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RICHNESS OF INVARIANT SUBSPACE LATTICES FOR A CLASS OF OPERATORS

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Abstract. In 1994, H. Mohebi and M. Radjabalipour proved that every operator in a certain class of operators on reflexive Banach spaces has infinitely many invariant subspaces. In this paper, we prove that the invariant subspace lattice for every operator in the class of operators on (general) Banach spaces is rich, and we give an example of an operator T that has infinitely many invariant subspaces, while the invariant subspace lattice $\mathrm{Lat}(T)$ for T is not rich. Here we call an invariant lattice subspace $\mathrm{Lat}(T)$ for the operator T rich if there exists an infinite dimensional Banach space E such that $\mathrm{Lat}(T)$ contains a sublattice that is order isomorphic to the lattice $\mathrm{Lat}(E)$ of all closed liner subspaces of E. Finally we show that the invariant subspace lattice $\mathrm{Lat}(T)$ for a bounded linear operator T on a reflexive Banach space X is reflexive-rich if and only if the invariant subspace lattice $\mathrm{Lat}(T^*)$ for T^* is reflexive-rich.

S. Brown [2] showed that if T is a hyponormal operator on a Hilbert space such that $\sigma(T)$ is dominating in some nonempty open subset G of the complex plane C, then T has a non-trivial invariant subspace. The work initiated by S. Brown has been generalized by J. Eschmeier and B. Prunaru [3].

To state the result of J. Eschmeier and B. Prunaru, we recall that, given a bounded linear operator B on a Banach space and a nonempty open set $G \subset C$, we say that B is decomposable in G if for every open subset $H \subset G$ there exists an invariant subspace M of B such that $\sigma(B|M) \subset \overline{H}$ and $\sigma(B/M) \subset C \setminus H$, where B/M is the quotient operator induced by B. In view of [9] (also see [2, p. 95]) and [4], every hyponormal operator on a Hilbert space is up to a similarity the restriction of an operator B that is decomposable in any open subset G of C.

J. Eschmeier and B. Prunaru [3] proved that if T is a bounded linear operator on a Banach space such that T is up to a similarity the restriction of

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an operator B decomposable in some nonempty open subset G of C and $\sigma(T)$ is dominating in the open set G, then T has a non-trivial invariant subspace. Moreover, if the essential spectrum $\sigma_{\mathrm{e}}(T)$ of T is dominating in the open set G, then the invariant subspace lattice $\mathrm{Lat}(T)$ for T is rich.

The results of [3] were, in turn, improved by H. Mohebi and M. Radjabalipour [8]. In particular, Mohebi and Radjabalipour showed that for reflexive Banach spaces the condition of decomposability of B can be replaced by a weaker condition. More precisely, they proved the following theorem:

Theorem A ([8, Theorem I.1]). *Assume the operators $T \in B(X)$ and $B \in B(Z)$ on the reflexive Banach spaces X and Z and the nonempty open set G in the complex plane C satisfy the following conditions:*

(1) *$qT = Bq$ for some injective $q \in B(X, Z)$ with a closed range qX.*
(2) *There exist sequences $\{G(n)\}$ of open sets and $\{M(n)\}$ of invariant subspaces of B such that $\overline{G}(n) \subset G(n+1)$, $G = \cup_n G(n)$, $\sigma(B|M(n)) \subset C \backslash G(n)$ and $\sigma(B/M(n)) \subset \overline{G}(n)$, $n = 1, 2, \ldots$.*
(3) *$\sigma(T) \backslash \sigma_p(B)$ is dominating in G.*

Then T has infinitely many invariant subspaces.

In this paper we give an extension of Theorem A. We show that, given two operators $T \in B(X)$ and $B \in B(Z)$ on (general) Banach spaces X and Z such that T is up to a similarity the restriction of B and there exist sequences $\{G(n)\}$ of open sets and $\{M(n)\}$ of invariant subspaces of B as in Theorem A, if $\sigma(T) \setminus \sigma_p(B^{**})$ is dominating in the open set G then T has infinitely many invariant subspaces. Moreover, if $\sigma_{\mathrm{e}}(T) \setminus \sigma(B^{**})$ is dominating in the open set G, then $\mathrm{Lat}(T)$ is rich. In particular, the reflexivity is not needed anymore. In addition, we give an example in which the operator T has infinitely many invariant subspaces, while the invariant subspace lattice $\mathrm{Lat}(T)$ for T is not rich. Finally we show that the invariant subspace lattice $\mathrm{Lat}(T)$ for a bounded linear operator T on a reflexive Banach space X is reflexive-rich if and only if the invariant subspace lattice $\mathrm{Lat}(T^*)$ for T^* is reflexive-rich.

To prove these results we first need to recall some basic terminology and facts and prove some lemmas.

Let E be a Banach space. Then $\mathrm{Lat}(E)$ denotes the lattice of all closed linear subspaces of E. If M is a nonempty subset of E, then $M^{\perp}$ denotes the annihilator of M. If N is a nonempty subset of E^*, then ${}^{\perp}N$ denotes the preannihilator of N. If $M \subset \mathrm{Lat}(E)$, $A \in B(E)$ and $AM \subset M$, then we denote by $A|M$ the restriction of A onto M, and by A/M the quotient operator induced by A on the quotient space E/M. It is well known that the essential spectrum is given by

$$\begin{aligned}\sigma_e(A) = \big\{\lambda \in C : \operatorname{ran}(\lambda - A) \text{ is not closed}\big\} \\ \cup \big\{\lambda \in C :\ \operatorname{ran}(\lambda - A) \text{ is closed and } \dim\ker(\lambda - A) = \infty\big\} \\ \cup \{\lambda \in C :\ \operatorname{ran}(\lambda - A) \text{ is closed and } \dim\ker(\lambda - A^*) = \infty\big\}.\end{aligned}$$

A subset σ of the complex plane C will be called dominating in the open set G if $\|f\| = \sup\{|f(\lambda)| : \lambda \in \sigma \cap G\}$ holds for all $f \in P^\infty(G)$, where $P^\infty(G)$ stands for the smallest w^*-closed subspace of $L^\infty(G)$ containing all polynomials. It is well-known that the point evaluation $E_\lambda : P^\infty(G) \to C$, $f \to f(\lambda)$, is a w^*-continuous linear functional.

Lemma 1. *Let X be a Banach space, $T \in B(X)$, and $\lambda \in C$. If $\dim\ker(\lambda - T) = \infty$, then the invariant subspace lattice $\operatorname{Lat}(T)$ for T is rich.*

Proof. Let M be a closed linear subspace in the Banach space $E = \ker(\lambda - T)$. Then $(\lambda - T)x = 0$ for each $x \in M$. Thus we obtain that $TM \subset M$ and $M \in \operatorname{Lat}(T)$. Therefore the lattice $\operatorname{Lat}(E)$ of all closed linear subspaces of the infinite dimensional Banach space $E = \ker(\lambda - T)$ is order isomorphic to a sublattice of $\operatorname{Lat}(T)$. □

Lemma 2. *Let X be a Banach space, $T \in B(X)$, and $\lambda \in C$. If $\dim(X/\overline{\operatorname{ran}(\lambda - T)}) = \infty$, then the invariant subspace lattice $\operatorname{Lat}(T)$ for T is rich.*

Proof. Assume without loss of generality that $\lambda = 0$. Let $\pi : X \longrightarrow X/\overline{\operatorname{ran} T}$ be the canonical quotient map. Let M be a given closed linear subspace in the Banach space $X/\overline{\operatorname{ran} T}$. Then there is a closed linear subspace $N_M = \pi^{-1}(M)$ in X such that $N_M \supset \overline{\operatorname{ran} T}$. Therefore $TN_M \subset \overline{\operatorname{ran} T} \subset N_M$. Consequently the map $\Phi : \operatorname{Lat}(X/\overline{\operatorname{ran} T}) \longrightarrow \operatorname{Lat}(T)$, $M \longmapsto \pi^{-1}(M)$, defines a lattice embedding. □

Lemma 3. *Let X be a Banach space, $T \in B(X)$, and $\lambda \in C$. Then we have:*

1. *If either $\dim\ker(\lambda - T^*)$ or $\dim(X/\overline{\operatorname{ran}(\lambda - T)})$ is equal to ∞, then the invariant subspace lattice $\operatorname{Lat}(T)$ for T and the invariant subspace lattice $\operatorname{Lat}(T^*)$ for T^* are rich.*
2. *If either $\dim\ker(\lambda - T)$ or $\dim(X^*/\operatorname{ran}(\lambda - T^*))$ is equal to ∞, and if $\operatorname{ran}(\lambda - T)$ is closed in X, then the invariant subspace lattice $\operatorname{Lat}(T)$ for T and the invariant subspace lattice $\operatorname{Lat}(T^*)$ for T^* are rich.*

Proof. Without loss of generality we may assume that $\lambda = 0$.

(1) Since $\ker T^* = \overline{\operatorname{ran} T}^{\perp} \cong (X/\overline{\operatorname{ran} T})^*$, it follows that $\dim\ker T^* = \infty$ if and only if $\dim(X/\overline{\operatorname{ran} T}) = \infty$. Thus the conclusion of (1) follows from Lemmas 1 and 2.

(2) Since $\operatorname{ran} T$ is closed in X, it follows from the Closed Range Theorem that $(\ker T)^{\perp} = \operatorname{ran}(T^*)$. Therefore $(\ker T)^* \cong X^*/\ker T^{\perp} = X^*/\operatorname{ran}(T^*)$. Consequently $\dim(X^*/\operatorname{ran}(T^*)) = \infty$ if and only if $\dim\ker T = \infty$. Thus the conclusion of (2) follows from Lemmas 1 and 2. $\square$

The following result improves Lemma I.3 in [8] and Proposition 1.4 in [3].

LEMMA 4. *Let $T \in B(X)$ and $B \in B(Z)$ be operators on Banach spaces X and Z, and let $q \in B(Z, X^*)$ be a surjection with $qB = T^*q$. Let $\lambda \in C$, and let* $\operatorname{ran}(\lambda - T)$ *be not closed in X. If either $\lambda - B$ or $(\lambda - B)|\ker q$ has a dense range, then for any finite codimensional subspace N in X^* there exists a sequence $\{z_n\}$ in Z such that $qz_n \in N$, $\|qz_n\| = 1$ for all n, and $\lim_{n\to\infty}\|(\lambda - B)z_n\| = 0$.*

Proof. Assume without loss of generality that $\lambda = 0$. It is obvious that $\ker q$ is an invariant subspace for B. Define the operator $\widetilde{q} : Z/\ker q \longrightarrow X^*$ by $\widetilde{q}(z + \ker q) = qz$ for all $z \in Z$. Since $\widetilde{q}$ is a bijection, we can assume without loss of generality that $X^* = Z/\ker q$. Therefore q is the canonical quotient map from Z into $Z/\ker q$ and T^* is the quotient operator induced by B on $Z/\ker q$.

Since N is a finite codimensional subspace of X^*, there exists a finite dimensional subspace M of X^* such that $X^* = M \oplus N$.

We first show that if $\overline{B(\ker q)} = \ker q$, then the conclusion of the lemma is valid. Define the operator $\widetilde{T^*} : N \longrightarrow X^*$ by $\widetilde{T^*}y = T^*y$ for all $y \in N$. Since $\dim M < \infty$ and $\operatorname{ran}(T^*)$ is not closed in X^*, $\operatorname{ran}(\widetilde{T^*})$ is not closed in X^*. Therefore there exists a sequence $\{y_n\}$ of unit vectors in N such that $\lim_{n\to\infty}\widetilde{T^*}y_n = 0$. Since $q \in B(Z, X^*)$ is a surjection, there exists $u_n \in Z$ such that $qu_n = y_n$. It is easy to see that $\lim_{n\to\infty}\|Bu_n + \ker q\| = 0$. It follows that for every natural number n there exists $z'_n \in \ker q$ such that $\lim_{n\to\infty}\|Bu_n - z'_n\| = 0$. Since $\overline{B(\ker q)} = \ker q$, there exists $v_n \in \ker q$ such that $\|z'_n - Bv_n\| < 1/n$. Thus we have $\lim_{n\to\infty}\|Bu_n - Bv_n\| = 0$. Set $z_n = u_n - v_n$. Then $z_n \in Z$, $qz_n \in N$, and $\|qz_n\| = 1$ for all n, and $\lim_{n\to\infty}\|Bz_n\| = 0$.

Next, we show that if $\overline{BZ} = Z$, then the conclusion of the lemma is still valid. Put $W = \overline{B(\ker q)} \subset \ker q$. Define the operator $\widetilde{B} : Z/\ker q \longrightarrow Z/W$ by $\widetilde{B}(u + \ker q) = Bu + W$ for all $u + \ker q \in Z/\ker q$. Then it is easy to see that $\widetilde{B}$ is a well-defined bounded linear operator from $Z/\ker q$ into Z/W.

We now show that $\overline{\widetilde{B}(Z/\ker q)} = Z/W$. In fact, for any $v + W \in Z/W$ and any $\epsilon > 0$ it follows from $\overline{BZ} = Z$ that there exists $v' \in Z$ such that $\|Bv' - v\| < \epsilon$. Consequently

$$\begin{aligned}\|\widetilde{B}(v' + \ker q) - (v + W)\| &= \|(Bv' + W) - (v + W)\| \\ &= \|(Bv' - v) + W\| < \epsilon.\end{aligned}$$

Next we prove that $\operatorname{ran}\widetilde{B}$ is not closed in Z/W. Indeed, assume $\operatorname{ran}(\widetilde{B})$ is closed. Then $\operatorname{ran}\widetilde{B} = \overline{\operatorname{ran}\widetilde{B}} = Z/W$. For every $u + \ker q \in Z/\ker q$, it follows from $u + W \in Z/W = \operatorname{ran}\widetilde{B}$ that there exists $v + \ker q \in Z/\ker q$ such that $u + W = \widetilde{B}(v + \ker q) = Bv + W$. Therefore $Bv - u \in W \subset \ker q$. Consequently

$$u + \ker q = Bv + \ker q = T^*(v + \ker q).$$

From the above argument it follows that T^* is a surjection. Hence $\operatorname{ran}(T^*)$ is closed. This contradicts the assumption that $\operatorname{ran} T$ is not closed.

Define the operator $\overline{B}$: $N \longrightarrow Z/W$ by $\overline{B}y = \widetilde{B}y$ for all $y \in N$. Since $\dim M < \infty$ and $\operatorname{ran}\widetilde{B}$ is not closed in Z/W, $\operatorname{ran}\overline{B}$ is not closed in Z/W. Therefore there exists a sequence $\{y_n\}$ of unit vectors in N such that $\lim_{n\to\infty}\overline{B}y_n = 0$. Assume that $y_n = u_n + \ker q$. It is easy to see that $\lim_{n\to\infty}\|Bu_n + W\| = 0$. It follows that for every natural number n there exists $z'_n \in W$ such that $\lim_{n\to\infty}\|Bu_n - z'_n\| = 0$. Since $W = \overline{B(\ker q)}$, there exists $v_n \in \ker q$ such that $\|z'_n - Bv_n\| < 1/n$. Thus we have $\lim_{n\to\infty}\|Bu_n - Bv_n\| = 0$. Set $z_n = u_n - v_n$. Then $z_n \in Z$, $qz_n \in N$, and $\|qz_n\| = 1$ for all n, and $\lim_{n\to\infty}\|Bz_n\| = 0$. $\square$

Theorem 1. *Assume the operators $T \in B(X)$ and $B \in B(Z)$ on Banach spaces X and Z and the nonempty open set G in the complex plane C satisfy the following conditions:*

(1) *$qB = T^*q$ for some surjective $q \in B(Z, X^*)$.*
(2) *There exist sequences $\{G(n)\}$ of open sets and $\{M(n)\}$ of invariant subspaces of B such that $\overline{G}(n) \subset G(n+1), G = \cup_n G(n)$, $\sigma(B|M(n)) \subset \overline{G}(n)$ and $\sigma(B/M(n)) \subset C\backslash G(n)$, $n = 1, 2, \ldots$.*

Then we have:

(a) *If the set $K = \sigma(T)\backslash\{\lambda \in C : \overline{(\lambda - B)Z} \neq Z$ and $\overline{(\lambda - B)\ker q} \neq \ker q\}$ is dominating in G, then T has infinitely many invariant subspaces.*
(b) *If $K \cap \sigma_e(T)$ is dominating in G, then the invariant subspace lattice* $\operatorname{Lat}(T)$ *for T and the invariant subspace lattice* $\operatorname{Lat}(T^*)$ *for T^* are rich.*

To prove Theorem 1 we first establish some lemmas. Throughout the proof of Theorem 1, we assume that X, Z, T, B, q, G, $G(n)$, $M(n)$, and K are as in Theorem 1. Put $M(G) = \cup_n M(n)$, and define

$$K_1 = \big\{\lambda \in C : \text{ for any finite codimensional subspace } N \text{ in } X^* \text{ there exists a sequence } \{z_n\} \text{ in } Z \text{ satisfying } qz_n \in N, \|qz_n\| = 1 \text{ for all } n, \text{ and } \lim_{n\to\infty}\|(\lambda - B)z_n\| = 0\big\}.$$

It is easy to see that $M(n) \subset M(n+1)$ for $n = 1, 2, \ldots$ (also see [8, Lemma I.2]). Therefore for every $x \in X$ and every $z \in M(G)$ there exists a natural number n such that $z \in M(n)$. Define a functional $x \otimes z : H^\infty(G) \longrightarrow C$ by

$$x \otimes z(f) = \langle x, qf(B|M(n))z\rangle, \quad f \in P^\infty(G),$$

where $f(B|M(n))$ is defined by the Riesz-Dunford functional calculus with analytic functions. It is easy to see that $x \otimes z$ is a well-defined w^*-continuous linear functional which is independent of the particular choice of n.

Lemma 5 (cf. [3, Proposition 2.8] or [6, Lemma 2]). *Let r, s be natural numbers. Consider non-negative real numbers $c_1, c_2, \dots, c_r$ with $c_1 + c_2 + \cdots + c_r = 1$ and complex numbers $\lambda_1, \lambda_2, \dots, \lambda_r \in K_1 \cap G$. If $a_1, a_2, \dots, a_s \in X$, $b_1, b_2, \dots, b_s \in M(G)$, and $\epsilon > 0$ are arbitrary, then there are vectors $x \in X$ and $z \in M(G)$ such that $\|x\| \leq 3$, $\|qz\| \leq 2$ and*

(1) $\|(c_1 E_{\lambda_1} + c_2 E_{\lambda_2} + \cdots + c_r E_{\lambda_r}) - x \otimes z\| < \epsilon$;
(2) $\max\{\|x \otimes b_j\| : j = 1, 2, \dots, s\} < \epsilon$, $\max\{\|a_j \otimes z\| : j = 1, 2, \dots, s\} < \epsilon$.

Proof. The proof of Lemma 5 is essentially the same as that of Proposition 2.8 in [3]. □

Let E be a nonempty set and m a natural number. We define

$$E^m = \{(x_1, x_2, \dots, x_m) : x_1, x_2, \dots, x_m \in E\},$$
$$M(m, E) = \{(x_{jk}) : x_{jk} \in E, j, k = 1, 2, \dots, m\}.$$

We write $M(\infty, E)$ for the set of all infinite matrices $(x_{jk})_{j,k\geq 1}$ with coefficients x_{jk} $(j, k = 1, 2, \dots)$ in E. Put $Q = L^1(G)/{}^\perp H^\infty(G)$. From Lemma 5 we derive the following result as in [3].

Lemma 6 (cf. [3, Proposition 2.6]). *If K_1 is dominating in G, then for each matrix $L = (L_{jk})_{j,k\geq 1} \in M(\infty, Q)$ there are sequences $\{x_m\}$ and $\{z_m\}$ such that*

(1) $x_m \in X^m, z_m \in [M(G)]^m$;
(2) *for each natural number j, the limits $x(j) = \lim_{m\to\infty} x_m(j) \in X$ and $x^*(j) = \lim_{m\to\infty} qz_m(j) \in X^*$ exist, where $x_m(j)$ and $z_m(j)$ denote the j-th components of x_m and z_m, respectively;*
(3) *for all natural numbers j and k we have $L_{jk} = \lim_{m\to\infty} x_m(j) \otimes z_m(k)$, where the limit is taken in Q.*

From Lemma 6 we deduce as in [3] the following result.

Lemma 7. *Assume the operators $T \in B(X)$ and $B \in B(Z)$ on Banach spaces X and Z and the nonempty open set G in the complex plane C satisfy the following conditions:*

(1) *$qB = T^* q$ for some surjective $q \in B(Z, X^*)$.*
(2) *There exist sequences $\{G(n)\}$ of open sets and $\{M(n)\}$ of invariant subspaces of B such that $\overline{G}(n) \subset G(n+1)$, $G = \cup_n G(n)$, $\sigma(B|M(n)) \subset \overline{G}(n)$ and $\sigma(B/M(n)) \subset C \backslash G(n)$, $n = 1, 2, \dots$.*

(3) *The set K_1 is dominating in G.*

Then the invariant subspace lattice $\mathrm{Lat}(T)$ *for T and the invariant subspace lattice* $\mathrm{Lat}(T^*)$ *for T^* are rich.*

We are now in a position to prove Theorem 1.

Proof of Theorem 1. If K is dominating in G and $K\backslash\sigma_e(T)$ is a finite set, then by the maximum modulus principle for analytic functions the set $K \cap \sigma_e(T)(= K\backslash(K\backslash\sigma_e(T)))$ is dominating in G. Therefore, if K is dominating in G and $K \cap \sigma_e(T)$ is not dominating in G, then $K\backslash\sigma_e(T)$ is an infinite set. Hence the set

$$\sigma(T) \setminus \{\lambda \in C : \ \mathrm{ran}(\lambda - T) \text{ is not closed}, \text{ or } \dim\ker(\lambda - T) = \infty\}$$

is also an infinite set. It is easy to see from this that either $\ker(\lambda - T)$ or $\overline{\mathrm{ran}(\lambda - T)}$ is a nontrivial invariant subspace of T for infinitely many complex numbers λ.

To prove Theorem 1 it therefore suffices to prove the second assertion of Theorem 1. Put

$$\begin{aligned} K_0 = K \cap \{\lambda \in C : \mathrm{ran}(\lambda - T) \text{ is closed}\} \\ \cap \big(\{\lambda \in C : \ \dim\ker(\lambda - T) = \infty\} \\ \cup \{\lambda \in C : \ \dim\ker(\lambda - T^*) = \infty\}\big). \end{aligned}$$

Let ϕ denote the empty set. If $K_0 \neq \phi$, then it follows from Lemma 3 that the invariant subspace lattice $\mathrm{Lat}(T)$ for T and the invariant subspace lattice $\mathrm{Lat}(T^*)$ for T^* are rich. If $K_0 = \phi$, then the set

$$\begin{aligned} K \cap \sigma_e(T) &= \big(K \cap \big\{\lambda \in C : \mathrm{ran}(\lambda - T) \text{ is not closed}\big\} \cap \sigma_e(T)\big) \cup K_0 \\ &= K \cap \big\{\lambda \in C : \mathrm{ran}(\lambda - T) \text{ is not closed}\big\} \end{aligned}$$

is dominating in G. On the other hand, by Lemma 4 we have

$$\big(K \cap \{\lambda \in C : \mathrm{ran}(\lambda - T) \text{ is not closed}\}\big) \subset K_1.$$

Consequently K_1 is dominating in G. Therefore it follows from Lemma 7 that the invariant subspace lattice $\mathrm{Lat}(T)$ for T and the invariant subspace lattice $\mathrm{Lat}(T^*)$ for T^* are rich. □

Theorem 2. *Assume the operators $T \in B(X)$ and $B \in B(Z)$ on Banach spaces X and Z and the nonempty open set G in the complex plane C satisfy the following conditions:*

(1) *$qT = Bq$ for some injective $q \in B(X, Z)$ with a closed range qX.*

(2) *There exist sequences $\{G(n)\}$ of open sets and $\{M(n)\}$ of invariant subspaces of B such that $\overline{G}(n) \subset G(n+1)$, $G = \cup_n G(n)$, $\sigma(B|M(n)) \subset C\backslash G(n)$ and $\sigma(B/M(n)) \subset \overline{G}(n)$, $n = 1, 2, \ldots$.*

Then we have:

(a) *If the set* $K_2 = \sigma(T)\backslash(\sigma_p(B^{**}) \cap \{\lambda \in C : \overline{(\lambda - B^*)\ker q^*} \neq \ker q^*\})$ *is dominating in* G*, then* T *has infinitely many invariant subspaces.*
(b) *If* $K_2 \cap \sigma_e(T)$ *is dominating in* G*, then the invariant subspace lattice* $\mathrm{Lat}(T)$ *for* T *and the invariant subspace lattice* $\mathrm{Lat}(T^*)$ *for* T^* *are rich.*

Proof. (1') By condition (1) of the theorem and the Closed Range Theorem we have $\mathrm{ran}(q^*) = (\ker q)^\perp = X^*$ and $q^*B^* = T^*q^*$.

(2') From condition (2) of the theorem we obtain $\sigma(B^*|M(n)^\perp) \subset \overline{G}(n)$ and $\sigma(B^*/M(n)^\perp) \subset C\backslash G(n)$, $n = 1, 2, \ldots$.

(3') Since $\overline{\mathrm{ran}\, A}^\perp = \ker(A^*)$ for any bounded linear operator A on any Banach space Y, it follows that $\overline{\mathrm{ran}(\lambda - B^*)}^\perp = \ker((\lambda - B^*)^*)$ for any $\lambda \in C$. Therefore,

$$\begin{aligned}\{\lambda \in C : \overline{(\lambda - B^*)Z^*} \neq Z^*\} &= \{\lambda \in C : \overline{\mathrm{ran}(\lambda - B^*)}^\perp \neq \{0\}\} \\ &= \{\lambda \in C : \ker(\lambda - B^{**}) \neq \{0\}\} = \sigma_p(B^{**}).\end{aligned}$$

Consequently,

$$K_2 = \sigma(T)\backslash\{\lambda \in C : \overline{(\lambda - B^*)Z^*} \neq Z^*, \text{ and } \overline{(\lambda - B^*)\ker q^*} \neq \ker q^*\}.$$

To prove Theorem 2 it therefore suffices to replace the quantities X, Z, T, B, q, G, $G(n)$ and $M(n)$ in Theorem 1 by the quantities X, Z^*, T, B^*, q^*, G, $G(n)$ and $M(n)^\perp$, respectively. □

Remark 1. It is clear that Theorem 2 contains as special case Theorem A, the main result of [8], and improves this result in several respects:

(1) The spaces X and Z in Theorem 2 are general Banach spaces, while X and Z in Theorem A are reflexive Banach spaces.
(2) The thickness condition of the spectrum in Theorem 2 is weaker than that in Theorem A.
(3) The conclusion of Theorem A is that the operator T has infinitely many invariant subspaces, while the conclusion of Theorem 2 is that the invariant subspace lattice $\mathrm{Lat}(T)$ for the operator T is rich. We give below an example in which the operator T has infinitely many invariant subspaces, while $\mathrm{Lat}(T)$ is not rich.

Corollary 1. *Assume the operators* $T \in B(X)$ *and* $B \in B(Z)$ *on the Banach space* X *and the reflexive Banach space* Z *and the nonempty open set* G *in the complex plane* C *satisfy the following conditions:*

(1) $qT = Bq$ *for some injective* $q \in B(X, Z)$ *with a closed range* qX*.*
(2) *There exist sequences* $\{G(n)\}$ *of open sets and* $\{M(n)\}$ *of invariant subspaces of* B *such that* $\overline{G}(n) \subset G(n+1)$, $G = \cup_n G(n)$, $\sigma(B|M(n)) \subset C\backslash G(n)$ *and* $\sigma(B/M(n)) \subset \overline{G}(n)$, $n = 1, 2, \ldots$.

Then we have:

(a) *If the set*

$$K_3 = \sigma(T)\backslash\big(\sigma_p(B) \cap \{\lambda \in C : \overline{(\lambda - B^*)\ker q^*} \neq \ker q^*\}\big)$$

is dominating in G, then T has infinitely many invariant subspaces.

(b) *If $K_3 \cap \sigma_e(T)$ is dominating in G, then the invariant subspace lattice* $\mathrm{Lat}(T)$ *for T and the invariant subspace lattice* $\mathrm{Lat}(T^*)$ *for T^* are rich.*

Example 1. Let S be the unilateral shift of multiplicity 1 on the Hardy space H^2. Let ϕ_1 be the inner function defined by $\phi_1(z) = \exp{(z+1)/(z-1)}$, and let P be the projection from H^2 onto $(\phi_1 H^2)^\perp$. Then it follows from [10, p. 61] that the operator $T = PS|(\phi_1 H^2)^\perp$ is unicellular and $\mathrm{Lat}(T)$ is order isomorphic to $[0,1]$. Therefore T has infinitely many invariant subspaces, and $\mathrm{Lat}(T)$ is totally ordered. But for an infinite dimensional Banach space E it is not possible that $\mathrm{Lat}(E)$ is totally ordered. Consequently, $\mathrm{Lat}(T)$ cannot be rich.

Remark 2. It is easy to see that the richness of $\mathrm{Lat}(T)$ in the theorems of [3] and in this paper can be strengthened to reflexive-richness. Here we call an invariant subspace lattice $\mathrm{Lat}(T)$ for the operator $T \in B(X)$ on the Banach space X reflexive-rich if there exists an infinite dimensional Banach space E such that $\mathrm{Lat}(T)$ contains a sublattice order isomorphic to the lattice $\mathrm{Lat}(E)$, and E is a reflexive Banach space whenever X is a reflexive Banach space.

Let X be a Banach space. Let M, N be linear subspaces of X and L a linear subspace of X^*. It is well known that $M \vee N = \overline{M+N}$, ${}^\perp(M^\perp) = \overline{M}$, $({}^\perp L)^\perp \supset \overline{L}$, and there exist examples for which $({}^\perp L)^\perp \neq \overline{L}$. But if X is a reflexive Banach space, then $({}^\perp L)^\perp = \overline{L}$.

Lemma 8. *Let E be a reflexive Banach space. Then $\varphi : L \longmapsto {}^\perp L$ is a bijection of* $\mathrm{Lat}(E^*)$ *onto* $\mathrm{Lat}(E)$.

Proof. We first show that φ is an injection. In fact, for any $L, H \in \mathrm{Lat}(E^*)$, if $L \neq H$, then ${}^\perp L \neq {}^\perp H$. Otherwise, ${}^\perp L ={}^\perp H$. Consequently $L = ({}^\perp L)^\perp) = ({}^\perp H)^\perp) = H$, which is a contradiction. Next we show that φ is a surjection. Indeed, for any $M \in \mathrm{Lat}(E)$, we have $M^\perp \in\mathrm{Lat}(E^*)$ and $({}^\perp(M^\perp) = M$. Therefore φ is a surjection of $\mathrm{Lat}(E^*)$ onto $\mathrm{Lat}(E)$. □

Lemma 9. *Let X be a reflexive Banach space and let $T \in B(X)$. Let* $[\mathrm{Lat}(T)]_s$ *be a sublattice of* $\mathrm{Lat}(T)$. *Put* $[\mathrm{Lat}(T^*)]_s = \{M^\perp, M \in [\mathrm{Lat}(T)]_s\}$. *Then* $[\mathrm{Lat}(T^*)]_s$ *is a sublattice of* $\mathrm{Lat}(T^*)$, *and the map $\psi : M \longmapsto M^\perp$ is a bijection of* $[\mathrm{Lat}(T)]_s$ *onto* $[\mathrm{Lat}(T^*)]_s$.

Proof. It is easy to see that $[\mathrm{Lat}(T^*)]_s \subset \mathrm{Lat}(T^*)$. Moreover for any $M^\perp, N^\perp \in [\mathrm{Lat}(T^*)]_s$ we have

$$M^\perp \vee N^\perp = (M \cap N)^\perp \in [\mathrm{Lat}(T^*)]_s,$$
$$M^\perp \cap N^\perp = (M \vee N)^\perp \in [\mathrm{Lat}(T^*)]_s.$$

Therefore $[\mathrm{Lat}(T^*)]_s$ is a sublattice of $\mathrm{Lat}(T^*)$. It can be proved as in Lemma 8 that ψ is a injection. Moreover, it is clear that ψ is a surjection. $\square$

THEOREM 3. *Let X be a reflexive Banach space and let $T \in B(X)$. Then the invariant subspace lattice* $\mathrm{Lat}(T)$ *for T is reflexive-rich if and only if the invariant subspace lattice* $\mathrm{Lat}(T^*)$ *for T^* is reflexive-rich.*

Proof. Necessity: Since $\mathrm{Lat}(T)$ is reflexive-rich, there exists an infinite dimensional reflexive Banach space E such that $\mathrm{Lat}(T)$ contains a sublattice $[\mathrm{Lat}(T)]_s$ that is order isomorphic to the lattice $\mathrm{Lat}(E)$. Assume that τ is the corresponding isometric isomorphism. It follows from Lemma 9 that $[\mathrm{Lat}(T^*)]_s$ is a sublattice of $\mathrm{Lat}(T^*)$. To prove the necessity it therefore suffices to show that $\mathrm{Lat}(E^*)$ is order isomorphic to $[\mathrm{Lat}(T^*)]_s$.

Define the map π of $\mathrm{Lat}(E^*)$ into $\mathrm{Lat}(T^*)]_s$ by

$$\pi(L) = [\tau({}^\perp L)]^\perp$$

for all $L \in \mathrm{Lat}(E^*)$. It follows from Lemmas 8 and 9 and the definition of τ that $\varphi : L \longmapsto {}^\perp L$ is a bijection of $\mathrm{Lat}(E^*)$ onto $\mathrm{Lat}(E)$, that $\tau : {}^\perp L \longmapsto \tau({}^\perp L)$ is a bijection of $\mathrm{Lat}(E)$ onto $[\mathrm{Lat}(T)]_s$, and that $\psi : \tau({}^\perp L) \longmapsto [\tau({}^\perp L)]^\perp$ is a bijection of $[\mathrm{Lat}(T)]_s$ onto $[\mathrm{Lat}(T^*)]_s$. Therefore $\pi : L \longmapsto [\tau({}^\perp L)]^\perp$ is a bijection of $\mathrm{Lat}(E^*)$ onto $[\mathrm{Lat}(T^*)]_s$.

It is easy to see that for any $L, H \in [\mathrm{Lat}(E^*)]_s$, if $L \subset H$, then $\pi(L) \subset \pi(H)$. Conversely, for any $L, H \in [\mathrm{Lat}(E^*)]_s$, if $\pi(L) \subset \pi(H)$, then $L \subset H$(since E is reflexive). Consequently π is a two-sided order-preserving map.

Sufficiency: Since $\mathrm{Lat}(T^*)$ is reflexive-rich, it follows from the necessity that $\mathrm{Lat}(T^{**})$ is reflexive-rich. Consequently $\mathrm{Lat}(T)$ is reflexive-rich. $\square$

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