h)$, then $\int \Phi(x) J(x, f_k)\,dx = \int |J(x,h)|^2\,dx \not\to 0$. A concentration of mass, which in this example takes place at the origin, is exactly why some bounded sequences in $\mathscr{L}^1(\Omega)$ do not contain any weakly convergent subsequence.

A way out of this anomaly is to cut out certain parts of Ω where the concentration of mass occurs. Those parts that are cut out will be called "*bites*". This procedure is well described by Chacon's Lemma [BC], [BM2]. To formulate Chacon's Lemma, we need to introduce a new concept.

Given a sigma finite measure space (X, μ), we denote by $\mathscr{M}(X, \mu)$ the space of all μ-measurable functions on X which are finite μ-almost everywhere.

Definition 1.3. A sequence $\{J_k\}_{k=1,2,\ldots}$ in $\mathscr{M}(X, \mu)$ is said to converge to $J \in \mathscr{M}(X, \mu)$ in a *biting sense* if X can be expressed as a union of countable number of μ-measurable subsets such that $J_k \rightharpoonup J$ weakly in $\mathscr{L}^1$ on each of these subsets. We write it as

$$\lim_{k\to\infty}^{\flat} = J\,, \qquad \text{or} \qquad J_k \xrightarrow{\flat} J\,. \tag{1.11}$$

Perhaps a few comments about how to read this definition are in order. Let $X = \bigcup_{\nu=1}^\infty X_\nu$ be the countable union of μ-measurable subsets mentioned in our definition. Although we did not say it explicitly, it is understood that all J_k and the limit J lie in $\mathscr{L}^1(X_\nu)$ for each $\nu = 1, 2, \ldots$. Thus, in contrast to the original definition, here we do not require that the functions J_k are integrable on the entire space X. Certainly, it involves no loss of generality in assuming that $X_1 \subset X_2 \subset \ldots$, since otherwise we could replace these sets by the sets $X_1 \subset X_1 \cup X_2 \subset X_1 \cup X_2 \cup X_3 \cup \ldots$. The weak limit remains unaffected. On several occasions it will be convenient to consider the complements $B_\nu = X \setminus X_\nu$. It is clear that, if $\mu(X) < \infty$, then $\lim \mu(B_\nu) = 0$. Therefore, we shall call them arbitrarily small bites from X. Outside those bites $\{J_k\}$ converges weakly in $\mathscr{L}^1$ to J. It is possible to make different bites. However, they all yield the same limit function. One should be a little cautious because some familiar properties of convergent sequences do not apply to convergence in the biting sense.

The notion of biting convergence owes much of its importance to the celebrated lemma of Chacon [BC] (see also [BZ]).

LEMMA 1.4 (Brooks-Chacon). *Every bounded sequence in $\mathscr{L}^1(X)$ contains a subsequence converging in a biting sense.*

Biting convergence is extremely useful in situations when the only available information about a family of functions is its boundedness in $\mathscr{L}^1(X)$. This typically occurs in the study of variational integrals [BZ], [Z] and in geometric function theory [GI], [IM], where we have a bounded sequence of Jacobians.

THEOREM 1.5 (K. Zhang). *Every bounded sequence $\{f_k\}$ in $\mathscr{W}^{1,n}(\Omega, \mathbb{R}^n)$ contains a subsequence $\{f_{k_i}\}$ converging weakly to a mapping $f \in \mathscr{W}^{1,n}(\Omega, \mathbb{R}^n)$ such that*

$$\overset{\flat}{\lim_{i \to \infty}} J(x, f_{k_i}) = J(x, f)\,. \tag{1.12}$$

Of course, the existence of a subsequence $\{J(x, f_{k_i})\}$ converging to a certain integrable function is immediate from Chacon's lemma. But the fact that the biting limit equals $J(x, f)$ is far from being obvious. Let us mention that the biting limit of the Jacobians may not exist for the entire sequence, even if we already know that $\{f_k\}$ converges weakly in $\mathscr{W}^{1,n}(\Omega, \mathbb{R}^n)$. What causes this anomaly is the lack of equi-integrability of the Jacobians. Recall that the equi-integrability condition is one of the *Dunford-Pettis Criteria* [E] for weak compactness in $\mathscr{L}^1$.

In this note we give a proof of Theorem 1.5 based on new estimates for the Jacobian. Perhaps the simplest example to illustrate such estimates is the following inequality for $f \in \mathscr{W}^{1,n-n\varepsilon}(\Omega, \mathbb{R}^n)$, with $0 \leqslant \varepsilon \leqslant 1/(n+1)$:

$$\int_\Omega \frac{\Phi(x) J(x,f)\,dx}{|J(x,f)|^\varepsilon} \leqslant C\,\|\nabla\Phi\|_\infty \|\mathfrak{D}F\|^{n-n\varepsilon}_{\frac{n^2}{n+1}} + \varepsilon\, C\,\|\Phi\|_\infty \|\mathfrak{D}F\|^{n-n\varepsilon}_{n-n\varepsilon}, \tag{1.13}$$

where Φ is an arbitrary test function in $\mathscr{C}_0^\infty(\Omega)$; see Lemma 3.1 for an even more general estimate. The case $\varepsilon = 0$ reduces to the well known inequality

$$\mathfrak{J}_f[\Phi] = \int_\Omega \Phi(x)\, J(x,f)\, dx \leqslant C(n)\, \|\nabla\Phi\|_\infty\, \|\mathfrak{D}f\|^n_{\frac{n^2}{n+1}}$$

The point to emphasize is that the first term on the right hand side of (1.13) only requires $\mathscr{L}^{\frac{n^2}{n+1}}$-integrability of the differential. This term, when applied to the mappings f_k, will pose no difficulty since the functions $|\mathfrak{D}f_k|^{\frac{n^2}{n+1}}$ are equi-integrable. The second term requires almost $\mathscr{L}^n$-integrability of $\mathfrak{D}f_k$ as $\varepsilon \to 0$, but the factor ε will come to the rescue.

There are also variants of (1.13) for differential forms [S]. In analogy to the differential map $df = (df^1, df^2, \ldots, df^n)$ we might consider the m-tuple of rather general differential forms

$$\boldsymbol{\omega} = (\omega^1, \omega^2, \ldots, \omega^m) \in \mathscr{L}^{p_1}(\Omega, \wedge^{l_1}) \times \cdots \times \mathscr{L}^{p_m}(\Omega, \wedge^{l_m}) \tag{1.14}$$

such that

$$d\boldsymbol{\omega} = (d\omega^1, d\omega^2, \ldots, d\omega^m) \in \mathscr{L}^{s_1}(\Omega, \wedge^{l_1+1}) \times \cdots \times \mathscr{L}^{s_m}(\Omega, \wedge^{l_m+1})\,. \tag{1.15}$$

Here we assume that the exponents $1 < p_1, p_2, \ldots, p_m < \infty$ satisfy Hölder's relation $1/p_1 + 1/p_2 + \cdots + p_m = 1$, while $1 \leqslant s_1, s_2, \ldots, s_m < \infty$ are coupled by the Sobolev relation $1/s_1 + 1/s_2 + \cdots + 1/s_m = 1 + 1/n$. Note that condition (1.15) trivially holds if $\boldsymbol{\omega}$ consists of closed or exact forms. This latter case is of great interest in the theory of null Lagrangians. In any case the set of all such m-tuples of differential forms is a Banach space with respect to the norm

$$\|\boldsymbol{\omega}\|_{\boldsymbol{p},\boldsymbol{s}} = \sum_{i=1}^m \left(\|\omega^i\|_{p_i} + \|d\omega^i\|_{s_i}\right), \qquad \boldsymbol{p} = (p_1, \ldots, p_m)\,, \quad \boldsymbol{s} = (s_1, \ldots, s_m)\,.$$

We denote this space by $\mathscr{L}^{\boldsymbol{p}}_{\boldsymbol{s}}(\Omega, \wedge^{l_1} \times \cdots \times \wedge^{l_m})$. Let us state without proof the following extension of K. Zhang's Theorem.

Theorem 1.6. *Every bounded sequence $\{\boldsymbol{\omega}_k\}$ in the space $\mathscr{L}^{\boldsymbol{p}}_{\boldsymbol{s}}(\Omega, \wedge^{l_1} \times \cdots \times \wedge^{l_m})$ has a subsequence $\{\boldsymbol{\omega}_{k_i}\}$ weakly converging to $\boldsymbol{\omega} = (\omega^1, \ldots, \omega^m) \in \mathscr{L}^{\boldsymbol{p}}_{\boldsymbol{s}}(\Omega, \wedge^{l_1} \times \cdots \times \wedge^{l_m})$ such that*

$$\overset{\flat}{\lim_{i\to\infty}} \omega^1_{k_i} \wedge \omega^2_{k_i} \wedge \cdots \wedge \omega^m_{k_i} = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^m\,. \tag{1.16}$$

In another direction, we consider functions which belong to $\mathscr{L}^p(\Omega)$ for every $1 \leqslant p < n$. The *n-modulus* of a function is defined by

$$\mathfrak{L}_n(\varepsilon, f) = \sup_{0<\tau\leqslant\varepsilon} \left(\tau \int_\Omega |f|^{n-\tau}\right)^{\frac{1}{n-\tau}}\,, \qquad 0 < \varepsilon \leqslant n-1\,. \tag{1.17}$$

Then the so-called *Grand Lebesgue space*, denoted by $\mathscr{GL}^n(\Omega)$, consists of functions with bounded n-modulus. This is a Banach space furnished with the norm

$$\|f\|_{\mathscr{GL}^n(\Omega)} = \sup_{0<\varepsilon\leqslant n-1} \mathfrak{L}_n(\varepsilon, f) = \sup_{0<\tau\leqslant n-1} \left(\tau \int_\Omega |f|^{n-\tau}\right)^{\frac{1}{n-\tau}} . \tag{1.18}$$

The closure of $\mathscr{L}^n(\Omega)$ with respect to the above norm consists of functions whose n-modulus of integrability vanishes at zero. We denote this closure by $\mathscr{GL}^n_0(\Omega)$. Thus,

$$\lim_{\varepsilon\to 0} \mathfrak{L}_n(\varepsilon, f) = 0 \qquad \text{for} \qquad f \in \mathscr{GL}^n_0(\Omega)\,. \tag{1.19}$$

The restriction on the parameter τ in (1.18) is immaterial, which is clear from the inequality

$$\|f\|_{\mathscr{GL}^n} \leqslant \mathfrak{L}_n(\varepsilon, f) + C_\varepsilon(n)\,\|f\|_1 \qquad \text{for every } 0 < \varepsilon \leqslant n-1\,.$$

The *Grand Sobolev space* $\mathscr{GW}^{1,n}(\Omega, \mathbb{R}^n)$ consists of mappings $f\colon \Omega \to \mathbb{R}^n$ whose differential has vanishing n-modulus. An example of such a mapping would be an f whose differential lies in the Zygmund class $\mathscr{L}^n \log^{-1} \mathscr{L}(\Omega)$, that is

$$\int_\Omega \frac{|\mathfrak{D}f(x)|^n\,dx}{\log\big(e + |\mathfrak{D}f(x)|\big)} < \infty\,. \tag{1.20}$$

For an exposition at greater length we refer the reader to [IS1], [IS2], [G], [IKO], [GIS]. In this category of mappings we have the following biting theorem.

THEOREM 1.7. *Let $\{f_k\}$ be a bounded sequence in $\mathscr{GW}^{1,n}(\Omega, \mathbb{R}^n)$. Suppose that the derivatives have uniformly vanishing n-modulus, that is,*

$$\lim_{\varepsilon\searrow 0} \varepsilon \sup_{k\geqslant 1} \int_\Omega |\mathfrak{D}f_k(x)|^{n-n\varepsilon}\,dx = 0\,, \tag{1.21}$$

and that the Jacobian determinants are bounded in $\mathscr{L}^1(\Omega)$,

$$\sup_{k\geqslant 1} \int_\Omega |J(x, f_k)|\,dx < \infty\,. \tag{1.22}$$

Then there exists a subsequence $\{f_{k_i}\}$ converging weakly in $\mathscr{W}^{1,p}(\Omega, \mathbb{R}^n)$ (for all $1 \leqslant p < n$) to a mapping $f \in \mathscr{GW}^{1,n}(\Omega, \mathbb{R}^n)$ such that

$$\overset{\flat}{\lim_{i\to\infty}} J(x, f_{k_i}) = J(x, f)\,. \tag{1.23}$$

It is worth noting that condition (1.22) is redundant when the mappings have nonnegative Jacobian, simply because for each compact $U \subset \Omega$ we have

$$\sup_{k\geqslant 1} \int_U |J(x, f_k)|\,dx \leqslant \sup_{k\geqslant 1} \|\mathfrak{D}f_k\|_{\mathscr{GL}^n(\Omega)} < \infty\,;$$

see [IS1], [W].

THEOREM 1.8. *Let $f_k\colon \Omega \to \mathbb{R}^n$ be mappings having nonnegative Jacobian and bounded in the space $\mathscr{GW}^{1,n}(\Omega,\mathbb{R}^n)$. Then there is a subsequence $\{f_{k_i}\}_{i=1,2,\dots}$ weakly converging to f in $\mathscr{GW}^{1,n}(\Omega,\mathbb{R}^n)$ such that*

$$\lim_{i\to\infty}\int_\Omega \Phi(x)J(x,f_{k_i})\,dx = \int_\Omega \Phi(x)J(x,f)\,dx \tag{1.24}$$

for every $\Phi \in \mathscr{L}^\infty(\Omega)$ with compact support.

Let us emphasize explicitly that this theorem tell us, in particular, that the Jacobian determinants are locally integrable.

There is another interesting biting convergence result that has not yet been noted in the literature. It concerns Sobolev mappings $f_k \in \mathscr{W}^{1,n-1}(\mathbb{R}^n,\mathbb{R}^n)$ converging to f weakly in $\mathscr{W}^{1,n-1}(\mathbb{R}^n,\mathbb{R}^n)$. We assume, in addition, that the cofactor matrices

$$\mathfrak{D}^\# f_k = \left\{\frac{\partial(f_k^1,\dots,f_k^{i-1},f_k^{i+1},\dots,f_k^n)}{\partial(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)}\right\}_{\substack{i=1,\dots,n\\ j=1,\dots,n}}$$

stay bounded in $\mathscr{L}^{\frac{n}{n-1}}(\mathbb{R}^n)$, that is,

$$\int_{\mathbb{R}^n} |\mathfrak{D}^\# f_k(x)|^{\frac{n}{n-1}}\,dx \leqslant M\,, \qquad k=1,2,\dots$$

In particular, we have

$$\int_{\mathbb{R}^n} |J(x,f_k)|\,dx \leqslant M\,, \qquad k=1,2,\dots$$

as well. It has been shown in [IO] that under these conditions the Jacobians belong to the Hardy space $\mathscr{H}^1(\mathbb{R}^n)$ and, in fact, converge to $J(x,f)$ in the weak star topology of $\mathscr{H}^1(\mathbb{R}^n)$. Then, using Corollary IV.1 in [CLMS], we conclude with the following result.

THEOREM 1.9. *Let f_k converge weakly to f in $\mathscr{W}^{1,n-1}(\mathbb{R}^n,\mathbb{R}^n)$ and suppose that the cofactors of $\mathfrak{D}f_k$ stay bounded in $\mathscr{L}^{\frac{n}{n-1}}(\mathbb{R}^n)$. Then there is a subsequence of the Jacobians $J(x,f_k)$ converging to $J(x,f)$ in a biting sense.*

2. Some preliminaries

A family $\mathcal{F} \subset \mathscr{L}^1(\Omega,\mu)$ is said to be equi-integrable if

$$\lim_{M\to\infty}\sup_{f\in\mathcal{F}}\int_{|f|\geqslant M}|f|\,d\mu = 0\,. \tag{2.1}$$

Clearly, we need only verify the limit to be zero for some unbounded sequence $\{M_j\}_{j=1,2,\dots}$, that is

$$\lim_{j\to\infty} \sup_{f\in\mathcal{F}} \int_{|f|\geqslant M_j} |f|\, d\mu = 0, \tag{2.2}$$

because the function $M \mapsto \sup\{\int_{|f|\geqslant M} |f|\, d\mu : f \in \mathcal{F}\}$ is decreasing.

The lemma below shows how to extract from $\mathcal{F}$ a sort of nearly equi-integrable sequence. The proof may be adopted from [AF]; see also [BM2].

LEMMA 2.1 (Good Selection). *Every bounded sequence $\{f_j\}$ in $\mathscr{L}^1(\Omega,\mu)$ contains a subsequence, again denoted by $\{f_j\}$, such that*

$$\lim_{M\to\infty} \sup_{j\geqslant 1} \int_{M\leqslant|f_j|<2^j} |f_j|\, d\mu = 0 \tag{2.3}$$

We shall need the following result:

COROLLARY 2.2. *Let $\{f_j\}$ be the sequence in Lemma 2.1. Define the sets (later referred to as bites)*

$$B_k = \bigcup_{j>k} E_j\,, \qquad \text{where } E_j = \{x\in\Omega; |f_j| \geqslant 2^j\}\,. \tag{2.4}$$

Then for every $k = 1, 2, \dots$ we have:

- $\mu(B_k) \leqslant 2^{-k}\, C$, *where* $C = \sup\{\|f_j\|_1 : j = 1, 2, \dots\}$.
- *The family $\{f_j\}_{j=1,2,\dots}$ is equi-integrable on $\Omega\setminus B_k$ for every k, that is, outside every bite.*

REMARK 2.3. At this point, the reader may wish to appeal to the Dunford-Pettis criterion [E] to conclude that our sequence$\{f_j\}$ contains a subsequence converging in a biting sense. But we are not going to pursue this approach here.

3. The $\mathscr{W}^{1,p}$-estimate, $1 \leqslant p < n$

The following strengthening of the inequality in (1.13) is critical for our approach to the biting convergence of Jacobians. The point to make here is that this inequality involves integration of the partial derivatives in powers less than the dimension n.

LEMMA 3.1. *Let f and g be mappings in the Sobolev class $\mathscr{W}^{1,n-n\varepsilon}(\Omega,\mathbb{R}^n)$, $0 \leqslant \varepsilon \leqslant 1/(n+1)$, defined on a ball $\Omega = \{x\in\mathbb{R}^n : |x| < R\}$. Then for every* *test function* $\phi \in \mathscr{C}_0^\infty(\Omega)$ *we have*

$$\begin{aligned}\int_\Omega \phi(x)\left[\frac{J(x,f)}{|J(x,f)|^\varepsilon}-\frac{J(x,g)}{|J(x,g)|^\varepsilon}\right]dx \\ \leqslant C(n)\,R^{(n+1)\varepsilon}\|\nabla\phi\|_\infty\|f-g\|_{n^2}\left(\|\mathfrak{D}f\|_{\frac{n^2}{n+1}}+\|\mathfrak{D}g\|_{\frac{n^2}{n+1}}\right)^{n-1-n\varepsilon} \\ +\varepsilon\,C(n)\|\phi\|_\infty\int_\Omega\left(|\mathfrak{D}f(x)|^{n-n\varepsilon}+|\mathfrak{D}g(x)|^{n-n\varepsilon}\right)dx\,.\end{aligned}$$

We will actually prove a slightly stronger variant of this estimate, namely

$$\begin{aligned}\int_\Omega \frac{\phi(x)\,[J(x,f)-J(x,g)]}{(|\mathfrak{D}f(x)|^2+|\mathfrak{D}g(x)|^2)^{n\varepsilon/2}}dx \\ \leqslant C(n)\,\|(f-g)\,\nabla\phi\|_{\frac{n^2}{1+\varepsilon n(n+1)}}\left(\|\mathfrak{D}f\|_{\frac{n^2}{n+1}}+\|\mathfrak{D}g\|_{\frac{n^2}{n+1}}\right)^{n-1-n\varepsilon} \\ +\varepsilon\,C(n)\,\|\phi\,(\mathfrak{D}f-\mathfrak{D}g)\|_{n-n\varepsilon}\big(\|\mathfrak{D}f\|_{n-n\varepsilon}+\|\mathfrak{D}g\|_{n-n\varepsilon}\big)^{n-1-n\varepsilon}\,.\end{aligned}\tag{3.1}$$

But let us first show how this implies the lemma. To this effect we only need to observe the following elementary pointwise inequality:

$$\left|\frac{J(x,f)}{|J(x,f)|^\varepsilon}-\frac{J(x,g)}{|J(x,g)|^\varepsilon}-\frac{J(x,f)-J(x,g)}{\big(|\mathfrak{D}f|^2+|\mathfrak{D}g|^2\big)^{n\varepsilon/2}}\right|\leqslant\varepsilon\,C\left(|\mathfrak{D}f|^{n-n\varepsilon}+|\mathfrak{D}g|^{n-n\varepsilon}\right).$$

This is a consequence of $t^{1-\varepsilon}(1-t^\varepsilon)\leqslant\varepsilon$, for $0\leqslant t\leqslant 1$. Indeed,

$$\begin{aligned}\left|\frac{J(x,f)}{|J(x,f)^\varepsilon|}-\frac{J(x,f)}{(|\mathfrak{D}f|^2+|\mathfrak{D}g|^2)^{n\varepsilon/2}}\right| = \big(|\mathfrak{D}f|^2+|\mathfrak{D}g|^2\big)^{(n-n\varepsilon)/2}\times \\ \times\left[\frac{|J(x,f)|}{\big(|\mathfrak{D}f|^2+|\mathfrak{D}g|^2\big)^{n/2}}\right]^{1-\varepsilon}\left\{1-\left[\frac{|J(x,f)|}{\big(|\mathfrak{D}f|^2+|\mathfrak{D}g|^2\big)^{n/2}}\right]^{\varepsilon}\right\} \\ \leqslant 2^n\,\varepsilon\left(|\mathfrak{D}f|^{n-n\varepsilon}+|\mathfrak{D}g|^{n-n\varepsilon}\right).\end{aligned}$$

Similarly we argue with the terms containing $J(x,g)$. All that remains to be proven is the estimate in (3.1). Before embarking upon this task, let us consider the pair of matrices $[A,B]$ as elements of a Hilbert space $\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}$. We shall consistently use the symbol $|A|$ as the Hilbert-Schmidt norm of a matrix $A\in\mathbb{R}^{n\times n}$, $|A|^2=\mathrm{Trace}(A^*A)$. The space $\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}$ is equipped with the inner product $[A,B]\cdot[A',B']=\langle A,A'\rangle+\langle B,B'\rangle=\mathrm{Trace}(A^*A'+B^*B')$ and the induced norm $\|[A,B]\|^2=|A|^2+|B|^2$. In this way we have defined the Lebesgue space $\mathscr{L}^p(\Omega,\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n})$ of pairs of matrix functions on Ω. We then may apply Hodge decomposition to such matrix fields; see [I1] for details. Accordingly,

$$\frac{[\mathfrak{D}f,\mathfrak{D}g]}{\big(|\mathfrak{D}f|^2+|\mathfrak{D}g|^2\big)^{n\varepsilon/2}}=[\mathfrak{D}F,\mathfrak{D}G]+[A,B],\tag{3.2}$$

where $F, G \in \mathscr{W}^{1,\frac{n-n\varepsilon}{1-n\varepsilon}}(\Omega, \mathbb{R}^{n\times n})$ and the pair $[A, B]$ consists of divergence free matrix fields A, B in the space $\mathscr{L}^{\frac{n-n\varepsilon}{1-n\varepsilon}}(\Omega, \mathbb{R}^{n\times n})$. This decomposition is unique once we impose certain Neumann type boundary conditions on A and B. An especially important consequence of the uniqueness is that the components $[\mathfrak{D}F, \mathfrak{D}G]$ and $[A, B]$ are expressed by singular integrals of the left hand side. We then arrive at various $\mathscr{L}^p$-estimates, in particular:

$$\|\mathfrak{D}F\|_{\frac{n-n\varepsilon}{1-n\varepsilon}} + \|\mathfrak{D}G\|_{\frac{n-n\varepsilon}{1-n\varepsilon}} \leqslant C(n) \left(\|\mathfrak{D}f\|_{n-n\varepsilon}^{1-n\varepsilon} + \|\mathfrak{D}g\|_{n-n\varepsilon}^{1-n\varepsilon}\right) , \tag{3.3}$$

and

$$\|\mathfrak{D}F\|_{\frac{n^2}{(n+1)(1-n\varepsilon)}} + \|\mathfrak{D}G\|_{\frac{n^2}{(n+1)(1-n\varepsilon)}} \leqslant C(n) \left(\|\mathfrak{D}f\|_{\frac{n^2}{n+1}}^{1-n\varepsilon} + \|\mathfrak{D}g\|_{\frac{n^2}{n+1}}^{1-n\varepsilon}\right) . \tag{3.4}$$

It is important to realize that there is no $[A, B]$ component for $\varepsilon = 0$, because of the uniqueness in the Hodge decomposition. With the aid of complex interpolation [I1], this fact is accounted for in the following inequality:

$$\|A\|_{\frac{n-n\varepsilon}{1-n\varepsilon}} + \|B\|_{\frac{n-n\varepsilon}{1-n\varepsilon}} \leqslant \varepsilon\, C(n) \left(\|\mathfrak{D}f\|_{n-n\varepsilon}^{1-n\varepsilon} + \|\mathfrak{D}g\|_{n-n\varepsilon}^{1-n\varepsilon}\right) . \tag{3.5}$$

The details can be found in [GIM], [I1], [IL], [IM], [IS2]. Before turning to the proof of (3.1), we shall introduce the notation $M = (|\mathfrak{D}f|^2 + |\mathfrak{D}g|^2)^{n\varepsilon/2}$, and make a telescoping decomposition of the numerator:

$$\begin{aligned}
[J(x,f) - J(x,g)]dx &= df^1 \wedge \cdots \wedge df^n - dg^1 \wedge \cdots \wedge dg^n \\
&= (df^1 - dg^1) \wedge df^2 \wedge \cdots \wedge df^n \\
&\quad + dg^1 \wedge (df^2 - dg^2) \wedge df^3 \wedge \cdots \wedge df^n \\
&\quad + \cdots + dg^1 \wedge \cdots \wedge dg^{n-1} \wedge (df^n - dg^n) .
\end{aligned}$$

Accordingly, the integral in the left hand side of (3.1) splits into n integrals. Each of them will be estimated by the two terms that appear in the right hand side of (3.1). Let us put on stage only the first integral

$$\begin{aligned}
I_1 &= \int_\Omega \phi(x)\,(df^1 - dg^1) \wedge \frac{df^2}{M} \wedge df^3 \wedge \cdots \wedge df^n \\
&= \int_\Omega \phi(x)\,(df^1 - dg^1) \wedge dF^2 \wedge df^3 \wedge \cdots \wedge df^n \\
&\quad + \int_\Omega \phi(x)\,(df^1 - dg^1) \wedge A^2 \wedge df^3 \wedge \cdots \wedge df^n .
\end{aligned}$$

Here we have used a part of the decomposition in (3.2), namely $M^{-1}df^2 = dF^2 + A^2$, where dF^2 and A^2 stand for the second column vector of $\mathfrak{D}F$ and A, respectively. Notice that we have a sufficient degree of regularity to integrate by parts in the first of the two integrals. We pass the exterior derivative from the term $d(f^1 - g^1)$ into the test function ϕ. This integration by parts yields

$$|I_1| \leqslant \int_\Omega |f-g|\,|\nabla\phi|\,\big(|\mathfrak{D}F| + |\mathfrak{D}G|\big)\,\big(|\mathfrak{D}f| + |\mathfrak{D}g|\big)^{n-2} + \int_\Omega |\mathfrak{D}f - \mathfrak{D}g|\,|\phi|\,\big(|A| + |B|\big)\,\big(|\mathfrak{D}f| + |\mathfrak{D}g|\big)^{n-2}.$$

Next we apply Hölder's inequalities. To simplify the writing, we introduce the following exponents involved in this computation:

$$p = \frac{n^2}{1+\varepsilon n(n+1)}\,, \quad q = \frac{n^2}{(n+1)(1-n\varepsilon)}\,, \quad r = \frac{n^2}{(n+1)(n-2)}\,,$$
$$\alpha = n - n\,\varepsilon\,, \quad \beta = \frac{n-n\,\varepsilon}{1-n\,\varepsilon}\,, \quad \gamma = \frac{n-n\,\varepsilon}{n-2}\,.$$

Accordingly, we obtain

$$|I_1| \leqslant \big\|\,(f-g)\,\nabla\phi\,\big\|_p\,\big\|\,|\mathfrak{D}F| + |\mathfrak{D}G|\,\big\|_q \big\|(\,|\mathfrak{D}f| + |\mathfrak{D}g|\,)^{n-2}\big\|_r + \big\|\phi(\mathfrak{D}f - \mathfrak{D}g)\big\|_\alpha\,\big\||A| + |B|\big\|_\beta\,\big\|(|\mathfrak{D}f| + |\mathfrak{D}g|)^{n-2}\big\|_\gamma\,.$$

Finally, the inequalities in (3.3), (3.4) and (3.5) imply

$$|I_1| \leqslant C(n)\,\big\|(f-g)\nabla\phi\big\|_{\frac{n^2}{1+\varepsilon n(n+1)}}\,\Big(\|\mathfrak{D}f\|_{\frac{n^2}{n+1}} + \|\mathfrak{D}f\|_{\frac{n^2}{n+1}}\Big)^{n-1-n\varepsilon} + \varepsilon\,C(n)\,\big\|\phi(\mathfrak{D}f - \mathfrak{D}g)\big\|_{n-n\varepsilon}\,\big(\|\mathfrak{D}f\|_{n-n\varepsilon} + \|\mathfrak{D}g\|_{n-n\varepsilon}\big)^{n-1-n\varepsilon}\,.$$

This is what we had to show. The proof of Lemma 3.1 is therefore complete.

4. Proof of biting theorem

We begin by selecting a suitable subsequence $\{f_{\nu_k}\}$ and arbitrarily small bites $B_1, B_2, \cdots$ from Ω. With the aid of Corollary 2.2, we are able to make a selection such that the Jacobians $J(x, f_{\nu_k}), k = 1, 2, \cdots$, form an equi-integrable family outside each of these bites. Note that although the hypotheses of Theorem 1.5 ensure $\mathscr{L}^1$-boundedness of the sequence $\{|\mathfrak{D}f_\nu|^n\}$, so far we have been using $\mathscr{L}^1$-boundedness of the Jacobians only. It is instructive to mention in advance that in the forthcoming arguments we shall not make use of the $\mathscr{L}^n$-integrability of the differentials, but we will use the boundedness of the sequence $\{\mathfrak{D}f_\nu\}$ in $\mathscr{L}^p(\Omega, \mathbb{R}^{n\times n})$ for some $n > p > n^2/(n+1)$. This is sufficient to claim compactness of the embedding $\mathscr{W}^{1,p}(\Omega, \mathbb{R}^n) \hookrightarrow \mathscr{L}^{n^2}(\Omega, \mathbb{R}^n)$. We can thereby assume without loss of generality that the sequence $\{f_{\nu_k}\}$ converges to f strongly in $\mathscr{L}^{n^2}(\Omega, \mathbb{R}^n)$. Moreover, if we confine ourselves to a further subsequence, again denoted by $\{f_{\nu_k}\}$, then we can also assume that

$$\lim_{k\to\infty} k^{n+1}\|f_{\nu_k} - f\|_{n^2} = 0 \tag{4.1}$$

To settle matters finally, we assume that $\nu_k = k$, for notational simplicity. This involves no loss of generality because we can always enumerate the indices as necessary. It will be convenient to assume that the complement of each bite is compactly contained in Ω. This is legitimate as we can always make additional bites near $\partial\Omega$. Our ultimate goal is to show that $J(x,f_k) \to J(x,f)$ weakly in $\mathscr{L}^1(\Omega\setminus B)$, where B stands for one of the bites $B_1, B_2, \cdots$ Fix a test function $\eta \in \mathscr{L}^\infty(\Omega\setminus B)$. We may assume that $|\eta(x)| \leqslant \chi_{\Omega\setminus B}(x)$, where $\chi_{\Omega\setminus B}$ stands for the characteristic function of $\Omega\setminus B$. In particular, η is supported in a compact subset of Ω. For each $0<\varepsilon<1$, we can write

$$\int_\Omega \eta(x)[J(x,f_k)-J(x,f)] = \int_\Omega \eta(x)\left[\frac{J(x,f_k)}{|J(x,f_k)|^\varepsilon} - \frac{J(x,f)}{|J(x,f)|^\varepsilon}\right] + \int_\Omega \eta(x)\left[J(x,f_k) - \frac{J(x,f_k)}{|J(x,f_k)|^\varepsilon}\right] - \int_\Omega \eta(x)\left[J(x,f) - \frac{J(x,f)}{|J(x,f)|^\varepsilon}\right]. \tag{4.2}$$

At this point, we need the following result:

Lemma 4.1. *Let $\mathcal{F} \subset \mathscr{L}^1(\Omega\setminus E)$ be an equi-integrable family. Then*

$$\lim_{\varepsilon\to 0}\sup_{J\in\mathcal{F}}\int_{\Omega\setminus B}\left|J-|J|^{-\varepsilon}J\right| = 0\,. \tag{4.3}$$

Proof. Here is perhaps the simplest way of seeing this result. Given any number $M \geqslant 1$, we define

$$\Theta(M) = \sup_{J\in\mathcal{F}}\int_{|J|>M}|J|\,.$$

Equi-integrability of $\mathcal{F}$ tells us that $\Theta(M)\to 0$ as M increases to infinity. For each $J\in\mathcal{F}$ we have the uniform estimate (independent of J)

$$\begin{aligned}\int_{\Omega\setminus B}\left|J-|J|^{-\varepsilon}J\right| &\leqslant \int_{|J|\leqslant M}\left|J-|J|^{-\varepsilon}J\right| + \int_{|J|>M}|J| \\ &\leqslant (\varepsilon + M - M^{1-\varepsilon})|\Omega| + \Theta(M)\,.\end{aligned}$$

This follows from the elementary inequality $|J-|J|^{-\varepsilon}J| \leqslant \varepsilon$, if $|J|\leqslant 1$ and $|J-|J|^{-\varepsilon}J| \leqslant M - M^{1-\varepsilon}$, if $1\leqslant |J| \leqslant M$. Letting ε go to zero yields

$$\overline{\lim_{\varepsilon\to 0}}\sup_{J\in\mathcal{F}}\int\left|J-|J|^{-\varepsilon}J\right| \leqslant \Theta(M)\,. \tag{4.4}$$

As M can be arbitrarily large, we conclude the proof of Lemma 4.1. □

Now we return to our proof. Lemma 4.1 shows that the last two integrals in (4.2) tend to zero uniformly in k as $\varepsilon\to 0$. The last two integrals of (4.2) will go to zero no matter what the ε_k are. Therefore it suffices to find positive numbers ε_k converging to zero and such that

$$\lim_{k\to\infty}\int_\Omega \eta(x)\left[\frac{J(x,f_k)}{|J(x,f_k)|^{\varepsilon_k}} - \frac{J(x,f)}{|J(x,f)|^{\varepsilon_k}}\right]dx = 0\,. \tag{4.5}$$

As we shall see the sequence ε_k will depend on the test function η. To this effect, we approximate η by the functions $\eta_k = \eta * \Phi_k \in \mathscr{C}_0^\infty(\Omega)$, where $\Phi_k(y) = k^n \Phi(ky)$, for large k. As always, the mollifying function $\Phi \in \mathscr{C}_0^\infty(\mathbb{B})$ is supported in the unit ball $\mathbb{B}$ and has integral 1. We then have

$$\|\eta_k\|_\infty \leqslant \|\eta\|_\infty \leqslant 1\,, \tag{4.6}$$

$$\|\nabla \eta_k\|_\infty \leqslant \|\nabla \Phi_k\|_\infty \|\eta\|_\infty \leqslant C\, k^{n+1}\,. \tag{4.7}$$

It is well known that, for every exponent $1 \leqslant p < \infty$, $\|\eta - \eta_k\|_p \to 0$. Now we are in a position to define the sequence $\{\varepsilon_k\}$. It must decrease to zero slowly enough to satisfy

$$\lim_{k\to\infty} \|\eta - \eta_k\|_{p_k} = 0\,, \qquad \text{where } p_k = 1/\varepsilon_k\,. \tag{4.8}$$

Let us split the integral in (4.2) in accordance with the decomposition $\eta = (\eta - \eta_k) + \eta_k$:

$$\int_\Omega (\eta - \eta_k)\left[\frac{J(x,f_k)}{|J(x,f_k)|^{\varepsilon_k}} - \frac{J(x,f)}{|J(x,f)|^{\varepsilon_k}}\right] dx + \int_\Omega \eta_k \left[\frac{J(x,f_k)}{|J(x,f_k)|^{\varepsilon_k}} - \frac{J(x,f)}{|J(x,f)|^{\varepsilon_k}}\right] dx\,.$$

By Hölder's inequality with the pair of exponents p_k and $1/(1-\varepsilon_k)$, the first integral is bounded by $\|\eta - \eta_k\|_{p_k}\big(\|J(x,f_k)\|_1^{1-\varepsilon_k} + \|J(x,f)\|_1^{1-\varepsilon_k}\big)$ and, therefore, goes to zero. It remains to estimate the second integral. Lemma 3.1 gives the following bound:

$$C(n)\, R^{n\varepsilon_k+\varepsilon_k} \|\nabla\eta_k\|_\infty \|f_k - f\|_{n^2} \left(\|\mathfrak{D} f_k\|_{\frac{n^2}{n+1}} + \|\mathfrak{D} f\|_{\frac{n^2}{n+1}}\right)^{n-1-n\varepsilon_k} + \varepsilon_k\, C(n)\, \|\eta_k\|_\infty \int_\Omega \big(|\mathfrak{D} f_k|^{n-n\varepsilon_k} + |\mathfrak{D} f|^{n-n\varepsilon_k}\big)\,.$$

The first term goes to zero, as the sequence $\{|\mathfrak{D} f_k|\}$ is bounded in $\mathscr{L}^{\frac{n^2}{n+1}}(\Omega)$ and $\|\nabla\eta_k\|_\infty \|f_k - f\|_{n^2} \leqslant Ck^{n+1}\|f_k - f\|_{n^2} \to 0$, by (4.7) and (4.1). To complete the proof, we need only observe that $\|\eta_k\|_\infty \leqslant 1$ and

$$\lim_{k\to\infty} \varepsilon_k \int_\Omega \big(|\mathfrak{D} f_k|^{n-n\varepsilon_k} + |\mathfrak{D} f|^{n-n\varepsilon_k}\big) = 0\,. \tag{4.9}$$

This is trivially true since the n-norms $\|\mathfrak{D} f_k\|_n$ stay bounded as $k \to \infty$. This completes the proof of Theorem 1.5.

These lines of reasoning can be modified to prove Theorem 1.7. In addition to what has already been said, we notice that it is not necessary to assume boundedness of $\{|\mathfrak{D} f_k|\}$ in $\mathscr{L}^n(\Omega)$. Equation (4.9) remains true for sequences with uniformly vanishing n-modulus, establishing Theorem 1.7.

References

[A] S.S. Antman, *Fundamental mathematical problems in the theory of nonlinear elasticity*, North Holland, Amsterdam, 1976.

[AF] E. Acerbi and N. Fusco, *Semicontinuity problems in the calculus of variations*, Arch. Rational Mech. Anal. **86** (1984), 125–145.

[B] J.M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal. **63** (1977), 337–403.

[BM1] J.M. Ball and F. Murat, *$\mathscr{W}^{1,p}$-quasiconvexity and variational problems for multiple integrals*, J. Funct. Anal. **58** (1984), 225–253.

[BM2] ———, *Remarks on Chacon's Biting Lemma*, Proc. Amer. Math. Soc. **107** (1989), 655–663.

[BC] J.K. Brooks and R.V. Chacon, *Continuity and compactness of measures*, Adv. in Math. **37** (1980), 16–26.

[BZ] J.M. Ball and K.W. Zhang, *Lower semicontinuity of multiple integrals and the biting lemma*, Proc. Roy. Soc. Edinburgh Sec. A **114** (1990), 367–379.

[C1] R. Caccioppoli, *Funzioni pseudo-analitiche e rappresentazioni pseudo-conformi delle superficie Riemanniane*, Ricerche Mat. **2** (1953), 104–127.

[C2] P.G. Ciarlet, *Mathematical elasticity*, North Holland, Amsterdam, 1987.

[CLMS] R. Coifman, P.L. Lions, Y. Meyer, and S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. **72** (1993), 247–286.

[E] R.E. Edwards, *Functional analysis*, Holt, Rinehart and Winston, New York, 1965.

[GI] F.W. Gehring and T. Iwaniec, *The limit of mappings with finite distortion*, Ann. Acad. Sci. Fenn. Math. **24** (1999), 253–264.

[G] L. Greco, *A remark on the equality* $\operatorname{Det} Df = \det Df$, Differential Integral Equations **6** (1993), 1089–1100.

[G2] ———, *Sharp integrability of nonnegative Jacobians*, Rend. Mat. Appl. (7) **18** (1998), 585–600.

[GIM] L. Greco, T. Iwaniec, and G. Moscariello, *Limits of the improved integrability of the volume forms*, Indiana Univ. Math. J. **44** (1995), 305–339.

[GIS] L. Greco, T. Iwaniec, and C. Sbordone, *Inverting the p-Harmonic Operator*, Manuscripta Math. **92** (1997), 249–258.

[GIOV] F. Giannetti, T. Iwaniec, J. Onninen, and A. Verde, *$\mathscr{L}^1$-estimates of the Jacobian by subdeterminants*, J. Geom. Anal. **12** (2002), 223–254.

[H] P. Hajłasz, *A note on weak approximation of minors*, Ann. Inst. Henri Poincaré Anal. Non Linéaire **12** (1995), 415–424.

[I1] T. Iwaniec, *Integrability theory of the Jacobians*, Lipschitz Lectures, preprint, Univ. Bonn, Sonderforschungsbereich 256, 1995.

[I2] ———, *On the concept of the weak Jacobian and Hessian*, Papers on analysis: A volume dedicated to Olli Martio, Rep. Univ. Jyväskylä, vol. 83, Univ. Jyväskylä, Jyväskylä, 2001, pp. 181–205.

[IKO] T. Iwaniec, P. Koskela, and J. Onninen, *Mappings of finite distortion: monotonicity and continuity*, Invent. Math. **144** (2001), 507–531.

[IL] T. Iwaniec and A. Lutoborski, *Integral estimates for null Lagrangians*, Arch. Rational Mech. Anal. **125** (1993), 25–79.

[IM] T. Iwaniec and E. Martin, *Geometric function theory and nonlinear analysis*, Oxford University Press, Oxford, 2001.

[IO] T. Iwaniec and J. Onninen, *$\mathscr{H}^1$-estimates of the Jacobian by subdeterminants*, Math. Ann. **324** (2002), 341–358.

[IS1] T. Iwaniec and C. Sbordone, *On the integrability of the Jacobian under minimal hypotheses*, Arch. Rational Mech. Anal. **119** (1992), 129–143.

[IS2] ———, *Weak minima of variational integrals*, J. Reine Angew. Math. **454** (1994), 143–161.

[IV] T. Iwaniec and A. Verde, *A study of Jacobians in Hardy-Orlicz spaces*, Proc. Roy. Soc. Edinburgh Sect. A **129** (1999), 539–570.

[KZ] P. Koskela and X. Zhong, *Minimal assumptions for the integrability of the Jacobian*, Ricerche Mat., to appear.

[L] P.-L. Lions, *Jacobians and Hardy spaces*, Ricerche Mat. suppl. **40** (1991).

[M1] C.B. Morrey, *Quasiconvexity and the semicontinuity of multiple integrals*, Pacific J. Math. **2** (1952), 25–53.

[M2] ———, *Multiple lntegrals in the calculus of variations*, Springer Verlag, New York, 1966.

[M3] G. Moscariello, *On the integrability of the Jacobian in Orlicz spaces*, Math. Japonica **40** (1992), 323–329.

[M4] S. Müller, *Weak continuity of determinants and nonlinear elasticity*, C.R. Acad. Sci. Paris **307** (1988), 501–506.

[M5] ———, Det = det*, a remark on the distributional determinant*, C.R. Acad. Sci. Paris **311** (1990), 13–17.

[M6] ———, *Higher integrability of determinants and weak convergence in* L^1, J. Reine Angew. Math. **412** (1990), 20–34.

[M7] ———, *A counter-example concerning formal integration by parts*, C.R. Acad. Sci. Paris **312** (1991), 45–49.

[M8] ———, *On the singular support of the distributional determinant*, Ann. Inst. H. Poincaré Anal. Non Linéaire **10** (1993), 657–698.

[MTS] S. Müller, Q. Tang, and B.S. Yan, *On a new class of elastic deformations not allowing for cavitation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **11** (1994), 217–243.

[R1] Yu. G. Reshetnyak, *On the stability of conformal mappings in multidimensional space*, Siberian Math. J. **8** (1967), 69–85.

[R2] ———, *Space mappings with bounded distortion*, Translations of Mathematical Monographs, vol. 73, American Mathematical Society, Providence, RI, 1989.

[RRT] J.W. Robin, R.C. Rogers, and B. Temple, *On weak continuity and Hodge decomposition*, Trans. Amer. Math. Soc. **303** (1987), 609–618.

[S] U. Subramanian, *Biting convergence of Null-Lagrangians*, Doctoral Dissertation, 2003.

[W] S. Wu, *On higher integrability of nonnegative Jacobian*, preprint.

[Z] K. Zhang, *Biting theorems for the Jacobians and their applications*, Ann. Inst. Henri Poicaré **7** (1990), 345–365.

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