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A WEAK QUALITATIVE UNCERTAINTY PRINCIPLE FOR COMPACT GROUPS

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ABSTRACT. For locally compact abelian groups it is known that if the product of the measures of the support of an L^1-function f and its Fourier transform is less than 1, then $f = 0$ almost everywhere. This is a weak version of the classical qualitative uncertainty principle. In this paper we focus on compact groups. We obtain conditions on the structure of a compact group under which there exists a lower bound for all products of the measures of the support of an integrable function and its Fourier transform, and conditions under which this bound equals 1. For several types of compact groups, we determine the exact set of values which the product can attain.

1. Introduction

Let G be a separable unimodular locally compact group of type I equipped with a left Haar measure m_G. Let $\widehat{G}$ denote the dual space of G, i.e., the set of all equivalence classes of irreducible unitary representations, and let μ_G be the Plancherel measure on $\widehat{G}$. For $\pi \in \widehat{G}$, we denote the associated representation space by $\mathcal{H}_\pi$, and let d_π be its dimension. The Fourier transform $\hat{f}$ of a function $f \in L^1(G)$ is defined by

$$\langle \hat{f}(\pi)\xi, \eta\rangle = \int_G f(x)\langle \pi(x^{-1})\xi, \eta\rangle dm_G(x),$$

where $\pi \in \widehat{G}$, $\xi, \eta \in \mathcal{H}_\pi$, and $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{H}_\pi$. For $f \in L^1(G)$, we let $A_f = \{x \in G : f(x) \neq 0\}$ and $B_f = \{\pi \in \widehat{G} : \hat{f}(\pi) \neq 0\}$.

In this paper we consider qualitative uncertainty principles for compact groups. Generally speaking, an uncertainty principle shows that a nonzero function and its Fourier transform cannot both be sharply localized. There exists an abundance of special types of uncertainty principles. For an excellent survey we refer to [5]. By *qualitative* uncertainty principle we mean one which,

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without giving quantitative estimates, shows that a function and its Fourier transform cannot both be too localized unless the function equals zero.

The first qualitative uncertainty principle of the type we want to discuss here was derived in 1973 by Matolcsi and Szücs [11] and states the following: Given a locally compact abelian group G, for $f \in L^2(G)$ we have

$$m_G(A_f)\mu_G(B_f) < 1 \quad \Longrightarrow \quad f = 0 \quad \text{a.e..}$$

For L^1-functions this result was established by Smith [15]. On $\mathbb{R}^n$ a much stronger result is true. In 1985 Benedicks [1] proved that for $f \in L^1(\mathbb{R}^n)$,

$$m_{\mathbb{R}^n}(A_f) < \infty \quad \text{and} \quad \mu_{\mathbb{R}^n}(B_f) < \infty \quad \Longrightarrow \quad f = 0 \quad \text{a.e..}$$

One formulation of the qualitative uncertainty principle which seems to be the right setting for a large class of locally compact groups G and which will be referred to as the QUP is the following: G is said to satisfy the *QUP* if, for all $f \in L^1(G)$,

$$m_G(A_f) < m_G(G) \quad \text{and} \quad \mu_G(B_f) < \mu_G(\widehat{G}) \quad \Longrightarrow \quad f = 0 \quad \text{a.e..}$$

Hogan [7] proved that the QUP holds for a non-compact non-discrete locally compact abelian group with connected component G_0 if and only if G_0 is non-compact. Hogan [8] also showed that an infinite compact group satisfies the QUP if and only if it is connected. There exists an abundance of generalizations of these results; see, e.g., [2], [13], [3], [9], [14].

It is natural to ask whether there exists a weaker version of the QUP, which is less restrictive. To this end, we consider the principle stated by Matolcsi and Szücs [11], which can be formulated for all separable unimodular locally compact groups G of type I. We say that such a group G satisfies the *weak QUP* if, for each $f \in L^1(G)$,

$$m_G(A_f)\mu_G(B_f) < 1 \quad \Longrightarrow \quad f = 0 \quad \text{a.e..}$$

The expectation is that this condition is satisfied by many more groups than the QUP. Indeed, each locally compact abelian group satisfies the weak QUP even though it may not satisfy the QUP (cf. [11], [7] and [8]). In this paper we focus on compact groups and study the weak QUP and related properties.

In Section 2 we state some basic results which will be needed in the sequel. In Section 3 we characterize exactly the weak QUP for a compact group G in terms of the group structure of G (Theorem 1). If G does not satisfy the weak QUP, it is an interesting question whether there still exists a lower bound for the product of the measures of the support of an integrable function and its Fourier transform. We give a sufficient condition for the existence of such a lower bound (Theorem 2), and we even obtain an explicit bound. Moreover, we show that this condition is also necessary under a certain hypothesis on the structure of G, and we describe a class of compact groups which satisfy this hypothesis (Proposition 3.1).

In Section 4 we investigate the question which values can be attained by the product $m_G(A_f)\mu_G(B_f)$, where G is a compact group and $f \in L^1(G)$. Knowing the exact set of these values would help us keep the time-frequency localization of the function under control. In Section 4.1 we consider the question of whether the lower bounds for $m_G(A_f)\mu_G(B_f)$ obtained in the two theorems are sharp, and in Section 4.2 we determine the exact set of possible values which are attained by this product for several types of compact groups.

2. Basic results

Let G be a compact group. We will always normalize m_G so that $m_G(G) = 1$. The Plancherel measure μ_G, which is the unique measure on $\widehat{G}$ such that for any $f \in L^1(G) \cap L^2(G)$

$$\int_G |f(x)|^2 dm_G(x) = \int_{\widehat{G}} \mathrm{tr}[\hat{f}(\pi)^* \hat{f}(\pi)] d\mu_G(\pi),$$

is then given by

$$\mu_G(F) = \sum_{\pi \in F} d_\pi \quad \text{for every subset } F \subseteq \widehat{G}.$$

Here $\mathrm{tr}[\cdot]$ denotes the trace of an operator.

We let $1_{\mathcal{H}_\pi}$ be the identity operator on a Hilbert space $\mathcal{H}_\pi$ and χ_E the characteristic function of a measurable subset E of G. If M is a finite set, the number of elements of M is denoted by $|M|$. Let G_0 denote the connected component of the identity in G. The annihilator of a closed subgroup H of G in $\widehat{G}$ is defined by

$$A(H, \widehat{G}) = \{\pi \in \widehat{G} : \pi(h) = 1_{\mathcal{H}_\pi} \text{ for all } h \in H\}.$$

If H is a closed normal subgroup, $A(H, \widehat{G})$ can be identified with $\widehat{G/H}$ (see [6, Corollary 28.10]). For more information on Fourier analysis on compact groups we refer to Folland [4].

In the sequel we will be often dealing with functions $f \in L^1(G)$ which are constant on cosets of some closed normal subgroup. In order to determine B_f we need to know the Fourier transform of f. The following lemma is folklore, but since we could not find a suitable reference, we provide a short proof.

Lemma 2.1. *Let G be a compact group, let H be a closed normal subgroup of G and let $\varphi : G \to G/H$ denote the quotient map. Further, let $f \in L^1(G)$ be such that there exists a function $g \in L^1(G/H)$ with $f(x) = g(\varphi(x))$. Then, for $\pi \in \widehat{G}$ and $\xi, \eta \in \mathcal{H}_\pi$ we have*

$$\langle \hat{f}(\pi)\xi, \eta\rangle = \chi_{A(H,\widehat{G})}(\pi)\langle \hat{g}(\pi)\xi, \eta\rangle.$$

Proof. Using Weil's formula, the Schur orthogonality relations and the fact that unitary representations of compact groups are direct sums of irreducible representations (see [4, Theorem 5.2]), we obtain

$$\langle \hat{f}(\pi)\xi, \eta\rangle = \int_{G/H} g(yH)\chi_{A(H,\widehat{G})}(\pi)\langle \pi(y^{-1})\xi, \eta\rangle dm_{G/H}(yH).$$

If $\pi \notin A(H, \widehat{G})$, we have $\hat{f}(\pi) = 0$. If $\pi \in A(H, \widehat{G})$, then

$$\langle \hat{f}(\pi)\xi, \eta\rangle = \langle \hat{g}(\pi)\xi, \eta\rangle. \qquad \square$$

The next two lemmas will be used throughout the proof of Theorems 1 and 2.

LEMMA 2.2. *Let G be a compact Lie group and let $f \in L^1(G)$, $f \neq 0$. Then there exists a function g on G/G_0, $g \neq 0$, such that*

$$m_G(A_f)\mu_G(B_f) \geq m_{G/G_0}(A_g)\mu_{G/G_0}(B_g).$$

Proof. Let $f \in L^1(G)$, $f \neq 0$, and let $\{x_i : i = 1, \ldots, [G : G_0]\}$ be a representative system for the G_0-cosets in G. We define g on G/G_0 by $g(x_i) = \int_{G_0} f(x_i h) dm_{G_0}(h)$ and $k \in L^1(G)$ by $k(x) = g(\varphi(x))$, where $\varphi : G \to G/G_0$ is the quotient map. Without loss of generality we can assume that $\mu_G(B_f) < \infty$. This means precisely that f equals a trigonometric polynomial almost everywhere. Since G is also a Lie group, such a function f must be analytic. Let $x \in G$ and consider the function $f|_{xG_0}$. This is also an analytic function, which is defined on a connected set. But nonzero analytic functions, defined on a connected set, cannot vanish on a set of positive measure. This shows that for each $x \in G$ we have either $f|_{xG_0} \neq 0$ a.e. or $f|_{xG_0} \equiv 0$. Thus, by the definition of the function k, $A_k \subseteq A_f$ and hence $m_G(A_f) \geq m_G(A_k)$. The normalization of the measures m_G and m_{G/G_0} implies that $m_G(A_k) = m_{G/G_0}(A_g)$.

To complete the proof, we now show that $\mu_G(B_f) \geq \mu_{G/G_0}(B_g)$. Using Weil's formula, we obtain, for each $\pi \in \widehat{G}$ and $\xi, \eta \in \mathcal{H}_\pi$,

$$\begin{aligned}
&\langle \hat{f}(\pi)\xi, \eta\rangle \\
&= \int_G f(x)\langle \pi(x^{-1})\xi, \eta\rangle dm_G(x) \\
&= \frac{1}{[G:G_0]} \sum_{i=1}^{[G:G_0]} \int_{G_0} f(x_i h)\langle \pi(h^{-1})\pi(x_i^{-1})\xi, \eta\rangle dm_{G_0}(h) \\
&= \frac{1}{[G:G_0]} \begin{cases} \sum_{i=1}^{[G:G_0]} \int_{G_0} f(x_i h) dm_{G_0}(h)\langle \pi(x_i^{-1})\xi, \eta\rangle & \text{if } \pi \in A(G_0, \widehat{G}), \\ \sum_{i=1}^{[G:G_0]} \langle \widehat{f(x_i\, \cdot)}(\pi)(\pi(x_i^{-1})\xi), \eta\rangle & \text{if } \pi \notin A(G_0, \widehat{G}). \end{cases}
\end{aligned}$$

This shows that $\langle \hat{f}(\pi)\xi, \eta\rangle = \langle \hat{k}(\pi)\xi, \eta\rangle$ for all $\pi \in A(G_0, \widehat{G})$ and $\xi, \eta \in \mathcal{H}_\pi$. Applying Lemma 2.1 yields that, for each $\pi \in \widehat{G}$ and $\xi, \eta \in \mathcal{H}_\pi$,

$$\langle \hat{k}(\pi)\xi, \eta\rangle = \chi_{A(G_0,\widehat{G})}(\pi)\langle \hat{g}(\pi)\xi, \eta\rangle.$$

Thus, by the structure of the Plancherel measure, we obtain $\mu_G(B_f) \geq \mu_{G/G_0}(B_g)$. □

LEMMA 2.3. *Let G be a compact group and let $f \in L^1(G)$, $f \neq 0$. Then there exist a closed normal subgroup H of G such that G/H is Lie and a function $g \in L^1(G/H)$ such that*

$$m_G(A_f)\mu_G(B_f) = m_{G/H}(A_g)\mu_{G/H}(B_g).$$

Proof. Each compact group is a projective limit of Lie groups (see [6, 28.61 (c)]), i.e., there exists a system $\mathcal{L}$ of closed normal subgroups H of G, which is downwards directed and satisfies $\bigcap_{H\in\mathcal{L}} H = \{e\}$, such that G/H is a compact Lie group for every $H \in \mathcal{L}$. Moreover, $\widehat{G}$ is the corresponding injective limit of the annihilators $A(H, \widehat{G})$, $H \in \mathcal{L}$. Let $f \in L^1(G)$, $f \neq 0$, with $\mu_G(B_f) < \infty$. By the Fourier inversion formula, f can be represented as follows:

$$f(x) = \sum_{i=1}^{n} d_{\pi_i} \mathrm{tr}[\hat{f}(\pi_i)\pi_i(x)].$$

Now there exists a subgroup $H \in \mathcal{L}$ such that $\pi_i \in A(H, \widehat{G})$ for all $1 \leq i \leq n$. For $h \in H$ we have $f(xh) = f(x)$ since $\pi_i(h) = 1_{\mathcal{H}_{\pi_i}}$. Let $\varphi : G \to G/H$ be the canonical quotient map and define $g \in L^1(G/H)$ by $g(\varphi(x)) = f(x)$. Then $m_G(A_f) = m_{G/H}(A_g)$, since m_G and $m_{G/H}$ were chosen to be normalized.

To prove that $\mu_G(B_f) = \mu_{G/H}(B_g)$, let $\pi \in \widehat{G}$ and let $\xi, \eta \in \mathcal{H}_\pi$. Lemma 2.1 implies that

$$\langle \hat{f}(\pi)\xi, \eta\rangle = \chi_{A(H,\widehat{G})}(\pi)\langle \hat{g}(\pi)\xi, \eta\rangle.$$

Employing now the structure of the Plancherel measure yields $\mu_G(B_f) = \mu_{G/H}(B_g)$. □

3. The weak QUP and related properties

Let G be a compact group. We first characterize the weak QUP in terms of the group structure of G. Our criterion for the weak QUP is satisfied by a larger set of compact groups than just the connected groups. Thus the weak QUP is indeed much less restrictive than the QUP.

THEOREM 1. *Let G be a compact group. The following conditions are equivalent.*

(i) *G satisfies the weak QUP.*
(ii) *G/G_0 is abelian.*

Proof. Let G be a compact group. To obtain a contradiction we assume that G/G_0 is non-abelian. Since G/G_0 is also totally disconnected, there exists an open normal subgroup C of G/G_0 such that $(G/G_0)/C$ is non-abelian. Let H be the pullback of C to G. Then G/H is finite and non-abelian. We define $f \in L^1(G)$ by $f = \chi_H$. Then, since $m_G(G) = 1$, we have $m_G(A_f) = [G : H]^{-1}$. In order to calculate $\mu_G(B_f)$, let $\pi \in \widehat{G}$ and $\xi, \eta \in \mathcal{H}_\pi$. Then, by Lemma 2.1,

$$\langle \hat{f}(\pi)\xi, \eta\rangle = \frac{1}{[G : H]} \chi_{A(H,\widehat{G})}(\pi)\langle \xi, \eta\rangle.$$

Let $A(H, \widehat{G})$ be identified with $\widehat{G/H}$. The definition of the Plancherel measure implies $\mu_G(B_f) = \sum_{\pi \in \widehat{G/H}} d_\pi$. Since G/H is non-abelian, there exists at least one element $\pi \in \widehat{G/H}$ with $d_\pi > 1$. Thus $\sum_{\pi \in \widehat{G/H}} d_\pi < \sum_{\pi \in \widehat{G/H}} d_\pi^2$. Since G/H is a finite group, we have $[G : H] = \sum_{\pi \in \widehat{G/H}} d_\pi^2$ (see [4, Proposition 5.27]). This shows that $\mu_G(B_f) < [G : H]$, which in turn implies $m_G(A_f)\mu_G(B_f) < 1$. This proves the implication (i) $\Rightarrow$ (ii).

Now suppose (ii) holds. We need to show that then G satisfies the weak QUP. This will be achieved by first reducing to the case of compact Lie groups and then to the case of finite groups.

Let G be an arbitrary compact group and let $f \in L^1(G)$, $f \neq 0$. Lemma 2.3 implies that there exist a closed normal subgroup H such that G/H is Lie and a function $g \in L^1(G/H)$ such that

$$m_G(A_f)\mu_G(B_f) = m_{G/H}(A_g)\mu_{G/H}(B_g).$$

Note that $G/G_0H = (G/H)/(G_0H/H)$ and, since G_0H/H is connected and open in G/H, we have $G_0H/H = (G/H)_0$. By hypothesis, G/G_0 is abelian. Thus $(G/H)/(G/H)_0$ is also abelian. Hence we can assume that G is a compact Lie group. In this situation we may apply Lemma 2.2, which shows the existence of a function $g \in L^1(G/G_0)$, $g \neq 0$, such that

$$m_G(A_f)\mu_G(B_f) \geq m_{G/G_0}(A_g)\mu_{G/G_0}(B_g).$$

Since G/G_0 is assumed to be abelian, applying [11] yields

$$m_{G/G_0}(A_g)\mu_{G/G_0}(B_g) \geq 1.$$

This finishes the proof. □

Let G be a compact group which does not satisfy the weak QUP. The following theorem deals with necessary and sufficient conditions for the existence of a lower bound for $m_G(A_f)\mu_G(B_f)$ for all $f \in L^1(G)$, $f \neq 0$. To this end, we define $\mathcal{H}$ to be the set of all compact open normal subgroups of G. Recall that an open subgroup of a locally compact group G always contains G_0.

A locally compact group G is called *almost abelian* if it contains an abelian normal subgroup of finite index. Moore [12] proved that for an arbitrary locally compact group G the existence of an abelian normal subgroup of finite index is equivalent to the condition $\max_{\pi\in\widehat{G}} d_\pi < \infty$.

THEOREM 2. *Let G be a compact group. Consider the following conditions.*

(i) *There exists $M > 0$ such that $m_G(A_f)\mu_G(B_f) \geq M$ for all $f \in L^1(G)$, $f \neq 0$.*

(ii) *G/G_0 is almost abelian.*

Then (ii) *implies* (i), *and M can be chosen as $(\max_{\pi\in\widehat{G/G_0}} d_\pi)^{-1}$. Conversely, if*

$$\inf_{H\in\mathcal{H}} \frac{\sum_{\pi\in\widehat{G/H}} d_\pi}{\sum_{\pi\in\widehat{G/H}} d_\pi^2} = 0,$$

then (i) *implies* (ii).

Proof. Let G be a compact group. Suppose first that G/G_0 is almost abelian. Let $f \in L^1(G)$, $f \neq 0$. By Lemma 2.2 and Lemma 2.3, there exist a closed normal subgroup H of G such that G/H is Lie and a function g on $(G/H)/(G/H)_0 = G/G_0H$, $g \neq 0$, such that

$$m_G(A_f)\mu_G(B_f) \geq m_{G/G_0H}(A_g)\mu_{G/G_0H}(B_g).$$

Moreover, we have

$$\max_{\pi\in\widehat{G/G_0H}} d_\pi \leq \max_{\pi\in\widehat{G/G_0}} d_\pi < \infty.$$

(For the second inequality see [12, Proposition 2.1].)

Now let G be a finite group. By the preceding paragraph and since G/G_0H is finite, it suffices to prove that $m_G(A_f)\mu_G(B_f) \geq (\max_{\pi\in\widehat{G}} d_\pi)^{-1}$ for each function f on G, $f \neq 0$. To this end, let f be a function on G, $f \neq 0$. For each $\pi \in \widehat{G}$, we may identify $\mathcal{H}_\pi$ with $\mathbb{C}^{d_\pi}$ and denote its canonical orthonormal basis by $\{\xi_i : i = 1, \ldots, d_\pi\}$. Then $\pi(x)$, where $x \in G$, can be represented by a matrix with respect to this basis, which we denote by $(\pi_{ij}(x))_{1\leq i,j\leq d_\pi}$. We then have

$$\begin{aligned}
\operatorname{tr}[\hat{f}(\pi)^*\hat{f}(\pi)] &= \sum_{i=1}^{d_\pi} \langle \hat{f}(\pi)\xi_i, \hat{f}(\pi)\xi_i\rangle \\
&= \frac{1}{|G|^2} \sum_{i=1}^{d_\pi} \sum_{x,y\in G} f(x)\overline{f(y)}\,\pi_{ii}(yx^{-1}) \\
&\leq \frac{1}{|G|^2} \sum_{i=1}^{d_\pi} \sum_{x,y\in G} |f(x)||f(y)||\pi_{ii}(yx^{-1})| \\
&\leq \frac{1}{|G|^2} \sum_{i=1}^{d_\pi} \sum_{x,y\in G} |f(x)||f(y)|.
\end{aligned}$$

Using the Plancherel formula, this inequality and Hölder's inequality, we obtain

$$\|f\|_2^2 \leq \mu_G(B_f) \max_{\pi \in \widehat{G}} \operatorname{tr}[\hat{f}(\pi)^* \hat{f}(\pi)] \tag{1}$$

$$\leq \mu_G(B_f)(\max_{\pi \in \widehat{G}} d_\pi)\|f\|_1^2 \tag{2}$$

$$\leq \mu_G(B_f) m_G(A_f)(\max_{\pi \in \widehat{G}} d_\pi)\|f\|_2^2. \tag{3}$$

This shows

$$m_G(A_f)\mu_G(B_f) \geq \frac{1}{\max_{\pi \in \widehat{G}} d_\pi},$$

and thus proves the first assertion of the theorem.

Now suppose that G/G_0 is not almost abelian. Let $H \in \mathcal{H}$. We define $f_H \in L^1(G)$ by $f_H = \chi_H$. Our choice of Haar measures on compact groups implies $m_G(A_{f_H}) = [G : H]^{-1}$. For the Fourier transform of f_H Lemma 2.1 shows that, for each $\pi \in \widehat{G}$ and $\xi, \eta \in \mathcal{H}_\pi$,

$$\langle \widehat{f_H}(\pi)\xi, \eta \rangle = \frac{1}{[G : H]} \chi_{A(H,\widehat{G})}(\pi)\langle \xi, \eta \rangle.$$

We identify $A(H, \widehat{G})$ with $\widehat{G/H}$. Then the definition of the Plancherel measure implies $\mu_G(B_{f_H}) = \sum_{\pi \in \widehat{G/H}} d_\pi$. Hence, using [4, Proposition 5.27] we get

$$m_G(A_{f_H})\mu_G(B_{f_H}) = \frac{1}{[G : H]} \sum_{\pi \in \widehat{G/H}} d_\pi = \frac{\sum_{\pi \in \widehat{G/H}} d_\pi}{\sum_{\pi \in \widehat{G/H}} d_\pi^2}.$$

By hypothesis, we have

$$\inf_{H \in \mathcal{H}} \frac{\sum_{\pi \in \widehat{G/H}} d_\pi}{\sum_{\pi \in \widehat{G/H}} d_\pi^2} = 0.$$

Hence

$$\inf_{H \in \mathcal{H}} m_G(A_{f_H})\mu_G(B_{f_H}) = 0.$$

This proves that, under the above hypothesis, (i) implies (ii) and completes the proof of the theorem. □

The next result gives an explicit class of compact groups for which conditions (i) and (ii) are equivalent.

PROPOSITION 3.1. *Let G be a compact group such that G/G_0 is a direct product of finite groups. Then the conditions* (i) *and* (ii) *of Theorem 2 are equivalent.*

Proof. Let G be a compact group such that G/G_0 is a direct product of finite groups. Suppose that G/G_0 is not almost abelian. This implies that there exist an abelian group A and infinitely many finite non-abelian groups F_j, $j \in \mathbb{N}$, with $G/G_0 = A \times \prod_{j=1}^{\infty} F_j$. By Theorem 2, it suffices to prove that

$$\inf_{H \in \mathcal{H}} \frac{\sum_{\pi \in \widehat{G/H}} d_\pi}{\sum_{\pi \in \widehat{G/H}} d_\pi^2} = 0.$$

Let H_n, $n \in \mathbb{N}$, denote those subgroups of G which satisfy $H_n/G_0 = A \times \prod_{j=n+1}^{\infty} F_j$, where we regard the direct product as a subgroup of G/G_0 in the canonical way. Then, for each $n \in \mathbb{N}$ we have $H_n \in \mathcal{H}$. We define G_n by $G_n = G/H_n = (G/G_0)/(H_n/G_0) = \prod_{j=1}^{n} F_j$. Also, for simplicity, we set

$$q(J) = |J|^{-1} \sum_{\pi \in \widehat{J}} d_\pi$$

for any finite group J.

We claim that

$$q(G_n) \to 0 \quad \text{as } n \to \infty.$$

To prove this, we first note that $J = B \times C$ implies $q(J) = q(B)q(C)$, since $\widehat{J} = \widehat{B} \times \widehat{C}$. Next, let J' denote the commutator subgroup of J, and set $k = |J'|$. Then

$$\begin{aligned} q(J) &= |J|^{-1} \left(|J/J'| + \sum_{\pi \in \widehat{J},\, d_\pi \geq 2} d_\pi \right) \\ &= \frac{1}{k} + |J|^{-1} \sum_{\pi \in \widehat{J},\, d_\pi \geq 2} d_\pi \\ &\leq \frac{1}{k} + \frac{1}{2}|J|^{-1} \sum_{\pi \in \widehat{J},\, d_\pi \geq 2} d_\pi^2 \\ &< \frac{1}{k} + \frac{1}{2}. \end{aligned}$$

Let $n \in \mathbb{N}$. Since F_{n+1} and F_{n+2} are both non-abelian, their commutator subgroups have order at least 2, so the commutator subgroup of $F_{n+1} \times F_{n+2}$ has order at least 4, whence $q(F_{n+1} \times F_{n+2}) < 3/4$ by the preceding calculation. Therefore we obtain

$$q(G_{n+2}) = q(G_n)q(F_{n+1} \times F_{n+2}) < \frac{3}{4} q(G_n).$$

This implies the above claim.

By the claim we have

$$\inf_{H \in \mathcal{H}} \frac{\sum_{\pi \in \widehat{G/H}} d_\pi}{\sum_{\pi \in \widehat{G/H}} d_\pi^2} \leq \inf_{n \in \mathbb{N}} q(G_n) = 0.$$

Hence the proof is complete. □

Theorem 2 and Proposition 3.1 lead to the following conjecture.

Conjecture. *Let G be a compact group. Then conditions* (i) *and* (ii) *of Theorem 2 are equivalent.*

4. Values of $m_G(A_f)\mu_G(B_f)$

After determining conditions under which the weak QUP holds and conditions which guarantee the existence of a lower bound for $m_G(A_f)\mu_G(B_f)$, we now consider the possible values of this product.

4.1. Lower bounds. Let G be a compact group and let $f \in L^1(G)$, $f \neq 0$. In this subsection we study lower bounds for the product $m_G(A_f)\mu_G(B_f)$.

First we consider the situation when G/G_0 is abelian. By Theorem 1, the value 1 is a lower bound. It is easy to show that this bound is always sharp. Let $f \in L^1(G)$ be defined by $f = \chi_G$. Then f satisfies

$$m_G(A_f)\mu_G(B_f) = 1.$$

Obviously, any function $f_H \in L^1(G)$ defined by $f_H = \chi_H$, where H is a compact open normal subgroup of G, satisfies this equation. It is interesting to note that, if G is an infinite compact group which does not satisfy the QUP, then for some closed normal subgroup H of G the function f_H not only attains the infimum, but even violates the QUP, i.e., satisfies $m_G(A_{f_H}) < m_G(G)$ and $\mu_G(B_{f_H}) < \mu_G(\widehat{G})$. It suffices to take any proper open compact normal subgroup H of G which is non-trivial. Such a subgroup exists, since the hypothesis implies that G is not connected (see [8, Theorem 2.6]) and hence G/G_0 is a non-trivial totally disconnected compact group. We can now apply [6, Theorem 7.7].

Let us mention that in the case of locally compact abelian groups we can completely classify all functions $f \in L^2(G)$ for which $m_G(A_f)\mu_G(B_f)$ attains the infimum, i.e., which satisfy $m_G(A_f)\mu_G(B_f) = 1$ (see [10, Theorem 2.4]).

Next, we examine the situation when G/G_0 is almost abelian. Theorem 2 shows that $(\max_{\pi \in \widehat{G/G_0}} d_\pi)^{-1}$ is a lower bound. Again the question arises whether this bound is sharp. We can easily construct functions satisfying

$$m_G(A_f)\mu_G(B_f) = \frac{\sum_{\pi \in \widehat{G/H}} d_\pi}{\sum_{\pi \in \widehat{G/H}} d_\pi^2},$$

where H is an open compact normal subgroup of G, by setting $f = \chi_H$. However, we now show that the bound $(\max_{\pi \in \widehat{G/G_0}} d_\pi)^{-1}$ is never attained.

PROPOSITION 4.1. *Let G be a compact group such that G/G_0 is almost abelian, but not abelian. Then, for each $f \in L^1(G)$, $f \neq 0$, we have*

$$m_G(A_f)\mu_G(B_f) > \frac{1}{\max_{\pi \in \widehat{G/G_0}} d_\pi}.$$

Proof. Let $f \in L^1(G)$, $f \neq 0$. Lemma 2.2 and Lemma 2.3 imply that there exists a function g on $(G/H)/(G/H)_0 = G/G_0H$, $g \neq 0$, such that

$$m_G(A_f)\mu_G(B_f) \geq m_{G/G_0H}(A_g)\mu_{G/G_0H}(B_g),$$

where H is a closed normal subgroup of G such that G/H is Lie. Without loss of generality we can assume that G/G_0H is non-abelian. Moreover, we have

$$\max_{\pi \in \widehat{G/G_0H}} d_\pi \leq \max_{\pi \in \widehat{G/G_0}} d_\pi < \infty.$$

Now let G be a finite non-abelian group. By the preceding paragraph it suffices to prove that $m_G(A_f)\mu_G(B_f) > (\max_{\pi\in\widehat{G}} d_\pi)^{-1}$ holds for all functions f on G, $f \neq 0$. To obtain a contradiction, assume that there exists a function f on G which satisfies

$$m_G(A_f)\mu_G(B_f) = \frac{1}{\max_{\pi \in \widehat{G}} d_\pi}.$$

Throughout the proof, for each $\pi \in \widehat{G}$, we identify $\mathcal{H}_\pi$ with $\mathbb{C}^{d_\pi}$ and denote its standard orthonormal basis by $\{\xi_i : i = 1, \ldots, d_\pi\}$. Furthermore, for $x \in G$, we let the matrix of $\pi(x)$, $(\pi_{ij}(x))_{1\leq i,j\leq d_\pi}$, be chosen with respect to this basis.

The assumption implies that we must have equality in the inequalities (1)–(3) above. This holds if and only if there exist $c, d > 0$ such that

(i) $\sum_{i=1}^{d_\pi} \langle \hat{f}(\pi)\xi_i, \hat{f}(\pi)\xi_i\rangle = d$ for all $\pi \in B_f$,
(ii) $d = (\max_{\rho\in\widehat{G}} d_\rho)m_G(A_f)^2c^2$,
(iii) $|\pi_{ii}(yx^{-1})| = 1$ for all $x, y \in A_f$, $\pi \in B_f$ and $1 \leq i \leq d_\pi$,
(iv) $|f(x)| = c\chi_{A_f}(x)$ for all $x \in G$.

More precisely, (i) is equivalent to equality in (1), (iv) is equivalent to equality in (3), and (ii) and (iii) hold if and only if we have equality in (2). Without loss of generality we can assume that $c = 1$.

Let $x, y \in A_f$. Using the Cauchy-Schwarz inequality, it follows from (iii) that ξ_i is an eigenvector of $\pi(yx^{-1})$ for all $1 \leq i \leq d_\pi$. By the choice of the basis $\{\xi_i : i = 1, \ldots, d_\pi\}$, this in turn implies that the matrix $(\pi_{ij}(yx^{-1}))_{1\leq i,j\leq d_\pi}$ is diagonal. Without loss of generality we can assume that $e \in A_f$, since otherwise we could choose an element $x_0 \in A_f$ and consider the function $g := f(x_0\cdot)$. Then we would have $e \in A_g$, $m_G(A_g) = m_G(A_f)$ and $\mu_G(B_g) = \mu_G(B_f)$, because $\hat{g}(\pi) = \hat{f}(\pi)\pi(x_0)$ for $\pi \in \widehat{G}$. Thus $(\pi_{ij}(y))_{1\leq i,j\leq d_\pi}$ is also diagonal, and $\pi_{ii}(yx^{-1}) = \pi_{ii}(y)\pi_{ii}(x^{-1})$ for all $1 \leq i \leq d_\pi$.

By conditions (i), (ii), (iii) and (iv) we have for all $\pi \in B_f$

$$\frac{1}{|G|^2} \sum_{i=1}^{d_\pi} \left| \sum_{x,y \in G} f(x)\overline{f(y)}\, \pi_{ii}(yx^{-1}) \right| = \sum_{i=1}^{d_\pi} \langle \hat{f}(\pi)\xi_i, \hat{f}(\pi)\xi_i \rangle$$
$$= (\max_{\rho \in \widehat{G}} d_\rho) m_G(A_f)^2 = (\max_{\rho \in \widehat{G}} d_\rho) \frac{1}{|G|^2} \sum_{x,y \in G} |f(x)||f(y)||\pi_{ii}(yx^{-1})|.$$

This yields immediately

$$d_\pi = \max_{\rho \in \widehat{G}} d_\rho \quad \text{for all } \pi \in B_f \tag{4}$$

and

$$\left| \sum_{x,y \in G} f(x)\overline{f(y)}\, \pi_{ii}(yx^{-1}) \right| = \sum_{x,y \in G} |f(x)||f(y)||\pi_{ii}(yx^{-1})|.$$

Thus, by [6, Theorem 12.4], there exists a constant $\lambda_{\pi_{ii}}$ such that

$$f(x)\overline{f(y)}\, \pi_{ii}(yx^{-1}) = f(x)\overline{f(y)}\, \pi_{ii}(y)\pi_{ii}(x^{-1}) = \lambda_{\pi_{ii}}$$

for all $x, y \in A_f$, $\pi \in B_f$ and $1 \le i \le d_\pi$. If we choose $x = y$ and use (iv), we obtain $\lambda_{\pi_{ii}} = 1$ for all $\pi \in B_f$, $1 \le i \le d_\pi$. This implies the existence of a constant λ with $|\lambda| = 1$ and

$$\overline{f(y)}\pi_{ii}(y) = \lambda \quad \text{for all } y \in A_f, \pi \in B_f \text{ and } 1 \le i \le d_\pi.$$

Let $(\hat{f}(\pi)_{ij})_{1 \le i,j \le d_\pi}$ denote the matrix of $\hat{f}(\pi)$ with respect to the basis $\{\xi_i : i = 1, \ldots, d_\pi\}$. Then, for each $\pi \in B_f$,

$$\hat{f}(\pi)_{ii} = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{\pi_{ii}(x)} = \overline{\lambda} m_G(A_f) \quad \text{for all } 1 \le i \le d_\pi.$$

Since $(\pi_{ij}(x))_{1 \le i,j \le d_\pi}$ is diagonal, $\hat{f}(\pi)_{ij} = 0$ for all $\pi \in B_f$, $i \ne j$.

Next we calculate f from the inverse Fourier transform. For all $x \in G$ we obtain

$$f(x) = \sum_{\pi \in \widehat{G}} d_\pi \sum_{i=1}^{d_\pi} \hat{f}(\pi)_{ii}\pi_{ii}(x) = \overline{\lambda} m_G(A_f)(\max_{\rho \in \widehat{G}} d_\rho) \sum_{\pi \in B_f} \sum_{i=1}^{d_\pi} \pi_{ii}(x), \tag{5}$$

where we have used (4).

Now let $x \in A_f$ and $\pi \in B_f$. Since, by assumption, the function f satisfies $m_G(A_f)\mu_G(B_f) = (\max_{\rho \in \widehat{G}} d_\rho)^{-1}$ and condition (iv), we obtain

$$\frac{1}{\mu_G(B_f)} \left| \sum_{\pi \in B_f} \sum_{i=1}^{d_\pi} \pi_{ii}(x) \right| = m_G(A_f)(\max_{\rho \in \widehat{G}} d_\rho) \left| \sum_{\pi \in B_f} \sum_{i=1}^{d_\pi} \pi_{ii}(x) \right| = |f(x)| = 1,$$

which in turn implies

$$\left| \sum_{\pi \in B_f} \sum_{i=1}^{d_\pi} \pi_{ii}(x) \right| = \mu_G(B_f).$$

However, since $\mu_G(B_f) = |B_f|(\max_{\rho \in \widehat{G}} d_\rho)$, this equality can only hold if $\pi_{ii}(x) = 1$ for all $1 \leq i \leq d_\pi$.

Next, we show that A_f is a normal subgroup of G. To this end, let $x \notin A_f$. Then, by (5), there exists an element $\pi \in B_f$ with $\pi(x) \neq 1_{\mathcal{H}_\pi}$. On the other hand, we just proved that for all $x \in A_f$ we have $\pi(x) = 1_{\mathcal{H}_\pi}$ for all $\pi \in B_f$. Thus

$$A_f = \{x \in G : \pi(x) = 1_{\mathcal{H}_\pi} \text{ for all } \pi \in B_f\},$$

which is a normal subgroup of G. Now Lemma 2.1 shows that $B_f = A(A_f, \widehat{G})$. However, this implies that B_f contains the trivial representation, which contradicts (4), since G was assumed to be non-abelian. □

4.2. Values attained. Let G be a compact group. In this subsection we consider the question which values the product $m_G(A_f)\mu_G(B_f)$, $f \in L^1(G)$, can attain. We note that the arguments below also show how to construct a function $f \in L^1(G)$ to obtain a given value.

PROPOSITION 4.2. *Let G be a compact group. For each $M \subseteq \{\pi \in \widehat{G} : \mathrm{tr}[\pi(x)] \neq 0$ for almost all $x \in G\}$ there exists a function $f \in L^1(G)$ such that*

$$m_G(A_f)\mu_G(B_f) = \sum_{\pi \in M} d_\pi.$$

Proof. Let $M \subseteq \widehat{G}$ be fixed. If $|M| = \infty$, we only have to choose $f \in L^1(G)$ such that $\mu_G(B_f) = \infty$. Such a function trivially exists.

It remains to deal with the case when $|M|$ is finite. To this end, let $f \in L^1(G)$ be defined by its Fourier transform

$$\hat{f} = \sum_{\pi \in M} a_\pi \chi_{\{\pi\}} 1_{\mathcal{H}_\pi},$$

where $a_\pi \neq 0$, $\pi \in M$, will be chosen later. Obviously, $\mu_G(B_f) = \sum_{\pi \in M} d_\pi$. Applying the inverse Fourier transform yields

$$f(x) = \sum_{\pi \in M} a_\pi d_\pi \mathrm{tr}[\pi(x)].$$

We have $\mathrm{tr}[\pi(x)] \neq 0$ for almost all $x \in G$. Moreover, G is compact and M is finite. Also, notice that, if X is a measure space with finite measure and $f, g : X \to \mathbb{C}$ are such that $f, g \neq 0$ almost everywhere, then there always exists a number $a \in \mathbb{C}$, $a \neq 0$, with $f \neq ag$ almost everywhere. Thus we may choose $a_\pi \neq 0$, $\pi \in M$, such that $f(x) \neq 0$ for almost all $x \in G$. Then f satisfies $m_G(A_f) = 1$, which finishes the proof. □

If, in addition, G is abelian, the above proposition reduces to the following result.

Corollary 4.3. *Let G be a compact abelian group. For each $n \in \{1, \ldots, |\widehat{G}|\}$ there exists a function $f \in L^1(G)$ such that*

$$m_G(A_f)\mu_G(B_f) = n.$$

Proof. Since G is abelian, we have $d_\pi = 1$ for all $\pi \in \widehat{G}$. Moreover, $\omega(x) \neq 0$ for all $x \in G$, $\omega \in \widehat{G}$. Hence the claim is an immediate consequence of Proposition 4.2. □

Remark 4.4. There exist compact groups G for which the product $m_G(A_f)\mu_G(B_f)$, $f \in L^1(G)$ can attain no values other than those described in Proposition 4.2. Indeed, let G be a compact connected group. Then G satisfies the QUP (see [8, Theorem 2.6]). Hence, for each $f \in L^1(G)$, $f \neq 0$, we have either $m_G(A_f) = 1$ or $\mu_G(B_f) = \infty$. Thus the numbers $\sum_{\pi \in M} d_\pi$, $M \subseteq \widehat{G}$, are the only possible values which $m_G(A_f)\mu_G(B_f)$, $f \in L^1(G)$, can attain. In addition, for all $\pi \in \widehat{G}$, we have $\mathrm{tr}[\pi(x)] \neq 0$ for almost all $x \in G$. This follows by standard arguments from the fact that G is connected.

Although Proposition 4.2 applies to finite groups, we can obtain a stronger result for finite abelian groups.

Proposition 4.5. *Let G be a finite abelian group. For each $1 \leq p \leq |G|$ and $q \in \{|G| - p + 1, \ldots, |G|\}$ there exists a function f on G such that*

$$m_G(A_f)\mu_G(B_f) = \frac{pq}{|G|}.$$

Proof. By the structure theorem, G is of the form $G = \mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_s}$ with integers $m_1, \ldots, m_s$ greater than 1, each of which is a power of a prime. We only treat the case $G = \mathbb{Z}_m$. The general case can be proven similarly.

Throughout this proof we identify G with $\widehat{G}$ in the canonical way (cf. [6, Example 23.27 (d)]). Let $p \in \{1, \ldots, |G| = m\}$ and $q \in \{m - p + 1, \ldots, m\}$ be fixed. We construct a function f on G which satisfies $m_G(A_f) = q/m$ and $\mu_G(B_f) = p$. To this end, we have to consider the matrix

$$T := \left(e^{2\pi i jk/m}\right)_{1 \leq j,k \leq m}.$$

If we set $d = e^{2\pi i(1/m)}$, we can write T in the form

$$T = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & d & d^2 & \ldots & d^{m-1} \\ 1 & d^2 & d^4 & \ldots & d^{2(m-1)} \\ \vdots & & & \ddots & \vdots \\ 1 & d^{m-1} & d^{2(m-1)} & \ldots & d^{(m-1)^2} \end{pmatrix}.$$

Notice that this is a Vandermonde matrix. Let $T_{r,s}$ denote the matrix consisting of the first r rows and s columns of T. Since $m-q<p$, the rank of $T_{m-q,p}$ is $m-q$ and the subspace $V_{p,q}=\{a\in\mathbb{C}^p : T_{m-q,p}a=0\}$ has dimension ≥ 1.

Next, we define q' by $q'=m-p+1$. Then $m-q'=p-1$. Hence the dimension of $V_{p,q'}$ equals 1. Let $b_0\in V_{p,q'}$, $b_0\neq 0$. To obtain a contradiction assume that, for some $1\leq i\leq p$, the ith component of b_0 is equal to zero. If the ith column of $T_{m-q',p}$ is deleted, the new matrix is a transpose of a Vandermonde matrix, and hence nonsingular. Then all other components of b_0 have to be equal to zero. This is a contradiction. Thus all components of b_0 are nonzero. We define $\tilde{b}_0\in\mathbb{C}^m$ by $\tilde{b}_0=(b_0,0,\ldots,0)^t$. We now claim that the last $m-p+1$ components of $T\tilde{b}_0$ are all nonzero. To this end, let $i\in\{p,\ldots,m\}$ be arbitrarily chosen and consider the $p\times p$-matrix consisting of $T_{m-q',p}$ and the first p components of the ith row of T as last row. This is again a Vandermonde matrix, and hence is nonsingular. Thus the ith component of $T\tilde{b}_0$ cannot equal zero. This proves the assertion.

Next, let $b\in V_{p,q}$ be such that the last $p-(m-q)$ components of $T_{p,p}b$ do not equal zero. Such a vector b always exists since $T_{p,p}$ is invertible. Choose $\lambda\in\mathbb{C}$ such that each component of λb_0+b is nonzero and that the last q components of Ta, where $a=(a_1,\ldots,a_m)^t\in\mathbb{C}^m$ is defined by $a:=(\lambda b_0+b,0,\ldots,0)^t$, are all nonzero. Note that the first $m-q$ components of Ta all equal 0.

Let us now define f by

$$f(j)=\sum_{k=0}^{m-1}a_{k+1}e^{2\pi ijk/m}.$$

An easy calculation shows that

$$\hat{f}(j)=\sum_{k=0}^{m-1}a_{k+1}\chi_{\{k\}}(j).$$

By the choice of a, we have $m_G(A_f)=q/m$ and $\mu_G(B_f)=p$. This completes the proof. □

Remark 4.6. We can easily extend Corollary 4.3 to general locally compact abelian groups. Indeed, let G be a non-compact non-discrete locally compact abelian group such that G_0 is compact. Let H be a compact open subgroup of G. Suppose there exist $g\in L^1(H)$ and $r>0$ such that $m_H(A_g)\mu_H(B_g)=r$. Then, for each $n\in\mathbb{N}$, we can construct a function $f\in L^1(G)$ such that

$$m_G(A_f)\mu_G(B_f)=nr.$$

This can be easily seen by choosing $x_i\in G$, $i=1,\ldots,n$, such that $x_iH\neq x_jH$ for all $i\neq j$ and defining $f\in L^1(G)$ by

$$f(x)=\sum_{i=1}^{n}a_ig(x_H)\chi_{x_iH}(x).$$

Here the decomposition $x = x_i x_H$ will be unique for all $x \in \bigcup_{i=1}^n x_i H$ and the values $a_i \neq 0$, $i = 1, \ldots, n$ have to be chosen appropriately.

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