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EQUIVARIANT VECTOR BUNDLES OVER THE UPPER HALF PLANE

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ABSTRACT. Holomorphic Hermitian vector bundles over the upper half plane that admit a lift of the action of $\mathrm{SL}(2,\mathbb{R})$ are considered. All such vector bundles are described and classified up to isomorphism.

1. Introduction

We consider holomorphic Hermitian vector bundles over the upper half $\mathbb{H}$ of the complex plane equipped with a lift of the action of the automorphism group $\mathrm{SL}(2,\mathbb{R})$ of $\mathbb{H}$. The lifted action is required to preserve both the holomorphic and the Hermitian structures. A vector bundle equipped with such an action of $\mathrm{SL}(2,\mathbb{R})$ is called an *equivariant vector bundle*. An equivariant vector bundle is called *irreducible* if it is not a direct sum of equivariant vector bundles of positive ranks. So any equivariant vector bundle is a direct sum of irreducible vector bundles.

We classify all irreducible bundles (Theorem 3.2). To explain this classification, we will first construct a class of equivariant vector bundles.

Consider the holomorphic tangent bundle $T\mathbb{H}$ of $\mathbb{H}$ equipped with the Poincaré metric. We recall that the Poincaré metric is $|dz|/\mathrm{Im}(z)$ and it is left invariant by the automorphisms of $\mathbb{H}$. Fix a pair $(L\,,\sigma)$, where L is a C^∞ complex line bundle over $\mathbb{H}$ and σ a C^∞ isomorphism of $L^{\otimes 2}$ with $T\mathbb{H}$. Using σ, the holomorphic and Hermitian structures of $T\mathbb{H}$ induce corresponding structures on L making it an equivariant line bundle. This equivariant line bundle defined by $(L\,,\sigma)$ will be denoted by $\mathcal{L}$. (The details of the construction of $\mathcal{L}$ are given in Section 3.)

Any equivariant line bundle is isomorphic to some tensor power of $\mathcal{L}$. For any integer c and positive integer n, consider the direct sum

$$\mathcal{V}(n,c) := \bigoplus_{i=0}^{n-1} \mathcal{L}^{\otimes(c+2i)}$$

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which has an obvious Hermitian structure obtained from the Hermitian structure of $\mathcal{L}$ and admits a natural action of $\mathrm{SL}(2,\mathbb{R})$—defined by the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{L}$—preserving this Hermitian structure. However, we need to modify its obvious holomorphic structure.

For any integer j, let $\overline{\partial}_{\mathcal{L}^{\otimes j}}$ denote the Dolbeault operator defining the holomorphic structure on $\mathcal{L}^{\otimes j}$ induced by the holomorphic structure on $\mathcal{L}$. Using the isomorphism σ and the Poincaré metric, the line bundle

$$\Omega^{0,1}_{\mathbb{H}} \otimes \mathrm{Hom}(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes (j-2)})$$

is identified with the trivial line bundle. For any $\underline{\delta} := (\delta_1, \delta_2, \cdots, \delta_{n-1}) \in \mathbb{C}^{n-1}$, consider the differential operator

$$\overline{\partial}_{\underline{\delta}} = \begin{pmatrix} \overline{\partial}_{\mathcal{L}^{\otimes c}} & \delta_1 & 0 & \cdots & 0 & 0 \\ 0 & \overline{\partial}_{\mathcal{L}^{\otimes (c+2)}} & \delta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \overline{\partial}_{\mathcal{L}^{\otimes (c+2n-4)}} & \delta_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & \overline{\partial}_{\mathcal{L}^{\otimes (c+2n-2)}} \end{pmatrix}$$

where δ_i is considered as a homomorphism from $\mathcal{L}^{\otimes (c+2i)}$ to $\Omega^{0,1}_{\mathbb{H}} \otimes \mathcal{L}^{\otimes (c+2i-2)}$.

The above operator $\overline{\partial}_{\underline{\delta}}$ defines a holomorphic structure on the C^∞ vector bundle $\mathcal{V}(n,c)$ and the equivariant vector bundle obtained this way would be denoted by $\mathcal{W}(\underline{\delta}, c)$. In Theorem 3.2 we prove the following:

The equivariant vector bundle $\mathcal{W}(\underline{\delta}, c)$ is irreducible if and only if $\delta_i \neq 0$ for each $i \in [1, n-1]$. The two irreducible vector bundles $\mathcal{W}(\underline{\delta}, c)$ and $\mathcal{W}(\underline{\delta}', c')$ are isomorphic if and only if $c = c'$ and $|\delta_i| = |\delta'_i|$ for each i. Any rank n irreducible vector bundle is isomorphic to some $\mathcal{W}(\underline{\delta}, c)$.

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2. Equivariant bundles

Let

$$\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

be the upper half of the complex plane. The group $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{H}$ using the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}.$$

For any $A \in \mathrm{SL}(2,\mathbb{R})$, the automorphism of $\mathbb{H}$ defined by it will be denoted by $\tau(A)$.

Let V be a holomorphic vector bundle over $\mathbb{H}$ equipped with a Hermitian structure. So there is a unique connection ∇ on V such that ∇ preserves the Hermitian structure and the $(0,1)$-part of ∇ coincides with the Dolbeault operator defining the holomorphic structure of V [1, p. 11, Proposition 4.9].

This unique connection on a holomorphic Hermitian vector bundle is known as the *Chern connection*.

Let V be a C^∞ vector bundle over $\mathbb{H}$. A $\mathrm{SL}(2,\mathbb{R})$-*linearization* of V is a C^∞ action of $\mathrm{SL}(2,\mathbb{R})$ on the total space of V such that the action of any $A \in \mathrm{SL}(2,\mathbb{R})$ is a vector bundle isomorphism of V with $\tau(A)^*V$.

DEFINITION 2.1. A holomorphic Hermitian vector bundle V will be called *equivariant* if it admits a $\mathrm{SL}(2,\mathbb{R})$-linearization such that the action of any element of $\mathrm{SL}(2,\mathbb{R})$ on V preserves both the holomorphic and the Hermitian structures. An equivariant vector bundle V will be called *irreducible* if there is no proper subbundle of V of positive rank preserved by the Chern connection ∇ on V.

LEMMA 2.2. *Let F be a complex subbundle of an equivariant vector bundle V preserved by the Chern connection on V. Then F is left invariant by the action of* $\mathrm{SL}(2,\mathbb{R})$ *on V.*

Proof. Let G denote the group of all automorphisms of V preserving its Chern connection. So G is the group of all holomorphic automorphisms of V preserving the Hermitian structure. Note that G is a compact group. Indeed, for any point $z \in \mathbb{H}$, the evaluation map $G \longrightarrow \mathrm{Iso}(E_z)$ makes G into a closed subgroup of $\mathrm{Iso}(E_z)$, the group of all unitary automorphisms of the fiber E_z.

Now, $\mathrm{SL}(2,\mathbb{R})$ acts on the group G through automorphisms as follows: $(g\,,T) \longmapsto gTg^{-1}$, where $g \in \mathrm{SL}(2,\mathbb{R})$ and $T \in G$. The group of all automorphisms of G connected to the identity automorphisms of G is identified with

$$G' := G/\mathrm{center}(G)$$

and the action of G' on G is the conjugation action.

Therefore, the above defined action of $\mathrm{SL}(2,\mathbb{R})$ on G defines a homomorphism

$$\rho : \mathrm{SL}(2,\mathbb{R}) \longrightarrow G'.$$

Now note that $\mathrm{SL}(2,\mathbb{R})$ does not have any nonconstant homomorphism to a compact group. Indeed, a homomorphism to a compact group must take the unipotent elements (for example, upper triangular matrices in $\mathrm{SL}(2,\mathbb{R})$ with 1 on the diagonal) to the identity element (as there are no nontrivial unipotent elements in a compact group). Since $\mathrm{SL}(2,\mathbb{R})$ is simple, such a homomorphism must be the trivial homomorphism. Therefore, the above homomorphism ρ is the trivial homomorphism.

Consider the automorphism

$$T := c_1\mathrm{Id}_F \oplus c_2\mathrm{Id}_{F^\perp} \in G$$

where c_1 and c_2 are *distinct* complex numbers with $|c_1| = 1 = |c_2|$, and $F^\perp \subset V$ is the orthogonal complement of F. Since ρ coincides with the trivial homomorphism, the automorphism T commutes with the action of $\mathrm{SL}(2,\mathbb{R})$. Therefore, F is left invariant by the action of $\mathrm{SL}(2,\mathbb{R})$, and the proof is complete. $\square$

If F is a complex subbundle of an equivariant vector bundle V preserved by the Chern connection ∇ on V and also left invariant by the action of $\mathrm{SL}(2,\mathbb{R})$ on V, then F has an induced structure of an equivariant vector bundle. Note that $F^{\perp}$ is also preserved by ∇, and the action of $\mathrm{SL}(2,\mathbb{R})$ leaves $F^{\perp}$ invariant. The equivariant vector bundle V is isomorphic (as an equivariant vector bundle) to the direct sum $F \oplus F^{\perp}$ of the two equivariant vector bundles. Therefore, in view of Lemma 2.2, an equivariant vector bundle is irreducible if and only if it is not holomorphically isometric (as an equivariant vector bundle) to a direct sum of equivariant vector bundles of positive ranks.

Lemma 2.3. *Let E be a holomorphic Hermitian vector bundle over $\mathbb{H}$ (no linearization on E is assumed) such that every automorphism of E preserving both the holomorphic and the Hermitian structures is a scalar multiplication. Then the number of possible equivariant structures on E is at most one.*

Proof. The condition on E implies that any two actions of $\mathrm{SL}(2,\mathbb{R})$ on E differ by a homomorphism from $\mathrm{SL}(2,\mathbb{R})$ to $U(1)$. But there are no such nontrivial homomorphisms. $\square$

Take any point $x \in \mathbb{H}$. The isotropy subgroup $H_x \subset \mathrm{SL}(2,\mathbb{R})$ for the point x is canonically identified with $U(1)$. With this identification, the action of $\exp(\sqrt{-1}\theta) \in H_x$ on the holomorphic tangent space $T_x\mathbb{H}$ is multiplication by $\exp(2\sqrt{-1}\theta)$. Since the action of $\mathrm{SL}(2,\mathbb{R})$ is transitive, the isotropy subgroups for any two points are conjugate. The isotropy subgroup for the point $\sqrt{-1}$ is

$$\mathrm{SO}(2,\mathbb{R}) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

with $a, b \in \mathbb{R}$ and $a^2 + b^2 = 1$.

Imitating the definition of $\mathrm{SL}(2,\mathbb{R})$-linearization, we will define H_x-linearization. Let V be a vector bundle over $\mathbb{H}$. A *H_x-linearization* of V is an action of H_x on the total space of V such that the action of any $A \in H_x$ on V is a vector bundle isomorphism of V with $\tau(A)^*V$.

Lemma 2.4. *Let E be a holomorphic Hermitian vector bundle over $\mathbb{H}$ satisfying the condition in Lemma 2.3 such that for each $A \in \mathrm{SL}(2,\mathbb{R})$, there is an isomorphism of $\tau(A)^*E$ with E that preserves both the holomorphic and the Hermitian structures. Assume that there is a point $x \in \mathbb{H}$ and a H_x-linearization of E that preserves both the holomorphic and the Hermitian structures of E. Then there is a unique equivariant structure on E.*

Proof. Let $\overline{f}$ be an action of H_x on the vector bundle E that defines a holomorphic and Hermitian structure preserving H_x-linearization. Given any $A \in \mathrm{SL}(2,\mathbb{R})$, fix an isomorphism

$$f(A) : E \longrightarrow \tau(A)^* E \tag{2.1}$$

with $f(1) = \mathrm{Id}_E$ and $f(B) = \overline{f}(B)$ for every $B \in H_x$. So for any pair $A, B \in \mathrm{SL}(2,\mathbb{R})$, the composition

$$F(A\,,B) := f(A) \circ f(B) \circ f(AB)^{-1} \tag{2.2}$$

is an automorphism of E, and hence is an element of $U(1)$. From this it follows that

(2.3)
$$F(B\,,C)F(A\,,BC)F(A\,,B)^{-1}F(AB\,,C)^{-1} = 1 = F(1\,,A) = F(A\,,1)$$

for all triples $A, B, C \in \mathrm{SL}(2,\mathbb{R})$. Indeed, $F(A\,,B)$ commutes with any automorphism of E. In particular, $F(A\,,B) = f(B) \circ f(AB)^{-1} \circ f(A) = f(AB)^{-1} \circ f(A) \circ f(B)$. Now the proof of (2.3) is straight-forward. Therefore, F in (2.2) defines a cohomology class

$$\beta \in H^2(\mathrm{SL}(2,\mathbb{R}), U(1)) \tag{2.4}$$

for the trivial action of $\mathrm{SL}(2,\mathbb{R})$ on $U(1)$ [2, p. 116].

The assertion that E is equivariant is equivalent to the condition that the cohomology class β in (2.4) vanishes. Indeed, if $\beta = 0$ then F is a coboundary which means that there is a map

$$\phi : \mathrm{SL}(2,\mathbb{R}) \longrightarrow U(1)$$

such that $F(A\,,B) = \phi(A)\phi(B)\phi(AB)^{-1}$. This immediately implies that sending any $A \in \mathrm{SL}(2,\mathbb{R})$ to $f(A)/\phi(A)$ we have an action of $\mathrm{SL}(2,\mathbb{R})$ on E that makes E an equivariant bundle.

So to complete the proof of the lemma it suffices to show that $\beta = 0$. The inclusion map $\iota : H_x \hookrightarrow \mathrm{SL}(2,\mathbb{R})$ defines a homomorphism

$$p : H^2(\mathrm{SL}(2,\mathbb{R}), U(1)) \longrightarrow H^2(H_x, U(1))\,.$$

Now, p is an isomorphism. In fact, both the cohomology groups are identified with $U(1)$ and p is the identity map of $U(1)$. This follows from the fact that the map ι is a homotopy equivalence.

For any pair $A, B \in H_x$, the automorphism $F(A\,,B)$ of E defined in (2.2) has the property that its evaluation at the point x is the identity automorphism of E_x. Since E is irreducible, such an automorphism must coincide with the identity map of E. Therefore, the restriction of the cocycle F to H_x takes the constant value Id_E. Consequently, we have $p(\beta) = 0$. Since p is injective, this completes the proof of the lemma. □

Take any point $x \in \mathbb{H}$. For an equivariant vector bundle E over $\mathbb{H}$, consider the action of the isotropy subgroup H_x on the fiber E_x. Since H_x $(= U(1))$

is abelian, there are *distinct characters*, say $\lambda_1, \lambda_2, \cdots, \lambda_n$, of H_x such that the eigenvalues for the action of any $g \in H_x$ on E_x are $\lambda_i(g)$, $i \in [1\,, n]$. Let

$$E_x = \bigoplus_{i=1}^{n} V_i^x$$

be the decomposition into eigenspaces, where V_i^x corresponds to the character λ_i. This decomposition of E_x is evidently orthogonal. Since H_x is naturally identified with $U(1)$, we may consider each λ_i as a character of $U(1)$. As $\mathrm{SL}(2, \mathbb{R})$ acts transitively on $\mathbb{H}$, the set of characters $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ of $U(1)$ is independent of x. Consequently, we have an orthogonal decomposition

$$E = \bigoplus_{i=1}^{n} V_i \tag{2.5}$$

into eigenbundles where the fiber of the subbundle V_i over any point y is the eigenspace V_i^y defined above. It may be noted that the subbundles V_i need not be holomorphic. However, clearly each of them is left invariant by the action of $\mathrm{SL}(2, \mathbb{R})$ on E.

The proof of the following proposition is straight-forward and we omit it.

Proposition 2.5. *If E is irreducible of (complex) rank at least two, then the number of characters—that is n—is at least two.*

Let c_i be the (unique) integer such that the character λ_i of $H_x = U(1)$ coincides with the one defined by $z \longmapsto z^{c_i}$. We arrange $\{c_i\}$ in ascending order. In other words,

$$c_1 < c_2 < c_3 < \cdots < c_{n-1} < c_n \tag{2.6}$$

(recall that the characters are distinct). For $j \in [1\,, n]$, set

$$\widetilde{V}_j := \bigoplus_{i=1}^{j} V_i \subset E \tag{2.7}$$

where V_i is the eigenbundle for the character $z \longmapsto z^{c_i}$ of $U(1)$.

We omit the proof of the following proposition since it is straight-forward.

Proposition 2.6. *Let E be an irreducible vector bundle and F a $\mathrm{SL}(2, \mathbb{R})$ invariant subbundle of V_1. Then F is a holomorphic subbundle of E. Also, each $\widetilde{V}_j$ defined in* (2.7) *is a holomorphic subbundle of E.*

Let ∇ denote the Chern connection on the irreducible vector bundle E. Consider the differential operator

$$\nabla^{1,0} : E \longrightarrow \Omega^{1,0}_{\mathbb{H}} \otimes E$$

defined by the $(1\,,0)$ part of ∇. For any C^∞ subbundle F of E, the Leibniz identity for ∇ ensures that the composition

$$F \hookrightarrow E \xrightarrow{\nabla^{1,0}} \Omega^{1,0}_{\mathbb{H}} \otimes E \longrightarrow \Omega^{1,0}_{\mathbb{H}} \otimes (E/F)$$

defines a vector bundle homomorphism

$$S'(F) \in \mathrm{Hom}(F\,,\Omega^{1,0}_{\mathbb{H}} \otimes (E/F))\,. \tag{2.8}$$

The following theorem is derived using the properties of this homomorphism.

THEOREM 2.7. *Let E as before be an irreducible vector bundle. For any $2 \le i \le n$, we have $c_i = c_{i-1} + 2$, where c_j are as in* (2.6). *Furthermore, for each $j \in [1\,,n]$ the rank of V_j is one.*

Proof. Let V^e (respectively, V^o) be the direct sum of all V_i such that i is an odd integer (respectively, even integer). We want to show that both V^e and V^o of E are left invariant by the Chern connection ∇ on E.

Let

$$S_d(F) \in \mathrm{Hom}(F\,,(E/F) \otimes \Omega^{0,1}_{\mathbb{H}})$$

be the second fundamental form of F for the Dolbeault operator and

$$S(F) : F \longrightarrow (T^{\mathbb{C}}\mathbb{H})^* \otimes F^\perp$$

the second fundamental form of F for the connection. Clearly we have

$$S(F) = S_d(F) + S'(F), \tag{2.9}$$

where $S'(F)$ is defined in (2.8). Let F be the subbundle V^e.

Take $\gamma = \exp(\sqrt{-1}\theta) \in H_x$. The action of γ on the line $(\Omega^{1,0}_{\mathbb{H}})_x$ (respectively, $(\Omega^{0,1}_{\mathbb{H}})_x$) is multiplication by γ^{-2} ((respectively, γ^2). This immediately implies that there is no H_x equivariant homomorphism from $(V^e)_x$ to $(V^o)_x \otimes (\Omega^{1,0}_{\mathbb{H}} \oplus \Omega^{0,1}_{\mathbb{H}})_x$. Indeed, the eigenvalues for the action of γ on $(V^e)_x$ are of the form γ^{even} and the eigenvalues for the action of γ on $(V^o)_x \otimes (\Omega^{1,0}_{\mathbb{H}} \oplus \Omega^{0,1}_{\mathbb{H}})_x$ are of the form γ^{odd}.

Therefore, from (2.9) it follows that $S(V^e) = 0$. In other words, V^e is left invariant by the connection ∇ on E. For exactly the same reason V^o is also left invariant by the connection ∇. Since E is irreducible, we conclude that either $V^e = 0$ or $V^o = 0$. This implies that $c_i \ge c_{i-1} + 2$ for all $i \in [2\,,n]$.

Assume that $c_i > c_{i-1} + 2$ with $i \in [2\,,n]$. Set $W_1 := \bigoplus_{j \le i-1} V_j$ and $W_2 := \bigoplus_{j \ge i} V_j$. We want to show that both the subbundles W_1 and W_2 of E are left invariant by the connection ∇.

The subbundle W_1 coincides with $\widetilde{V}_{i-1}$ defined in (2.7) and in Proposition 2.6 we saw that $\widetilde{V}_{i-1}$ is a holomorphic subbundle of E. In view of (2.9), to prove that W_1 is left invariant by ∇ it suffices to show that $S'(W_1) = 0$, where $S'(W)$ is defined in (2.8).

As before, take any $\gamma = \exp(\sqrt{-1}\theta) \in H_x$. Since $E \cong W_1 \oplus W_2$ as C^∞ bundles, the quotient vector bundle E/W_1 is naturally identified with W_2 in a $\mathrm{SL}(2,\mathbb{R})$ equivariant fashion. The action of γ on the fiber $(\Omega^{1,0}_{\mathbb{H}} \otimes W_2)_x$ has eigenvalues of the form γ^{c_j-2}, where $c_j \geq c_i$. On the other hand, the action of γ on the fiber $(W_1)_x$ has eigenvalues of the form γ^{c_j}, where $c_j \leq c_{i-1}$. Since $c_i > c_{i-1}+2$, there is no common eigen–character for the actions of H_x on $(\Omega^{1,0}_{\mathbb{H}} \otimes W_2)_x$ and $(W_1)_x$. This immediately implies that $S'(W_1) = 0$. Consequently, W_1 is left invariant by the connection ∇.

Since W_1 is left invariant by ∇ and E is irreducible, we conclude that $W_1 = E$. This contradicts the fact that $V_i \neq 0$ (as $i \in [2\,, n]$). Therefore, we have $c_i = c_{i-1}+2$ for all $2 \leq i \leq n$.

We will now show that each V_j is of rank one.

Let $F^1_x \subset (V_1)_x$ be a linear subspace of dimension one. Using the action of $\mathrm{SL}(2,\mathbb{R})$, the line F^1_x generates a line subbundle F^1 of V_1. From Proposition 2.6 we know that F^1 is a holomorphic subbundle of E. Consider the homomorphism

$$S'(F^1) : F^1 \longrightarrow \Omega^{1,0}_{\mathbb{H}} \otimes (E/F^1)$$

constructed in (2.8). Since any $\lambda \in H_x = U(1)$ acts on $\Omega^{1,0}_{\mathbb{H}}$ as multiplication by λ^{-2}, the image $S'(F^1)(F_1)$ is contained in the subbundle

$$\Omega^{1,0}_{\mathbb{H}} \otimes (\widetilde{V}_2/F^1) \subset \Omega^{1,0}_{\mathbb{H}} \otimes (E/F^1)$$

where $\widetilde{V}_2$ is defined in (2.7).

If the homomorphism $S'(F^1)$ is identically zero, then F^1 is left invariant by the Chern connection ∇. In that case we have $F^1 = E$ as E is irreducible. If $S'(F^1) \neq 0$, let F^2 denote the (unique) line subbundle of V_2 such that

$$S'(F^1)(F_1) \subset \Omega^{1,0}_{\mathbb{H}} \otimes ((F^1 \oplus F^2)/F^1) \subset \Omega^{1,0}_{\mathbb{H}} \otimes (\widetilde{V}_2/F^1)\,.$$

Now set $G^2 = F^1 \oplus F^2$. From the construction G^2 it is easy to see that G^2 is a holomorphic subbundle of E. Let F in (2.8) be G^2. Clearly $S'(G^2)(G^2)$ is contained in $\Omega^{1,0}_{\mathbb{H}} \otimes (\widetilde{V}_3/G^2)$. Let F^3 be the line subbundle of V_3 such that the image $S'(G^2)(G^2)$ is contained in the subbundle

$$\Omega^{1,0}_{\mathbb{H}} \otimes ((G^2 \oplus F^3)/G^2) \subset \Omega^{1,0}_{\mathbb{H}} \otimes (\widetilde{V}_3/G^2)\,.$$

Now set $G^3 = G^2 \oplus F^3$ and proceed inductively. More precisely, let $G^{i+1} = G^i \oplus F^{i+1}$, where F^{i+1} is the line subbundle of V_{i+1} such that $S'(G^i)(G^i)$ is contained in $\Omega^{1,0}_{\mathbb{H}} \otimes ((G^i \oplus F^{i+1})/G^i)$. Note that from the construction it follows that G^{i+1} is a holomorphic subbundle of E.

The subbundle G^n of E is clearly left invariant by the Chern connection. Since E is irreducible, G^n must coincide with E. But this immediately implies that the rank of V_j is one for each $j \in [1\,, n]$. This completes the proof of the theorem. $\square$

In the next section we will show existence of irreducible vector bundles of a given type and will classify them.

3. Construction of irreducible bundles

Take any integer $c \in \mathbb{Z}$. We will first show that there is exactly one equivariant line bundle with $c_1 = c$, where c_1 is defined in (2.6).

Let V be a complex vector space of dimension two; fix a nonzero vector $\omega \in \bigwedge^2 V$. Let $\mathcal{O}_{\mathbb{P}(V)}(1)$ be the tautological line bundle on the projective line $\mathbb{P}(V)$ parametrizing all one–dimensional quotients of V. Using ω, the holomorphic tangent bundle $T\mathbb{P}(V)$ gets identified with

$$\mathcal{O}_{\mathbb{P}(V)}(2) := \mathcal{O}_{\mathbb{P}(V)}(1) \otimes \mathcal{O}_{\mathbb{P}(V)}(1).$$

The group $\mathrm{SL}(V)$ acts on $\mathbb{P}(V)$, and the action lifts to $\mathcal{O}_{\mathbb{P}(V)}(1)$. The isomorphism of $T\mathbb{P}(V)$ with $\mathcal{O}_{\mathbb{P}(V)}(2)$ is $\mathrm{SL}(V)$-equivariant. Now fix a basis of V compatible with ω, i.e., such that $\omega = e_1 \wedge e_2$, where e_1, e_2 is the basis. So we have $\mathbb{P}(V) \cong \mathbb{CP}^1 \cong \mathbb{P}(\mathbb{C}^2)$. Let L denote the restriction of $\mathcal{O}_{\mathbb{P}(V)}(1)$ to $\mathbb{H} \subset \mathbb{CP}^1$. Restricting the identification of $\mathcal{O}_{\mathbb{P}(V)}(2)$ with $T\mathbb{P}(V)$ to $\mathbb{H}$ we get a holomorphic isomorphism

$$\sigma : L^{\otimes 2} \longrightarrow T\mathbb{H} \tag{3.1}$$

which is compatible with the $\mathrm{SL}(2, \mathbb{R})$-linearizations (as the isomorphism of $T\mathbb{P}(V)$ with $\mathcal{O}_{\mathbb{P}(V)}(2)$ is $\mathrm{SL}(V)$-equivariant).

There is a unique Hermitian structure on L such that if we equip $L^{\otimes 2}$ with the induced Hermitian structure, then σ is a Hermitian structure preserving isomorphism between $L^{\otimes 2}$ and $T\mathbb{H}$ equipped with the Poincaré metric. We will denote by $\mathcal{L}$ the equivariant line bundle defined by L equipped with this holomorphic Hermitian structure.

For any integer c, consider the equivariant line bundle $\mathcal{L}^{\otimes c}$. The holomorphic and the Hermitian structures on $\mathcal{L}^{\otimes c}$ are induced by the corresponding structures on $\mathcal{L}$. It is easy to see that $c_1 = c$ for this equivariant line bundle $\mathcal{L}^{\otimes c}$, where c_1 is defined in (2.6). This follows immediately from the fact that $c_1 = 1$ for the equivariant line bundle $\mathcal{L}$.

It is easy to check that up to isomorphism, there is exactly one equivariant line bundle with $c_1 = c$. Indeed, if $\mathcal{L}'$ is another equivariant line bundle with $c_1 = c$, then consider the equivariant line bundle $\mathrm{Hom}(\mathcal{L}^{\otimes c}, \mathcal{L}')$. Since the isotropy group H_x acts trivially on the fiber $\mathrm{Hom}(\mathcal{L}^{\otimes c}, \mathcal{L}')_x$ for all $x \in \mathbb{H}$, there is a unique isomorphism, up to multiplication by a global constant $\exp(2\pi\sqrt{-1}r)$ for some $r \in \mathbb{R}$, between the equivariant line bundles $\mathcal{L}'$ and $\mathcal{L}^{\otimes c}$.

Next we will construct equivariant rank two vector bundles. Consider the C^∞ complex vector bundle

$$\mathcal{V}(2, c) := \mathcal{L}^{\otimes c} \oplus \mathcal{L}^{\otimes (c+2)} . \tag{3.2}$$

The Hermitian structure on $\mathcal{L}$ induces a Hermitian structure on the rank two bundle $\mathcal{V}(2,c)$. The Hermitian structures on $\mathcal{L}^{\otimes c}$ and $\mathcal{L}^{\otimes(c+2)}$ are the obvious ones obtained from the Hermitian structure on $\mathcal{L}$ and the decomposition in (3.2) is orthogonal. This determines the Hermitian structure of $\mathcal{V}(2,c)$. The $\mathrm{SL}(2,\mathbb{R})$-linearizations of $\mathcal{L}^{\otimes c}$ and $\mathcal{L}^{\otimes(c+2)}$ induce a $\mathrm{SL}(2,\mathbb{R})$-linearization of $\mathcal{V}(2,c)$ that preserves the Hermitian structure. We will construct a holomorphic structure on $\mathcal{V}(2,c)$. First note that the holomorphic structures of $\mathcal{L}^{\otimes c}$ and $\mathcal{L}^{\otimes(c+2)}$ induce a holomorphic structure on the direct sum $\mathcal{V}(2,c)$. But we need to modify this holomorphic structure.

Consider the line bundle

$$\xi_c := \Omega^{0,1}_{\mathbb{H}} \otimes \mathrm{Hom}(\mathcal{L}^{\otimes(c+2)}, \mathcal{L}^{\otimes c}) . \tag{3.3}$$

The section σ in (3.1) identifies $\mathrm{Hom}(\mathcal{L}^{\otimes(c+2)}, \mathcal{L}^{\otimes c}) = \mathcal{L}^{-2}$ with the holomorphic cotangent bundle $\Omega^{1,0}_{\mathbb{H}}$. Therefore, the line bundle ξ_c is canonically trivialized, with the trivialization given by the section of $\Omega^{1,1}_{\mathbb{H}}$ defined by the Poincaré metric on $\mathbb{H}$.

The holomorphic structure of $\mathcal{L}$ induces a holomorphic structure on any tensor power of it. For any integer j, let $\overline{\partial}_{\mathcal{L}^{\otimes j}}$ denote the Dolbeault operators defining the holomorphic structure of $\mathcal{L}^{\otimes j}$. For any $\delta \in \mathbb{C}$, consider the differential operator

$$\overline{\partial}_\delta : \mathcal{V}(2,c) \longrightarrow \Omega^{0,1}_{\mathbb{H}} \otimes \mathcal{V}(2,c)$$

defined by

$$\overline{\partial}_\delta = \begin{pmatrix} \overline{\partial}_{\mathcal{L}^{\otimes c}} & \delta \\ 0 & \overline{\partial}_{\mathcal{L}^{\otimes(c+2)}} \end{pmatrix} . \tag{3.4}$$

Using the canonical identification of the line bundle ξ_c in (3.3) with the trivial line bundle with fiber $\mathbb{C}$, the complex number δ in (3.4) gives a homomorphism from $\mathcal{L}^{\otimes(c+2)}$ to $\Omega^{0,1}_{\mathbb{H}} \otimes \mathcal{L}^{\otimes c}$. It is easy to see that the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{V}(2,c)$ preserves the holomorphic structure defined by $\overline{\partial}_\delta$. Indeed, this is an immediate consequence of the fact that $\overline{\partial}_{\mathcal{L}^{\otimes c}}$, $\overline{\partial}_{\mathcal{L}^{\otimes(c+2)}}$ and the above mentioned homomorphism defined by δ are all invariant under the action of $\mathrm{SL}(2,\mathbb{R})$.

We already constructed an invariant Hermitian structure on $\mathcal{V}(2,c)$ for the action of $\mathrm{SL}(2,\mathbb{R})$. Since $\overline{\partial}_\delta$ is also invariant the action of $\mathrm{SL}(2,\mathbb{R})$,

$$\mathcal{W}(\delta,c) := (\mathcal{V}(2,c), \overline{\partial}_\delta),$$

is an equivariant vector bundle.

The following proposition describes all irreducible rank two vector bundles in terms of the above equivariant vector bundles $\mathcal{W}(\delta,c)$.

Proposition 3.1. *The equivariant vector bundle $\mathcal{W}(\delta,c)$ is irreducible if and only if $\delta \neq 0$. The two irreducible vector bundles $\mathcal{W}(\delta,c)$ and $\mathcal{W}(\delta',c')$*

are isomorphic if and only if $c = c'$ as well as $|\delta| = |\delta'|$. Any rank two irreducible vector bundle is isomorphic to some $\mathcal{W}(\delta, c)$.

Proof. If ζ is a line subbundle of $\mathcal{W}(\delta, c)$ left invariant by the Chern connection ∇, then its orthogonal complement $\zeta^{\perp}$ is also left invariant by ∇ (as ∇ preserves the Hermitian structure). This implies that either ζ_x or $(\zeta^{\perp})_x$ must coincide with $(\mathcal{L}^{\otimes(c+2)})_x$. Consequently, $\mathcal{L}^{\otimes(c+2)}$ coincides with either ζ or $\zeta^{\perp}$. Now, it is easy to see that $\mathcal{L}^{\otimes(c+2)}$ is *not* a holomorphic subbundle unless $\delta = 0$. This proves that $\mathcal{W}(\delta, c)$ is irreducible if and only if $\delta \neq 0$.

If $\mathcal{W}(\delta', c')$ is isomorphic to $\mathcal{W}(\delta, c)$ as an equivariant vector bundle, then considering the action of H_x on their fibers over x we see that $c = c'$ and the isomorphism of $\mathcal{W}(\delta', c')$ with $\mathcal{W}(\delta, c)$ must preserve the decomposition of the (common) underlying C^∞ vector bundle $\mathcal{V}(2, c)$ given in (3.2). In other words, the isomorphism individually preserves the subbundles $\mathcal{L}^{\otimes c}$ and $\mathcal{L}^{\otimes(c+2)}$. Furthermore, since $\mathrm{SL}(2,\mathbb{R})$ acts transitively on $\mathbb{H}$, the induced isomorphisms of $\mathcal{L}^{\otimes c}$ and $\mathcal{L}^{\otimes(c+2)}$ must be constant scalar multiplications.

Consider the C^∞ automorphism

$$T = \exp(\sqrt{-1}\theta)\mathrm{Id}_{\mathcal{L}^{\otimes c}} \oplus \exp(\sqrt{-1}\theta')\mathrm{Id}_{\mathcal{L}^{\otimes(c+2)}}$$

of $\mathcal{V}(2, c)$, where θ and θ' are real numbers. We observed above that any $\mathrm{SL}(2,\mathbb{R})$ action preserving unitary automorphisms of $\mathcal{V}(2, c)$ must be of this type. It is straight-forward to check that the conjugation $T^{-1} \circ \overline{\partial}_\delta \circ T$ of the Dolbeault operator defined in (3.4) satisfies the identity

$$T^{-1} \circ \overline{\partial}_\delta \circ T = \begin{pmatrix} \overline{\partial}_{\mathcal{L}^{\otimes c}} & \exp(\sqrt{-1}(\theta' - \theta))\delta \\ 0 & \overline{\partial}_{\mathcal{L}^{\otimes(c+2)}} \end{pmatrix}$$

Consequently, $\mathcal{W}(\delta, c)$ and $\mathcal{W}(\delta', c')$ are isomorphic if and only if the two conditions in the statement, namely $c = c'$ and $|\delta| = |\delta'|$, are satisfied.

Let E be an irreducible vector bundle of rank two. From Theorem 2.7 we know that $c_1 = c$ and $c_2 = c + 2$ for some $c \in \mathbb{C}$. Consequently, the $\mathrm{SL}(2,\mathbb{R})$-linearized bundle E is isomorphic to $\mathcal{V}(2, c)$. Now it is easy to check that E is isomorphic to some $\mathcal{W}(\delta, c)$, and the proof of the proposition is complete. $\square$

We will now extend Proposition 3.1 to higher ranks. This extension is rather straight-forward. We first proved the rank two case since it is notationally simpler.

Let

$$\mathcal{V}(n, c) := \mathcal{L}^{\otimes c} \oplus \mathcal{L}^{\otimes(c+2)} \oplus \cdots \oplus \mathcal{L}^{\otimes(c+2n-4)} \oplus \mathcal{L}^{\otimes(c+2n-2)} \tag{3.5}$$

be the C^∞ rank n vector bundle over $\mathbb{H}$. The vector bundle $\mathcal{V}(n, c)$ has a $\mathrm{SL}(2,\mathbb{R})$-linearization and a Hermitian structure defined by the direct sum of the corresponding structures on the individual direct summands in the decomposition (3.5); as before, the Hermitian structure on any tensor power of $\mathcal{L}$ is induced by the Hermitian structure on $\mathcal{L}$. The action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{V}(n,c)$ clearly preserves the Hermitian structure.

Take $\underline{\delta} := (\delta_1, \delta_2, \cdots, \delta_{n-1}) \in \mathbb{C}^{n-1}$. Consider the holomorphic structure on $\mathcal{V}(n,c)$ defined by the Dolbeault operator

$$\overline{\partial}_{\underline{\delta}} = \begin{pmatrix} \overline{\partial}_{\mathcal{L}^{\otimes c}} & \delta_1 & 0 & \cdots & 0 & 0 \\ 0 & \overline{\partial}_{\mathcal{L}^{\otimes(c+2)}} & \delta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \overline{\partial}_{\mathcal{L}^{\otimes(c+2n-4)}} & \delta_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & \overline{\partial}_{\mathcal{L}^{\otimes(c+2n-2)}} \end{pmatrix}. \tag{3.6}$$

In other words, the (i,i)-th entry is the Dolbeault operator $\overline{\partial}_{\mathcal{L}^{\otimes(c+2i-2)}}$ on $\mathcal{L}^{\otimes(c+2i-2)}$, the $(i,i+1)$-th entry is δ_i and the rest of the entries are zero (recall that $\overline{\partial}_{\mathcal{L}^{\otimes j}}$ denotes the Dolbeault operator for the homomorphic structure induced by the one on $\mathcal{L}$). Note that using σ in (3.1) the line bundle

$$\Omega^{0,1}_{\mathbb{H}} \otimes \mathrm{Hom}(\mathcal{L}^{\otimes(c+2i)}, \mathcal{L}^{\otimes(c+2i-2)})$$

is identified with the trivial line bundle. Using this identification, δ_i defines a homomorphism from $\mathcal{L}^{\otimes(c+2i)}$ to $\Omega^{0,1}_{\mathbb{H}} \otimes \mathcal{L}^{\otimes(c+2i-2)}$.

This Dolbeault operator $\overline{\partial}_{\underline{\delta}}$ is preserved by the action of $\mathrm{SL}(2,\mathbb{R})$. Therefore,

$$\mathcal{W}(\underline{\delta}, c) := (\mathcal{V}(n,c), \overline{\partial}_{\underline{\delta}})$$

is an equivariant vector bundle of rank n.

Define $|\underline{\delta}| := (|\delta_1|, |\delta_2|, \cdots, |\delta_{n-1}|) \in \mathbb{R}^{n-1}$. Now we are in a position to generalize Proposition 3.1.

THEOREM 3.2. *The equivariant vector bundle $\mathcal{W}(\underline{\delta}, c)$ is irreducible if and only if $\delta_i \neq 0$ for every $i \in [1, n-1]$. The two irreducible vector bundles $\mathcal{W}(\underline{\delta}, c)$ and $\mathcal{W}(\underline{\delta}', c')$ are isomorphic if and only if $c = c'$ as well as $|\underline{\delta}| = |\underline{\delta}'|$. Any rank n irreducible vector bundle is isomorphic to some $\mathcal{W}(\underline{\delta}, c)$.*

Proof. For any $j \in [1, n-1]$, define the subbundles

$$\mathcal{V}_j := \bigoplus_{i=0}^{j-1} \mathcal{L}^{\otimes(c+2i)} \subset \mathcal{V}(n,c)$$

and $\mathcal{W}_j := (\mathcal{V}_j)^{\perp} = \bigoplus_{i=j}^{n-1} \mathcal{L}^{\otimes(c+2i)} \subset \mathcal{V}(n,c)$, the orthogonal complement.

If $\delta_i = 0$, then clearly both $\mathcal{V}_i$ and $\mathcal{W}_i$ are holomorphic subbundles of $\mathcal{W}(\underline{\delta}, c)$. Therefore, if $\delta_i = 0$, then $\mathcal{W}(\underline{\delta}, c)$ is not irreducible.

Conversely, if $\mathcal{W}(\underline{\delta}, c)$ is not irreducible, then we have two homomorphic subbundles W and $W^{\perp}$ of positive rank left invariant by the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{W}(\underline{\delta}, c)$. Considering the action of an isotropy subgroup H_x we see that both W and $W^{\perp}$ must be of the form $\bigoplus_i \mathcal{L}^{\otimes(c+2i)}$ with respect to the decomposition in (3.5) (since the components in (3.5) are precisely the eigenbundles for characters of the isotropy subgroups).

Therefore, there is $i \in [0, n-1]$ such that the component $\mathcal{L}^{\otimes(c+2i-2)}$ in (3.5) is contained in W and the component $\mathcal{L}^{\otimes(c+2i)}$ is contained in $W^{\perp}$. But this would imply that $\delta_i = 0$. Indeed, for a local section s of $\mathcal{L}^{\otimes(c+2i)}$, the component of $\overline{\partial}_{\underline{\delta}}(s)$ in $\mathcal{L}^{\otimes(c+2i-2)}$ coincides with $\delta_i s$. So, if $\delta_i \neq 0$, then $W^{\perp}$ is *not* closed under $\overline{\partial}_{\underline{\delta}}$. This contradicts that fact that $W^{\perp}$ is a holomorphic subbundle of $\mathcal{W}(\underline{\delta}, c)$.

Therefore, $\mathcal{W}(\underline{\delta}, c)$ is irreducible if and only if $\delta_i \neq 0$ for each $i \in [1, n-1]$.

Any isometry of $\mathcal{V}(\underline{\delta}, c)$ commuting with the action of $\mathrm{SL}(2,\mathbb{R})$ must be a diagonal matrix (with respect to the decomposition in (3.5)) with all the diagonal entries of absolute value one. Let T denote the diagonal matrix with (i,i)-th entry $\exp(\sqrt{-1}\theta_i)$ with $\theta_i \in \mathbb{R}$. It is easy to check that

$$T^{-1} \circ \overline{\partial}_{\underline{\delta}} \circ T = \overline{\partial}_{\underline{\eta}}$$

where $\eta_i = \exp(\sqrt{-1}(\theta_{i+1} - \theta_i))\delta_i$ and $\overline{\partial}_{\underline{\delta}}$, $\overline{\partial}_{\underline{\eta}}$ are as in (3.6). From this identity it follows that $\mathcal{W}(\underline{\delta}, c)$ and $\mathcal{W}(\underline{\delta}', c')$ are isomorphic if and only if $c = c'$ as well as $|\underline{\delta}| = |\underline{\delta}'|$.

Let E be an irreducible vector bundle. Using Theorem 2.7 we see that the there is a unitary C^{∞} isomorphism of E with some $\mathcal{V}(\underline{\delta}, c)$ that intertwines the actions of $\mathrm{SL}(2,\mathbb{R})$. The $\mathrm{SL}(2,\mathbb{R})$-linearized line bundle $\mathcal{L}^{\otimes j}$ has a nonzero section invariant under the action of $\mathrm{SL}(2,\mathbb{R})$ if and only if $j = 0$. Using this and the earlier observation that any equivariant line bundle is isomorphic to some power of $\mathcal{L}$ it follows that the holomorphic structure on E is of the form (3.6). This completes the proof of the theorem. □

References

[1] S. Kobayashi, *Differential geometry of complex vector bundles*, Publications of the Mathematical Society of Japan, vol. 15, Iwanami Shoten, Tokyo, and Princeton University Press, Princeton, NJ, 1987.

[2] S. Mac Lane, *Homology*, Grundlehren der mathematischen Wissenschaften, vol. 114, Springer-Verlag, New York-Heidelberg-Berlin, 1975.

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