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DOUBLE ERGODICITY OF NONSINGULAR TRANSFORMATIONS AND INFINITE MEASURE-PRESERVING STAIRCASE TRANSFORMATIONS

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Abstract. A nonsingular transformation is said to be doubly ergodic if for all sets A and B of positive measure there exists an integer $n > 0$ such that $\lambda(T^{-n}(A) \cap A) > 0$ and $\lambda(T^{-n}(A) \cap B) > 0$. While double ergodicity is equivalent to weak mixing for finite measure-preserving transformations, we show that this is not the case for infinite measure preserving transformations. We show that all measure-preserving tower staircase rank one constructions are doubly ergodic, but that there exist tower staircase transformations with non-ergodic Cartesian square. We also show that double ergodicity implies weak mixing but that there are weakly mixing skyscraper constructions that are not doubly ergodic. Thus, for infinite measure-preserving transformations, double ergodicity lies properly between weak mixing and ergodic Cartesian square. In addition we study some properties of double ergodicity.

1. Introduction

We say that a nonsingular transformation T is doubly ergodic if for all sets A and B of positive measure there is an integer $n > 0$, such that $\lambda(T^{-n}(A) \cap A) > 0$ and $\lambda(T^{-n}(A) \cap B) > 0$. In the case of finite measure-preserving transformations double ergodicity is equivalent to weak mixing, and this was shown in [Fu]. We show that for infinite measure-preserving transformations the situation is quite different.

Weak mixing was studied in the context of nonsingular transformations by Aaronson, Lin and Weiss [ALW], who showed that there exists an infinite measure-preserving transformation T such that T is weakly mixing (i.e., $T \times S$ is ergodic for all ergodic finite measure preserving S) but $T \times T$ is not ergodic. It is easy to see that if $T \times T$ is conservative ergodic then T is doubly ergodic. We show that double ergodicity does not imply ergodic Cartesian square, and while double ergodicity implies weak mixing, the converse is not

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true. We also study some properties of double ergodicity for nonsingular, not necessarily invertible, transformations. Our examples are all rank one infinite measure-preserving invertible transformations (i.e., infinite measure-preserving transformations constructed by the cutting and stacking technique using only one Rohlin column).

There is a class of rank one transformations, called staircase constructions, that have recently garnered increasing attention. It was shown by Adams [A] that finite measure-preserving infinite staircases with an additional technical condition (see Section 5.1 for the precise statement) are mixing. In this paper we introduce a class of transformations called tower staircases, which include the staircases of Adams above, and show that tower staircases are doubly ergodic, but that there exist infinite measure preserving tower staircases with non-conservative, hence non-ergodic, Cartesian square. (Tower staircases may be of finite or infinite measure.) This results in one of our counterexamples. For the other example we show that a skyscraper construction that was shown in [AFS] to be weakly mixing is not doubly ergodic. Thus double ergodicity lies properly between weak mixing and ergodic Cartesian square in the case of nonsingular invertible transformations. We end with an example of infinite measure-preserving tower staircases with conservative Cartesian square.

We also show that double ergodicity implies a stronger k-fold version that we call k-conservative ergodicity. This is interesting as in infinite measure many properties do not imply their k-fold analogue. For example, it was shown by Kakutani and Parry [KP] that there exist infinite transformations with $T\times T$ ergodic but $T\times T\times T$ not ergodic; and more recently it was shown in [AFS2] that there exist infinite transformations with $T\times\cdots\times T$ ergodic for all k-fold products, but $T\times T^2$ not ergodic; these counterexamples were later constructed in [D] for actions of countable abelian groups.

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2. Preliminaries

2.1. Definitions. We will let $(X,\mathcal{B},\lambda)$ denote a finite or σ-finite Lebesgue measure space, where we will assume that λ is non-atomic. In all of our examples X will be the unit interval or the positive reals and λ the Lebesgue measure. A *nonsingular endomorphism* is a map $T\colon (X,\mathcal{B},\lambda)\to(X,\mathcal{B},\lambda)$ such that is T is measurable and $\lambda(A)=0$ if and only if $\lambda(T^{-1}(A))=0$.

A *nonsingular automorphism* is a nonsingular endomorphism with a measurable inverse T^{-1}. T is *ergodic* if for all $A \in \mathcal{B}$ with $T^{-1}(A) = A$ we have $\lambda(A)\lambda(A^c) = 0$. T is *conservative* if for every A with $\lambda(A) > 0$ there is an integer $n > 0$ such that $\lambda(A \cap T^{-n}(A)) > 0$. Therefore T is conservative ergodic if and only if for all measurable sets A and B of positive measure, there is an integer $n > 0$ such that $\lambda(T^{-n}(A) \cap B) > 0$. As our measure is non-atomic, ergodic automorphisms are conservative.

An endomorphism T is *measure-preserving* if $\lambda(T^{-1}(A)) = \lambda(A)$; it is finite measure-preserving if X is of finite measure and infinite measure-preserving if X is of infinite measure. Given two measurable sets $A, B \subset X$, with A of finite measure, and $\epsilon > 0$, we shall say that A is (at least) $(1-\epsilon)$-*full* of B if $\lambda(A \cap B) > (1-\epsilon)\lambda(A)$.

A nonsingular endomorphism $T\colon X \to X$ is *weakly mixing* if for all finite measure-preserving ergodic endomorphisms $S\colon Y \to Y$, where (Y, ν) is a Lebesgue probability space, the transformation defined by the Cartesian product $T \times S\colon X \times Y \to X \times Y$ is ergodic. Using the natural extension of S we may assume that S is invertible.

Definition 1. We say that a nonsingular endomorphism T is *doubly ergodic* if for all $A, B \subset X$ of positive measure there exists an integer $n > 0$ such that

$$\lambda\left(T^{-n}(A) \cap A\right) > 0 \quad \text{and} \quad \lambda\left(T^{-n}(A) \cap B\right) > 0.$$

Definition 2. Let $k \geq 1$ be an integer. We say that a nonsingular endomorphism T is *k-conservative ergodic* if for all $A_i, B_i \subset X$, $1 \leq i \leq k$, of positive measure, there exists an integer $n > 0$ such that

$$\lambda\left(T^{-n}(A_i) \cap B_i\right) > 0 \quad \text{for} \quad i = 1, \ldots, k.$$

Remark 1. For a finite measure-preserving transformation T, Furstenberg [Fu] defines the sets

$$N(A, B) = \{n : \lambda(A \cap T^{-n}(B)) > 0\},$$

and shows that

$$N(A, B) \cap N(A, A) \neq \emptyset$$

for all A, B of positive measure (the condition we call double ergodicity) is equivalent to weak mixing, and he also obtains the equivalence with 2-conservative ergodicity. The proof uses the spectral characterization of weak mixing for finite measure-preserving transformations. In [E], Eigen considers a property, which he calls property-one, for pairs of nonsingular automorphisms T, S that is equivalent to 2-conservative ergodicity when $T = S$. Eigen observes that if $T \times S$ is ergodic then T, S satisfy property-one, but does not prove any other statements about property-one. In [KSW], the authors studied double ergodicity, but only in the finite measure-preserving case (unpublished).

2.2. k-Conservative ergodicity.

PROPOSITION 2.1. *Let T be a nonsingular endomorphism on a σ-finite space X. T is doubly ergodic if and only if T is 2-conservative ergodic. Furthermore, if T is 2-conservative ergodic then T is k-conservative ergodic for all $k \geq 1$.*

Proof. Suppose T is doubly ergodic. Let A, B, C, $D \subset X$ be any four sets of positive measure. It is clear that T must be conservative ergodic (which is the same as 1-conservative ergodic). Then there exists an integer $j > 0$ such that $\lambda(T^{-j}(C) \cap A) > 0$, and an integer $k > 0$ such that $\lambda(T^{-k}(T^{-j}(C) \cap A) \cap B) > 0$. As $\lambda(T^{-j}(D)) > 0$, there exists some $\ell > 0$ such that

$$\lambda\left(T^{-\ell}\left[T^{-k}\left(T^{-j}(C) \cap A\right) \cap B\right] \cap \left[T^{-k}\left(T^{-j}(C) \cap A\right) \cap B\right]\right) > 0,$$
$$\lambda\left(T^{-\ell}\left[T^{-k}\left(T^{-j}(C) \cap A\right) \cap B\right] \cap T^{-j}(D)\right) > 0.$$

In particular,

$$\lambda\left(T^{-\ell-k}(A) \cap B\right) > 0 \quad \text{and} \quad \lambda\left(T^{-\ell-k-j}(C) \cap T^{-j}(D)\right) > 0.$$

Letting $m = \ell + k$ and using the nonsingularity of T, we get

$$\lambda\left(T^{-m}(A) \cap B\right) > 0 \quad \text{and} \quad \lambda\left(T^{-m}(C) \cap D\right) > 0.$$

Thus T is 2-conservative ergodic. The converse is clear. Therefore the two properties are equivalent.

We now show that k-conservative ergodicity implies $(k+1)$-conservative ergodicity for all $k \geq 2$. Let $A_i, B_i \subset X$, $i = 1, \ldots, k+1$, be sets of positive measure. We know that there exists an integer $m > 0$ such that $\lambda(T^{-m}(A_1) \cap A_2) > 0$ and $\lambda(T^{-m}(B_1) \cap B_2) > 0$. Now apply the fact that T is k-conservative ergodic to the k pairs of sets

$$\begin{aligned} T^{-m}(A_1) \cap A_2, &\quad T^{-m}(B_1) \cap B_2 \\ A_3, &\quad B_3 \\ &\vdots \\ A_{k+1}, &\quad B_{k+1} \end{aligned}$$

to obtain an integer $n > 0$ such that

$$\lambda\left(T^{-n}\left(T^{-m}\left(A_1\right) \cap A_2\right) \cap \left(T^{-m}\left(B_1\right) \cap B_2\right)\right) > 0 \tag{1}$$
$$\lambda\left(T^{-n}\left(A_3\right) \cap B_3\right) > 0$$
$$\vdots$$
$$\lambda\left(T^{-n}\left(A_{k+1}\right) \cap B_{k+1}\right) > 0.$$

Equation (1) implies

$$\lambda\left(T^{-n-m}\left(A_1\right) \cap T^{-n}\left(A_2\right) \cap T^{-m}\left(B_1\right) \cap B_2\right) > 0,$$

which implies in particular that

$$\lambda\left(T^{-n-m}\left(A_1\right) \cap T^{-m}\left(B_1\right)\right) > 0 \quad \text{and} \quad \lambda\left(T^{-n}\left(A_2\right) \cap B_2\right) > 0. \tag{2}$$

Applying nonsingularity to equation (2) gives $\lambda\big(T^{-n}(A_1) \cap B_1\big) > 0$, and this completes the proof. □

The proof of the following corollary is left to the reader.

Corollary 2.1. *Let T be a nonsingular automorphism on a σ-finite space X. T is doubly ergodic if and only if T^{-1} is doubly ergodic.*

2.3. Double ergodicity of T^k.

Proposition 2.2. *If T is a doubly ergodic nonsingular endomorphism on a σ-finite space X, then T^k is doubly ergodic for all $k > 0$.*

Proof. Let $A, B \subset X$ be sets of positive measure. Using Proposition 2.1, we know that there is some $n \geq k$ such that $\lambda(T^{-n}(A) \cap C) > 0$ for all $C \in \{A, T^{-1}(A), \ldots, T^{-k+1}(A), B, T^{-1}(B), \ldots, T^{-k+1}(B)\}$. There exist integers a, b such that $n = ka + b$, with $0 \leq b < k$ and $ka > 0$. Thus, in particular,

$$\lambda\left(T^{-n}(A) \cap T^{-b}(A)\right) > 0, \qquad \lambda\left(T^{-n}(A) \cap T^{-b}(B)\right) > 0.$$

Using that $-n = -ka - b$ and that T is nonsingular we obtain

$$\lambda\left(T^{-ka}(A) \cap A\right) > 0, \qquad \lambda\left(T^{-ka}(A) \cap B\right) > 0.$$

Thus T^k is doubly ergodic. □

3. Double ergodicity implies weak mixing

When T is a finite measure-preserving transformation, it is known that if T is doubly ergodic then for all finite measure preserving ergodic transformations S, $T \times S$ is ergodic [Fu]. We prove this for the case when T is a nonsingular transformation on a finite or infinite space. The beginning of the proof is a modification of the argument that a finite measure preserving transformation with no isometric factors must be weakly mixing (cf. [Ru], Theorem 4.10).

Theorem 1. *Let (X, λ) be a σ-finite Lebesgue space and (Y, ν) a probability Lebesgue space. Let $T\colon X \to X$ be a doubly ergodic nonsingular endomorphism. Then, for all invertible ergodic finite measure-preserving transformations $S\colon Y \to Y$, $T \times S$ is ergodic.*

Proof. We prove this by contradiction. Suppose $T \times S$ is not ergodic. Then given an invariant subset for $T \times S$, we will construct a T-invariant pseudo-metric and use this to show that T is not doubly ergodic. Let $A \subset X \times Y$ be such that $(T \times S)^{-1}(A) = A$ with $\lambda \times \nu(A)\lambda \times \nu(A^c) > 0$. We define the *fiber* of A over x to be $A_x = \{y \in Y \mid (x, y) \in A\}$. Define a function $d\colon X \times X \to \mathbb{R}$ by $d(x, x') = \nu(A_x \Delta A_{x'})$. It is easy to verify that d is nonnegative, reflexive, and satisfies the triangle inequality. As A is $(T \times S)$-invariant, one can show that $S(A_x) = A_{T(x)}$. This can be used to show that d is a T-invariant pseudo-metric.

With respect to d, we can find two balls B_1, B_2 of radius ϵ whose centers are separated by more than 4ϵ such that both have positive measure. For suppose this does not happen for any ϵ. Now fix ϵ. We can cover X with a countable number of ϵ-balls, centered at $\{x_i\}_{i=1}^{\infty}$, under the pseudo-metric d. (This is a consequence of the fact that Y is finite, so $d(x, x') \leq \nu(Y)$. In a finite measure space, any measurable set can be approximated arbitrarily well by a finite union of elements of a countable sufficient class.)

At least one of the ϵ-balls in our cover will have positive measure, and without loss of generality we may assume it is the one centered at x_1. We claim that the ball around x_1 of radius 5ϵ has full measure, that is, all the measure is concentrated around x_1. By assumption, the union of balls of radius ϵ centered around those x_i which are more than 4ϵ away from x_1 must have zero measure. Any x in the complement of this set must be a distance at most ϵ from some x_i that satisfies $d(x_i, x_1) \leq 4\epsilon$, so $d(x, x_1) \leq 5\epsilon$.

Intersecting over a sequence of $\epsilon \to 0$ and noting that countable intersections of sets of full measure have full measure, we see that there is an $x \in X$ such that $U = \{x' \in X \mid d(x, x') = 0\}$ has full measure. Now, by Fubini's theorem,

$$(A \cap (U \times Y))\, \Delta\, (U \times A_x) = (A \cap (U \times Y))\, \Delta \bigcup_{x' \in U} (\{x'\} \times A_{x'})$$

has measure 0. Thus,

$$A = A \cap (U \times Y) = U \times A_x = X \times A_x \pmod{\lambda \times \nu}.$$

Therefore, by the ergodicity of S, A_x must have full measure; hence A has full measure and we have a contradiction.

We now know that there is an ϵ such that there are two ϵ-balls B_1, B_2 that are separated by more than 4ϵ. $T^{-n}(B_1)$ will, by the T-invariance of d, always have diameter at most 2ϵ, so it is impossible to have both $\lambda(T^{-n}(B_1) \cap B_1) > 0$ and $\lambda(T^{-n}(B_1) \cap B_2) > 0$. Therefore, T is not doubly ergodic. $\square$

4. Weakly mixing but not doubly ergodic

PROPOSITION 4.1. *There exists an infinite measure-preserving automorphism T such that T is weakly mixing but not doubly ergodic.*

4.1. Construction. We begin with the construction of the $2h_n+1$ skyscraper transformation of [AFS]. The construction is by the technique of cutting and stacking (cf. [F]). We first set up some notation for cutting and stacking constructions that will be used in all of our examples; the construction will yield a piecewise linear transformation in $X=[0,\infty)$. One constructs inductively a sequence of *columns*. A *column* C_n consists of a set of h_n disjoint intervals of the same measure, denoted by $(B_n^{(0)}, B_n^{(1)}, \ldots, B_n^{(h_n-1)})$, where we think of $B_n^{(i+1)}$ as sitting above $B_n^{(i)}$. The elements of C_n are called *levels* and h_n is called the *height* of C_n. A column C_n partially defines a transformation T_n on all levels of C_n, except the top level, by the unique orientation preserving translation that sends interval $B_n^{(k)}$ to interval $B_n^{(k+1)}$, so that $T_n(B_n^{(k)}) = B_n^{(k+1)}$, for $k=0,\ldots,h_n-2$. In the construction one has to make sure that T_{n+1} agrees with T_n on C_n, and that the collection of all levels generates the measurable sets in X, where $X=\bigcup_{n\geq 0} C_n$. (By abuse of notation here C_n denotes the union of the levels of C_n.) In our examples $X=[0,\infty)$, so that we obtain an infinite measure preserving transformation. The transformation T is defined in the limit by $T=\lim_{n\to\infty} T_n$.

To define the $2h_n+1$ skyscraper transformation we start with $C_0=[0,1)$ and $h_0=1$. Given column C_n, to obtain column C_{n+1} first cut each level of C_n into two sublevels, denoted $B_{n,i}^{(k)}$, for $i=0,1$ and $k=0,\ldots,h_{n-1}$. This results in the two *subcolumns* of C_n, consisting of $C_{n,i}=\{B_{n,i}^{(0)},\ldots,B_{n,i}^{(h_n-1)}\}$ for $i=0,1$. Next consider a collection of $2h_n+1$ disjoint intervals chosen outside of (the union of the levels in) C_n, and denoted by $S_{n,j}$, for $j=0,\ldots,2h_n$; we call these intervals the *spacers* of C_n. We choose $S_{n,j}$ so that its left endpoint is the right endpoint of $S_{n,j-1}$, for $j=1,\ldots,2h_n$, and $S_{n,0}$ so that its left endpoint is the right endpoint of $S_{n-1,2h_n}$. This ensures that X will be an interval. (Whether one obtains a finite or infinite measure preserving transformations depends on whether the sum of all the spacers adds up to finite or infinite measure.) Place the spacers on top of subcolumn $C_{n,1}$ and stack this new subcolumn with spacers on top of $C_{n,0}$ to obtain C_{n+1} with $h_{n+1}=4h_n+1$ levels. This last operation extends the definition of T_n by sending the top of the left subcolumn of C_n to the bottom of the right subcolumn of C_n, so that $T_{n+1}(B_{n,0}^{(h_n-1)}) = B_{n,1}^{(0)}$, and sending the top subcolumn of C_n to the bottom spacer, so that $T_{n+1}(B_{n,1}^{(h_n-1)}) = S_{n,0}$. Also, T_{n+1} is defined on all but the top spacer by the usual translation sending each spacer to the one above it.

One can verify that $h_n = \sum_{j=0}^{n} 4^j$. Since T_n is defined on the levels $B_n^{(0)}, \ldots, B_n^{(h_n-2)}$ of C_n, and each $B_n^{(k)}$ has measure 2^{-n}, T_n is thus defined on a set of measure

$$\frac{1}{2^n}(h_n - 1) = \frac{1}{2^n}\left(\sum_{j=1}^{n} 4^j\right) \geq 2^n$$

for $n \geq 1$. In the limit as $n \to \infty$, $T\colon X \to X$ is indeed defined on $[0, \infty)$.

4.2. Proof of Proposition 4.1. Let T be defined as above. In [AFS], T was shown to be weakly mixing. (In fact, there it is shown that T has no nonconstant L^∞ eigenfunctions, and [ALW] is used to obtain the equivalence with our definition.) It suffices to show that there exist levels I, J with $\lambda(I) > 0$, $\lambda(J) > 0$, such that there does not exist an integer $k > 0$ that satisfies $\lambda(T^k(I) \cap I) > 0$ and $\lambda(T^k(I) \cap J) > 0$.

Let $I = B_0^{(0)}$, $J = B_1^{(3)}$. Suppose there exists $k > 0$ such that $\lambda(T^k(I) \cap I) > 0$ and $\lambda(T^k(I) \cap J) > 0$. Then there exists a smallest such k. Choose n such that $k < 2h_n + 1$. The image of I in C_{n+1} consists of several levels, called the copies of I. The image of J in C_{n+1} also consists of several levels, called the copies of J.

We define the distance between two levels $B_n^{(i)}$ and $B_n^{(j)}$ of column C_n to be $d(B_n^{(i)}, B_n^{(j)}) = |i - j|$. The assumption implies that there are two copies of I separated by a distance k, as well as a copy of I a distance k from a copy of J. This contradicts the following lemma. $\square$

LEMMA 4.1. *With I and J as above, the set of all distances between two copies of I, denoted D_n, is disjoint from the set of all distances between a copy of I and a copy of J, denoted D'_n.*

Proof. We take C_1 as our base case. Then $D_1 = \{0, 1\}$, and $D'_1 = \{2, 3\}$. In general, if we know D_n and D'_n, we can find D_{n+1} and D'_{n+1} as follows: C_{n+1} consists of two copies of C_n stacked one on top of the other, with $2h_n + 1$ spacers at the top; the lower copy is $C_{n,0}$ and the upper copy is $C_{n,1}$. Then the set of distances between copies of I within $C_{n,0}$ is D_n, as is the set of distances between copies of I within $C_{n,1}$. The set of distances between a copy of I and a copy of J in each is D'_n. Therefore, $D_n \subset D_{n+1}$ and $D'_n \subset D'_{n+1}$. Any elements of D_{n+1} or D'_{n+1} that are not elements of D_n or D'_n, respectively, are distances between one element in $C_{n,0}$ and one element in $C_{n,1}$. We note that if $B_n^{(i)} \in C_{n,0}$ is a copy of I, then $B_n^{(h_n+i)} \in C_{n,1}$ is also a copy of I.

The set of these "new" distances, $D_{n+1} \backslash D_n$, is equal to the set $A = \{\ell \mid \ell = h_n \pm a,\ a \in D_n\}$. For suppose $\ell \in D_{n+1} \backslash D_n$; then $T^\ell(B_{n+1}^{(i)}) = B_{n+1}^{(i+\ell)}$, where $B_{n+1}^{(i)} \in C_{n,0}$ and $B_{n+1}^{(i+\ell)} \in C_{n,1}$ are both copies of I. The level $B_{n+1}^{(i+h_n)}$ in

$C_{n,1}$ is copy of I, so the distance, $d(B_{n+1}^{(i+h_n)}, B_{n+1}^{(i+\ell)}) = |h_n - \ell|$, is an element of D_n. Therefore, $\ell = h_n \pm a$, $a \in D_n$. Hence, $D_{n+1} \backslash D_n \subset A$. Conversely, if $a \in D_n$, then a is the distance between two copies of I, $B_n^{(i)}$ and $B_n^{(i+a)}$, both in $C_{n,0}$. Then $h_n + a = d(B_{n+1}^{(i)}, B_{n+1}^{(i+a+h_n)})$, and $h_n - a = d(B_{n+1}^{(i+a)}, B_{n+1}^{(i+h_n)})$. Thus $A \subset D_{n+1} \backslash D_n$. Similarly, $D'_{n+1} \backslash D'_n = \{\ell \mid \ell = h_n \pm a, a \in D'_n\}$. The largest element in D_n, therefore, is

$$\begin{aligned}\max\{D_n\} &= h_{n-1} + \max\{D_{n-1}\} \\ &= h_{n-1} + h_{n-2} + \cdots + \max\{D_1\} = \left(\sum_{i=1}^{n-1} h_i\right) + 1.\end{aligned}$$

Similarly, the largest element in D'_n is

$$\max\{D'_n\} = h_{n-1} + \cdots + h_1 + \max\{D'_1\} = \left(\sum_{i=1}^{n-1} h_i\right) + 3.$$

The smallest element in $D_{n+1} \backslash D_n$ is

$$\min\{D_{n+1} \backslash D_n\} = h_n - \max\{D_n\} = h_n - \left(\left(\sum_{i=1}^{n-1} h_i\right) + 1\right);$$

the smallest element in $D'_{n+1} \backslash D'_n$ is

$$\min\{D'_{n+1} \backslash D'_n\} = h_n - \max\{D'_n\} = h_n - \left(\left(\sum_{i=1}^{n-1} h_i\right) + 3\right).$$

Now for $n \geq 2$,

$$\begin{aligned}\min\{D_{n+1} \backslash D_n\} - \max\{D'_n\} &= h_n - 2\left(\sum_{i=1}^{n-1} h_i\right) - 4 > h_n - 2\left(\sum_{i=2}^{n} \frac{h_n}{4^i}\right) - 4 \\ &> h_n - 2h_n\left(\sum_{i=2}^{\infty} \frac{1}{4^i}\right) - 4 \\ &= h_n - 2h_n\left(\frac{1}{12}\right) - 4 = \frac{5}{6}h_n - 4 > 0,\end{aligned}$$

so $(D_{n+1} \backslash D_n) \cap D'_n = \emptyset$ for $n \geq 2$. Similarly,

$$\min\{D'_{n+1} \backslash D'_n\} - \max\{D_n\} = h_n - 2\sum_{i=1}^{n-1} h_i - 4 > 0,$$

so $(D'_{n+1} \backslash D'_n) \cap D_n = \emptyset$ for $n \geq 2$.

Now, we know that $D_2 \cap D'_2 = \emptyset$. Suppose for a contradiction that $D_n \cap D'_n = \emptyset$ but $D_{n+1} \cap D'_{n+1} \neq \emptyset$ for some $n \geq 2$. As $(D_{n+1} \backslash D_n) \cap D'_n = (D'_{n+1} \backslash D'_n) \cap D_n = \emptyset$, this implies that $(D_{n+1} \backslash D_n) \cap (D'_{n+1} \backslash D'_n)$ is nonempty. So there is some $d_1 \in D_{n+1} \backslash D_n$ and $d_2 \in D'_{n+1} \backslash D'_n$ such that $d_1 = d_2$. But

$d_1 = h_n + a_1$ or $d_1 = h_n - a_1$, with $a_1 \in D_n$, and $d_2 = h_n + a_2$ or $d_2 = h_n - a_2$, with $a_2 \in D'_n$. Then we must have either plus signs in both equations or minus signs in both equations, since $a_1, a_2 > 0$, and so $a_1 = a_2$. Thus $D_n \cap D'_n$ is nonempty, a contradiction. □

5. Double ergodicity of staircase transformations

5.1. Staircases. In [A], Adams defines a transformation T acting on X as a *staircase construction* if, in the cutting and stacking construction, column C_{n+1} is obtained by cutting C_n into r_n subcolumns of equal width and placing i spacers over subcolumn i, $0 \le i \le r_n - 1$. The sequence $\{r_n\}_{n=1}^{\infty}$ is a sequence of natural numbers. If $r_n \to \infty$, T is called an *infinite staircase construction*. (Infinite staircases may be either finite or infinite measure-preserving.)

We will call such infinite staircase constructions *pure staircases*. We also define *tower staircases*, in which the staircase is placed as usual on all but the final subcolumn. In the final subcolumn we may place any number of spacers, usually a large number. Note that pure staircases are a special case of tower staircases, with $r_n - 1$ spacers over the final subcolumn in C_n, and that tower staircases may be of finite or infinite measure.

Adams [A] also proves the following result for finite measure preserving infinite staircases.

Theorem (Mixing of Staircases, [A]). *Let T be a finite measure-preserving pure staircase transformation. If $\lim_{n\to\infty} r_n^2/h_n = 0$ then $T = T_{(r_n)}$ is mixing.*

It is not known if the condition $\lim_{n\to\infty} r_n^2/h_n = 0$ is necessary for mixing; in particular it is not known if under the absence of this condition finite measure preserving infinite staircases might be partially rigid. (If $\limsup r_n < \infty$ then the staircase must be partially rigid.) In Section 6, we construct an infinite measure-preserving tower staircase with certain restrictions on r_n which has non-conservative Cartesian square, and hence, by [AFS], is not partially rigid.

5.2. Double ergodicity. We now prove that all tower staircases are doubly ergodic. In the proof of double ergodicity, we will use the following approximation lemma from analysis. This lemma can be obtained from the Martingale Convergence Theorem, and an elementary proof of a more general version may be found in the Double Approximation Lemma [DGMS].

Lemma 5.1. *Let $\epsilon > 0$, $\delta > 0$, and $0 < \tau < 1$. Let I be an interval that is τ-full of a measurable set A, and $\{r_n\}$ an infinite sequence such that $r_n \in \mathbb{N}$ and $r_n > 1$ for sufficiently large n. There exists $N \in \mathbb{N}$ such that if we let $I_k = [\frac{k}{r_1 \dots r_N}, \frac{k+1}{r_1 \dots r_N}]$ for $0 \le k \le r_1 \dots r_N - 1$ there exists a subset*

$K \subset \{0, 1, \ldots, r_1 \ldots r_N - 1\}$, with $|K| > (\tau - \delta) r_1 \ldots r_N$, such that each I_j, $j \in K$, is $(1-\epsilon)$-full of A.

Lemma 5.2. *Suppose $r_n \geq 2$ for all sufficiently large n. Given $\epsilon > 0$ and any sets $A, B \subset X$, both of positive measure, there exist intervals I and J in some column C_n, with I above J, such that I is $(1-\epsilon)$-full of A and J is $(1-\epsilon)$-full of B.*

Proof. The only difficulty is insuring that I is above J. We can find levels I', J' in some $C_{n'}$ that are $\frac{3}{4}$-full of A and B, respectively (and with $r_k \geq 2$ for all $k \geq n'$). By Lemma 5.1, there exists an $n \geq n' + 2$ such that at least $\frac{5}{8}$ of the copies of I' in C_n are $(1-\epsilon)$-full of A, and similarly for J'. Looking at the preimages of these levels in $C_{n'}$, at least $\frac{1}{4}$ of the $(1-\epsilon)$-full copies of I' must lie in the same subcolumn as a $(1-\epsilon)$-full copy of J'. By our choice of n there are at least 2 such I', J'-copy pairs. Let I be the copy of I' in the right pair and J be the copy of J' in the left pair. Then I and J are levels in C_n, with I above J, and I is $(1-\epsilon)$-full of A and J is $(1-\epsilon)$-full of B. □

Lemma 5.3. *Let $A, B \subset X$ be sets of positive measure, and let levels $I, J \subset C_m$ be such that $\lambda(I \cap A) + \lambda(J \cap B) > \delta\lambda(I)$, with I a distance $\ell \geq 0$ above J. If we cut I and J into r_m equal pieces $I_0, \ldots, I_{r_m - 1}$ and $J_0, \ldots, J_{r_m-1}$, respectively (numbered from left to right), then there is some k such that*

$$\lambda\left(I_k \cap A\right) + \lambda\left(J_k \cap B\right) > \delta\lambda\left(I_k\right),$$

and I_k will be ℓ above J_k in C_{m+1}.

Proof. Write $I = \bigcup_{i=0}^{r_m-1} I_i$ and $J = \bigcup_{i=0}^{r_m-1} J_i$. Then

$$\begin{aligned}\sum_{i=0}^{r_m-1} \lambda\left(I_i \cap A\right) + \sum_{i=0}^{r_m-1} \lambda\left(J_i \cap B\right) &= \lambda\left(I \cap A\right) + \lambda\left(J \cap B\right) \\ &> \delta\lambda(I) = \delta \sum_{i=0}^{r_m-1} \lambda\left(I_i\right).\end{aligned}$$

Thus $\sum_{i=0}^{r_m-1}(\lambda(I_i \cap A) + \lambda(J_i \cap B) - \delta\lambda(I_i)) > 0$, so at least one of the summands must be positive. Therefore, for some k, $\lambda(I_k \cap A) + \lambda(J_k \cap B) > \delta\lambda(I_k)$. Note that if we look at I_k and J_k as levels in C_{m+1}, I_k is ℓ levels above J_k. □

Theorem 2. *Let T be any tower staircase. Then T is doubly ergodic.*

Proof. Let $\epsilon = 1/16$. As $r_n \to \infty$, by Lemma 5.2, for any sets $A, B \subset X$ of positive measure we can find intervals I' and J' in some column C_n, with I' above J', so that I' is $(1 - \frac{\epsilon}{2})$-full of A and J' is $(1 - \frac{\epsilon}{2})$-full of B. Let ℓ be the distance between I' and J'. Choose N so that $r_N > 16(\ell+1)$. We have that

$$\begin{aligned}\lambda\left(I' \cap A\right) + \lambda\left(J' \cap B\right) &> \left(1 - \frac{\epsilon}{2}\right)\lambda\left(I'\right) + \left(1 - \frac{\epsilon}{2}\right)\lambda\left(J'\right) \\ &= 2\left(1 - \frac{\epsilon}{2}\right)\lambda\left(I'\right) = (2-\epsilon)\,\lambda\left(I'\right).\end{aligned}$$

By applying Lemma 5.3 $N - n$ times, we find intervals $I, J \subset C_N$ such that $\lambda(I \cap A) + \lambda(J \cap B) > (2-\epsilon)\lambda(I)$ and I is a distance ℓ above J. Thus

$$\begin{aligned}\lambda\left(I \cap A\right) &\geq (2-\epsilon)\lambda(I) - \lambda\left(J \cap B\right) \\ &> (2-\epsilon)\lambda(I) - \lambda(J) \\ &= (1-\epsilon)\lambda(I),\end{aligned}$$

so I is $(1-\epsilon)$-full of A. Similarly, J is $(1-\epsilon)$-full of B.

Let $I_0, \ldots, I_{r_N-1}$ and $J_0, \ldots, J_{r_N-1}$ denote the r_N subintervals of I and J that we obtain by cutting C_N into subcolumns (numbered from left to right).

Then for all j, $0 \leq j \leq r_N - \ell - 1$, we have

$$\begin{aligned}T^{h_N+j}\left(I_j\right) &= I_{j+1}, \\ T^{h_N+j}\left(I_{j+\ell}\right) &= T^{-\ell}\left(I_{j+\ell+1}\right) = J_{j+\ell+1}.\end{aligned}$$

Let $K = \{0, 1, \ldots, r_N - \ell - 1\}$. Let $G = \{I_k\}_{k \in K}$, $G' = \{I_{k+1}\}_{k\in K}$, $H = \{I_{k+\ell}\}_{k\in K}$, and $H' = \{J_{k+\ell+1}\}_{k\in K}$. Let G_m denote I_m, G'_m denote I_{m+1}, H_m denote $I_{m+\ell}$, and H'_m denote $J_{m+\ell+1}$.

The sets G, G', H, H', and K all have the same number of elements. We have, by our choice of N,

$$|G| = r_N - \ell - 1 \geq r_N - \frac{r_N}{16} = \frac{15}{16} r_N.$$

Thus, $r_N \geq |G| \geq \frac{15}{16} r_N$.

Recall that I is $(1-\epsilon)$-full of A and $\epsilon = 1/16$. The set G is composed of distinct intervals from $\{I_0, \ldots, I_{r_N-1}\}$, each of which has measure $\lambda(I)/r_N$, so $\lambda(I) \geq \lambda(\bigcup_m G_m) \geq \frac{15}{16}\lambda(I)$. Therefore, $\bigcup_m G_m$ is at least

$$\frac{(1-\epsilon) - \left(1 - \lambda\left(\cup_m G_m\right)\right)}{\lambda\left(\cup_m G_m\right)} \geq \frac{\left(1 - \frac{1}{16}\right) - \left(1 - \frac{15}{16}\right)}{1} = \frac{7}{8}\text{-full of } A.$$

By the same argument, $\bigcup_m G'_m$ and $\bigcup_m H_m$ are each $\frac{7}{8}$-full of A, and $\bigcup_m H'_m$ is $\frac{7}{8}$-full of B. It follows that more than $\frac{3}{4}$ of the intervals in each of G, G', and H must be at least $\frac{1}{2}$-full of A, and more than $\frac{3}{4}$ of the intervals in H' must be $\frac{1}{2}$-full of B.

Now let $K_G = \{k \in K \mid$ there exists j such that $T^j(G_k) = G'_k$, G_k and G'_k both at least $\frac{1}{2}$-full of $A\}$ and $K_H = \{k \in K \mid$ there exists j such that $T^j(H_k) = H'_k$, H_k at least $\frac{1}{2}$-full of A and H'_k at least $\frac{1}{2}$-full of $B\}$. The number of elements in K_G is $|K_G| > \frac{3}{4}|G| - \frac{1}{4}|G'| = \frac{1}{2}|K|$, and the number of elements in K_H is $|K_H| > \frac{3}{4}|H| - \frac{1}{4}|H'| = \frac{1}{2}|K|$. Therefore, we must have $K_G \cap K_H \neq \emptyset$.

Choose some $t \in K_G \cap K_H$. For this t, there exists a j such that $T^j(G_t) = G'_t$ where both G_t and G'_t are $\frac{1}{2}$-full of A. Thus G'_t is at least $\frac{1}{2}$-full of A and at least $\frac{1}{2}$-full of $T^j(A)$. Hence,

$$\lambda\left(T^j(A) \cap A\right) \geq \lambda\left(T^j\left(A \cap G_t\right) \cap \left(A \cap G'_t\right)\right) > 0.$$

Similarly, $\lambda(T^j(A) \cap B) > 0$, using the same j. Thus, T is doubly ergodic. □

5.3. Doubly ergodic on intervals but not doubly ergodic. A transformation T is *doubly ergodic on intervals* if for all intervals I and all sets A of positive measure there exists an integer $n > 0$ such that $\lambda(T^{-n}(I) \cap I) > 0$ and $\lambda(T^{-n}(I) \cap A) > 0$.

In this section we construct an ergodic infinite measure-preserving automorphism that is doubly ergodic on intervals but is not doubly ergodic. The proof is a modification of the construction in [MRSZ] of an ergodic finite measure-preserving automorphism that is lightly mixing on intervals but not lightly mixing.

It is well known that there exists a set $K^* \subset [0,1]$ of positive measure, such that for any interval $I \subset [0,1]$ with $\lambda(I) > 0$, $\lambda(I \cap K^*) > 0$ and $\lambda(I \cap K^{*c}) > 0$ (see, e.g., [MRSZ]). Let K_i^* and K_i^{*c} be the translations of K^* and K^{*c}, respectively, to $[i, i+1]$. Now let $K = \bigcup_{i=0}^{\infty} K_i^* \subset [0,\infty)$, and $K^c = \bigcup_{i=0}^{\infty} K_i^{*c} \subset [0,\infty)$. Then K and K^c will have the property that for any interval $I \subset [0,\infty)$ with $\lambda(I) > 0$, $\lambda(I \cap K) > 0$ and $\lambda(I \cap K^c) > 0$. Define the functions $\phi\colon K \to [0,\infty)$ and $\psi\colon K^c \to [0,\infty)$ by $\phi(x) = \lambda(K \cap [0,x))$ and $\psi(x) = \lambda(K^c \cap [0,x))$. The functions ϕ and ψ have well-defined inverses a.e. and are measure-preserving.

Proposition 5.1. *There exists an ergodic infinite measure-preserving automorphism that is doubly ergodic on intervals but not doubly ergodic.*

Proof. Let T be the infinite measure-preserving pure staircase transformation with $r_n = 2^{2^n}$, $n = 1, 2, \ldots$.

Define functions $V_1\colon K \to K^c$ by $V_1(x) = \psi^{-1} \circ T \circ \phi(x)$ and $V_2\colon K^c \to K$ by $V_2(x) = \phi^{-1} \circ T \circ \psi(x)$. Define $V\colon [0,\infty) \to [0,\infty)$ by

$$V(x) = \begin{cases} V_1(x), & \text{if } x \in K, \\ V_2(x), & \text{if } x \in K^c. \end{cases}$$

The proof that V is ergodic is similar to that in [DGMS], and is left to the reader. Let $A' \subset [0,\infty)$ be any set of positive measure, and let $J \subset [0,\infty)$ be any interval. There exists a level I' in some column C_n that is contained entirely in J. Suppose $\lambda(A' \cap K) > 0$, and let $A = A' \cap K$ and $I = I' \cap K$. (If this condition is not satisfied, then $\lambda(A' \cap K^c) > 0$, and a similar argument can be applied by letting $A = A' \cap K^c$ and $I = I' \cap K^c$.) Note that $\lambda(A) > 0$ and $\lambda(I) > 0$. We will show that there exists an integer $\ell \neq 0$ such that

$$\lambda\left(V^{\ell}(I) \cap A\right) > 0 \quad \text{and} \quad \lambda\left(V^{\ell}(I) \cap I\right) > 0.$$

By Proposition 2.2, T^2 is doubly ergodic. Thus, there exists an integer $m > 0$ such that

$$\lambda\left(\left(T^2\right)^m (\phi(I)) \cap \phi(A)\right) > 0 \quad \text{and} \quad \lambda\left(\left(T^2\right)^m (\phi(I)) \cap \phi(I)\right) > 0.$$

We let $\ell = 2m$. Then, restricted to K, $V^{\ell} = V^{2m} = \phi^{-1} \circ T^{2m} \circ \phi = \phi^{-1} \circ T^{\ell} \circ \phi$. Therefore,

$$\lambda\left(V^{\ell}(I) \cap A\right) = \lambda\left(T^{\ell} \circ \phi(I) \cap \phi(A)\right) > 0.$$

Similarly, $\lambda(V^{\ell}(I) \cap I) > 0$. Hence, V is doubly ergodic on intervals. However, there does not exist any integer ℓ such that

$$\lambda\left(V^{\ell}(K) \cap K^c\right) > 0 \quad \text{and} \quad \lambda\left(V^{\ell}(K) \cap K\right) > 0.$$

Therefore, V is not doubly ergodic. □

6. A staircase transformation whose Cartesian product is not conservative, and hence not ergodic

6.1. Columns in the staircase transformations. In this section we find methods for studying intervals in higher columns of the tower staircase transformation. Let h_n be the height of C_n and let s_n be the height of the tower. Then $h_{n+1} = r_n h_n + (1 + 2 + \cdots + (r_n - 2)) + s_n$.

Given a level I in C_n, we first study I in C_{n+1}. In C_{n+1}, I is a union of r_n certain levels, called *copies*. The kth copy, counting from the bottom, corresponds to the piece of I which came from the kth subcolumn of C_n. Because we put $k-1$ spacers on the kth subcolumn, we see that the separation distance in C_{n+1} between this kth copy and the $(k+1)$th copy is $h_n + k - 1$.

This describes I in C_{n+1}; we extend this to C_{n+j} for $j \geq 1$. In each such column, I is a union of copies. There is one copy in C_n (I itself), r_n copies in C_{n+1}, and, in general, $r_n r_{n+1} \ldots r_{n+j-1}$ copies in C_{n+j}. As C_{n+j} was made by cutting C_{n+j-1} into r_{n+j-1} subcolumns and stacking them, the copies in C_{n+j} are made up of r_{n+j-1} identical *clusters*. In turn, each cluster has the same structure as the set of copies in C_{n+j-1}, and so is made up of r_{n+j-2} subclusters, each of which resembles the set of copies in C_{n+j-2}. We get a self-similar pattern.

We introduce notation for referencing the copies in C_{n+j}. Each copy lies in one of the r_{n+j-1} largest clusters. Within this cluster, it lies in one of the r_{n+j-2} next largest clusters, and so forth until the jth step, where we reach single levels. Hence we can represent a copy $a \subset C_{n+j}$ of I by a j-tuple of numbers:

$$[a_1, a_2, \ldots, a_j], 0 \leq a_i \leq r_{n+j-i} - 1.$$

To fully understand the structure of the set of copies in C_{n+j}, it is necessary to know the distances between them. The following proposition gives us a formula for the distance between two copies a and b, where by definition a is a distance $|\ell|$ from b if $a = T^{\ell}(b)$.

PROPOSITION 6.1. *The distance between levels* $a = [a_1, a_2, \ldots, a_j]$ *and* $b = [b_1, b_2, \ldots, b_j]$ *is*

$$d(a,b) = \sum_{k=1}^{j} \left((b_k - a_k)\, h_{n+j-k} + \frac{(b_k - a_k)\,(b_k + a_k - 1)}{2} \right).$$

This result is negative if a *is above* b*; the usual positive distance is given by* $|d(a,b)|$.

Proof. The proof is by induction. Suppose this formula is true for C_{n+j-1}. Looking at the r_{n+j-1} largest clusters, a is in cluster number a_1, and b is in b_1. If we translate cluster number a_1 by $(b_1 - a_1)h_{n+j-1} + \frac{(b_1-a_1)(b_1+a_1-1)}{2}$ steps (upwards if this number is nonnegative, downwards otherwise), it will overlap precisely with cluster number b_1. The image of $[a_1, a_2, \ldots, a_j]$ under this translation is $[b_1, a_2, \ldots, a_j]$. Since each of the largest clusters has the same structure as C_{n+j-1}, to get the total distance we can use the induction hypothesis to obtain the distance between these two intervals and add the amount we translated by:

$$\begin{aligned}
d\left([a_1, a_2, \ldots, a_j], [b_1, b_2, \ldots, b_j]\right)& \\
&= d\left([a_2, a_3, \ldots, a_j], [b_2, b_3, \ldots, b_j]\right) \\
&\qquad + (b_1 - a_1)\, h_{n+j-1} + \frac{(b_1 - a_1)\,(b_1 + a_1 - 1)}{2} \\
&= \sum_{k=2}^{j} \left((b_k - a_k)\, h_{n+j-k} + \frac{(b_k - a_k)\,(b_k + a_k - 1)}{2} \right) \\
&\qquad + (b_1 - a_1)\, h_{n+j-1} + \frac{(b_1 - a_1)\,(b_1 + a_1 - 1)}{2} \\
&= \sum_{k=1}^{j} \left((b_k - a_k)\, h_{n+j-k} + \frac{(b_k - a_k)\,(b_k + a_k - 1)}{2} \right). \qquad \square
\end{aligned}$$

Suppose that, in column C_{n+j}, a is a copy of level $I \subset C_n$ and b is a copy of level $J \subset C_n$. We say that (a,b) is a *unique distance pair* if there does not exist a pair $\{a', b'\} \neq \{a, b\}$ in C_{n+j} with a' a copy of I and b' a copy of J, such that $d(a', b') = d(a, b)$.

6.2. The staircase construction. We describe the construction of a tower staircase by specifying r_n and the height s_n of the final spacer column, and then we show that it is not $T \times T$ conservative, and hence not $T \times T$

ergodic. The fact that it (as well as any other tower staircase) is doubly ergodic was proven in Section 5.

Start with $C_0 = [0,1)$ and $r_0 = 1$ as the base case for our inductive definition. In general, given C_{n-1} and r_{n-1}, we determine s_{n-1}, and r_n as follows: Let h_n denote the height of column C_n, and choose $r_n > n^2(2r_{n-1}h_{n-1} + (r_{n-1}-1)(r_{n-1}-2))$. Then choose s_{n-1} large enough so that whenever

$$\left|\left(kh_n + \frac{k(k-1)}{2} + kf\right) - \left(jh_n + \frac{j(j-1)}{2} + jg\right)\right| \leq 2r_{n-1}h_{n-1} + (r_{n-1}-1)(r_{n-1}-2),$$

with $0 \leq k, j, f, g \leq r_n - 1$, we must have $j = k$ (note that $h_n > s_{n-1}$). This completes the construction.

6.3. Definitions and lemmas. In this section and the next, let $I = [0,1)$. From the vector notation of Section 6.1 ($a = [a_1, a_2, \ldots, a_j]$), we see that a_1 specifies the largest cluster containing a in C_{n+j}.

LEMMA 6.1. *Suppose a, b, x, y are copies of I in C_{n+1}, a is above b and x is above y, and $d(a,b) = d(x,y)$. Suppose further that $|a_1 - b_1| > r_n/n^2$ and $|x_1 - y_1| > r_n/n^2$. Then $(x_1, y_1) = (a_1, b_1)$.*

Proof. Let $k = |a_1 - b_1|$ and $j = |x_1 - y_1|$. Let a' be the level in C_n containing the preimage of a, and define b', x', and y' similarly. Then let $v = d(a', b')$ and $w = d(x', y')$. We note that

$$v \leq r_{n-1}h_{n-1} + \frac{(r_{n-1}-1)(r_{n-1}-2)}{2}$$

and

$$w \leq r_{n-1}h_{n-1} + \frac{(r_{n-1}-1)(r_{n-1}-2)}{2}.$$

The distance formula implies that

$$d(a,b) = kh_n + \frac{k(k-1)}{2} + kb_1 \pm v,$$
$$d(x,y) = jh_n + \frac{j(j-1)}{2} + jy_1 \pm w.$$

By construction, h_n is large enough to ensure $k = j$. This gives us the relation $k(b_1 - y_1) = \mp v \pm w$. Suppose $b_1 \neq y_1$; then we have $r_n/n^2 < k \leq v + w \leq 2r_{n-1}h_{n-1} + (r_{n-1}-1)(r_{n-1}-2)$. However, this contradicts the choice of r_n in the construction. Thus $b_1 = y_1$, and so $a_1 = x_1$. $\square$

LEMMA 6.2. *Let p_n denote the fraction of pairs of copies of I (a,b) in C_n, with $a \neq b$, which are unique distance pairs. Let $R_n = r_1 r_2 \ldots r_n$. Then for all $n \geq 2$, $p_{n+1} \geq (1 - \frac{2}{n^2})(1 - \frac{1}{R_{n-1}})p_n$.*

Proof. Let G_i and G_j be any two clusters in C_{n+1}. If G is the set of copies of I in C_n, then there are natural correspondences $\psi_i \colon G_i \to G$ and $\psi_j \colon G_j \to G$. Consider the pairs (a, b), with $a \in G_i$, $b \in G_j$, and $\psi_i(a) \neq \psi_j(b)$, and the pairs (a', b'), $a', b' \in C_n$, with $a' \neq b'$. Note that a_1 identifies G_i, and b_1 identifies G_j. We have a bijection by letting $a' = \psi_i(a)$ and $b' = \psi_j(b)$.

Now suppose $|a_1 - b_1| > \frac{r_n}{n^2}$. If the pair (a, b) is not a unique distance pair in C_{n+1}, then by Lemma 6.1 we can find $x \in G_i, y \in G_j$, such that $d(a, b) = d(x, y)$, and $x \neq a$. Then $d(\psi_i(a), \psi_j(b)) = d(\psi_i(x), \psi_j(y))$, and $(\psi_i(a), \psi_j(b)) \neq (\psi_i(x), \psi_j(y))$. Note that either $\psi_i(a)$ is above $\psi_j(b)$ and $\psi_i(x)$ is above $\psi_j(y)$, or $\psi_i(a)$ is below $\psi_j(b)$ and $\psi_i(x)$ is below $\psi_j(y)$. Therefore, $(\psi_i(a), \psi_j(b))$ is not a unique distance pair in C_n. Conversely, if $(\psi_i(a), \psi_j(b))$ is not a unique distance pair in C_n, we easily see that (a, b) cannot be a unique distance pair in C_{n+1}. Thus our correspondence between pairs induces a correspondence between unique distance pairs.

Since the fraction of pairs (c, d) in C_n, $c \neq d$, which are unique distance pairs is p_n, the fraction of pairs (a, b), with $a \in G_i$, $b \in G_j$, and $\psi_i(a) \neq \psi_j(b)$, which are unique distance pairs in C_{n+1} is also p_n. The number of elements in G_i is the number of copies of I in C_n, which is equal to R_{n-1}, and similarly for G_j. We see that the fraction of all pairs (a, b), $a \in G_i$, $b \in G_j$, which are unique distance pairs is at least $p_n(1 - \frac{1}{R_{n-1}})$. Furthermore, the fraction of pairs of copies of I (a, b) in C_{n+1}, $a \neq b$, for which the relation $|a_1 - b_1| > \frac{r_n}{n^2}$ holds is at least $(1 - \frac{2}{n^2})$. Multiplying these results, we obtain $p_{n+1} \geq (1 - \frac{2}{n^2})(1 - \frac{1}{R_{n-1}})p_n$ as desired. $\square$

6.4. Non-ergodicity of the product.

THEOREM 3. *Let T be one of the staircase transformations described in Section* 6.2. *T is doubly ergodic, but $T \times T$ is not conservative, and hence not ergodic.*

Proof. Recall that $I = [0, 1)$. Let

$$S = \left(\bigcup_{n \neq 0} (T \times T)^n (I \times I) \right) \cap (I \times I) = \bigcup_{n \neq 0} \big((T^n(I) \cap I) \times (T^n(I) \cap I) \big).$$

It is sufficient to show that $(I \times I) \backslash S$ has positive measure, for since $(T \times T)^n((I \times I) \backslash S) \cap ((I \times I) \backslash S) = \emptyset$ for all n, this will imply that $T \times T$ is not conservative.

Using Lemma 6.2 and induction, we obtain, for all n,

$$p_{n+1} \geq \left(1 - \frac{2}{n^2}\right)\left(1 - \frac{2}{(n-1)^2}\right)\dots\left(1 - \frac{2}{2^2}\right)$$
$$\times\left(1 - \frac{1}{R_{n-1}}\right)\left(1 - \frac{1}{R_{n-2}}\right)\dots\left(1 - \frac{1}{R_1}\right)p_2.$$

Let

$$\gamma = \prod_{k=1}^{\infty}(1 - \frac{2}{k^2})\prod_{k=2}^{\infty}(1 - \frac{1}{R_k}) > 0.$$

Then $p_n > \gamma$ for all n. For each n there exists a partition of I into a disjoint union of intervals of equal measure, namely the copies of I in C_n. This induces a partition of $I \times I$ into squares. Now, for $0 < |k| < s_n$, we define

$$S_n = \bigcup_k \left((T \times T)^k(I \times I)\right) \cap (I \times I).$$

Noting that pairs of copies of I correspond to the squares in $I \times I$, we see that S_n is a union of at most $1 - p_n(1 - \frac{1}{R_{n-1}})$ squares. Thus $\lambda((I \times I)\backslash S_n) \geq p_n(1 - \frac{1}{R_{n-1}})$. Since $S = \bigcup_{n=1}^{\infty} S_n$ is an increasing union of the sets S_n, $\lambda((I \times I)\backslash S_n) \geq \gamma > 0$. Thus T is doubly ergodic (Theorem 2), but $T \times T$ is not ergodic. □

In [AN] it was shown that if T is of positive type then $T \times T$ is conservative. Thus we obtain the following corollary.

Corollary 6.1. *Let T be one of the staircase transformations described in Section* 6.2. *T is not of positive type, and hence not partially rigid.*

7. Staircase transformations with conservative Cartesian product

7.1. Conservativity of $T \times T$. We will use the following well-known lemma about Diophantine equations, which may be found, e.g., in [S].

Lemma 7.1 (Siegel's Lemma). *Suppose that we have a system of linear equations with integer coefficients:*

$$a_{11}x_1 + \cdots + a_{1n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = 0$$

Further, suppose that $|a_{ij}| \leq A$ $(1 \leq i, j \leq n)$ where A is a positive integer. Then there is a nontrivial solution in the integers with

$$|x_i| < 1 + (nA)^{\frac{m}{n-m}} \quad (i = 1, \dots, n).$$

Corollary 7.1. *Given* $\{r_k\}_{k=1}^{\infty}$ *with* ${r_k}^{1/k} < M$, *and given* $n \geq 1$, *there exists* j *such that for any* $\{d_k\}_{k=1}^{j}$ *with* $|d_k| \leq r_{n+k}$ *there exists a solution* $\{x_k\}_{k=1}^{j}$ *to* $\sum_{k=1}^{j} d_k x_k = 0$ *with* $|x_k| \leq M+2$.

Proof. By Siegel's Lemma with $m = 1$, for any j there always exists a solution with $|x_k| < 1 + (kr_{n+k})^{\frac{1}{k-1}} = 1 + (k^{\frac{1}{k-1}})(r_{n+k}{}^{\frac{1}{n+k}})^{\frac{n+k}{k-1}}$, which tends to $1 + (1)(M)$ as $j \to \infty$. Thus for some j there is a solution with $|x_k| < 2 + M$. □

Proposition 7.1. *Let* $\{r_k\}$ *be a sequence such that* ${r_k}^{1/k} < M$ *for all* k *and* $\prod_{k=\ell}^{\infty}(1 - \frac{1}{r_k}) > 0$ *for some* ℓ. *Let* T *be a tower staircase construction with* r_k *cuts in column* C_k. *Then* $T \times T$ *is conservative.*

Proof. First we show that $T \times T$ satisfies a type of conservativity on sets of the form $I \times J$, where I, J are levels in some column C_n. Specifically, we show that for large enough n, $(I \times J) \backslash \big(\bigcup_{k\neq 0}(I \times J) \cap (T \times T)^k (I \times J)\big)$ has arbitrarily small measure independently of I and J. To do this, we study the images of I and J in column C_{n+j}, with j as in Lemma 7.1. In each such column both I and J are a union of levels; this partition of I and J into intervals induces a partition of $I \times J$ into squares. Let $F_I = \{$copies of I in $C_{n+j}\}$ and $F_J = \{$copies of J in $C_{n+j}\}$. The squares are indexed by pairs (a, b), $a \in F_I$, $b \in F_J$.

Without loss of generality, $J = T^k(I), k \geq 0$. We have a correspondence between pairs (a, b), $a, b \in F_I$, and (a', b'), $a' \in F_I, b' \in F_J$, given by $(a, b) \to (a, T^k(b))$. This correspondence preserves the property of being a unique distance pair. So we need only to show that, for j high enough, sufficiently many pairs (a, b), $a, b \in F_I$ are not unique distance pairs.

We can choose ℓ large enough so that $\alpha = \prod_{k=\ell}^{\infty} \frac{1-2(M+2)}{r_k}$ is arbitrarily close to 1. This is possible as $\prod_{k=\ell}^{\infty}(1 - \frac{1}{r_k}) > 0$ for some ℓ is equivalent to $\prod_{k=\ell'}^{\infty}(1 - \frac{2(M+2)}{r_k}) > 0$ for some ℓ'. Now, suppose we have a pair of copies in C_{n+j}, $a = [a_1, a_2, \ldots, a_j] \in F_I$ and $b = [b_1, b_2, \ldots, b_j] \in F_J$. Suppose additionally that $M + 2 \leq a_k, b_k \leq r_{n+j-k} - (M + 2)$. The fraction of pairs that satisfy this is at least α^2. We treat a and b as vectors. Let $[c_1, c_2, \ldots, c_j] = c = b - a$. We let j be large enough so that by Lemma 7.1

$$\sum_{k=1}^{j} c_k x_k = 0$$

has a solution with $|x_k| < M + 2$ for all k. Let $x = [x_1, x_2, \ldots, x_j]$, and set $a' = a + x$, $b' = b + x$. Now, by the choice of a and b, we know that $a' \in F_I$

and $b' \in F_J$. We claim that $d(a', b') = d(a, b)$. Indeed,

$$\begin{aligned} d\left(a', b'\right) &= \sum_{k=1}^{j} \left(b'_k - a'_k\right) \left(h_{n+j-k} + \frac{b'_k + a'_k - 1}{2}\right) \\ &= \sum_{k=1}^{j} \left(b_k - a_k\right) \left(h_{n+j-k} + \frac{b_k + x_k + a_k + x_k - 1}{2}\right) \\ &= \sum_{k=1}^{j} \left(b_k - a_k\right) \left(h_{n+j-k} + \frac{b_k + a_k - 1}{2}\right) + \sum_{k=1}^{j} c_k x_k \\ &= \sum_{k=1}^{j} \left(b_k - a_k\right) \left(h_{n+j-k} + \frac{b_k + a_k - 1}{2}\right) = d(a, b). \end{aligned}$$

Any square that corresponds to a non-unique distance pair lies in

$$\bigcup_{k \neq 0} \left((I \times J) \cap (T \times T)^k (I \times J)\right),$$

and $\lambda \times \lambda(\bigcup_{k \neq 0}((I \times J) \cap (T \times T)^k (I \times J))) \geq \alpha^2 \lambda \times \lambda(I \times J)$. Therefore, $I \times J \backslash \left(\bigcup_{k \neq 0}(I \times J) \cap (T \times T)^k (I \times J)\right)$ has measure less than $(1-\alpha^2)\lambda \times \lambda(I \times J)$.

Now that we have this preliminary version of conservativity, we prove the conservativity of $T \times T$ on arbitrary measurable sets $A \subset X \times X$. We can find levels of C_n I, J such that $I \times J$ is at least $\frac{7}{8}$-full of A, where n is as large as we want. Consider the subdivision of $I \times J$ induced by the subdivisions of I and J into intervals which are levels in C_{n+j}. Let $F = F_I \times F_J$. Let $s \in F$, $s = a \times b$. If j satisfies the criteria of Lemma 7.1, then for a fraction at least α^2 of these s, (a, b) is not a unique distance pair. Suppose $(T \times T)^k (a \times b) = a' \times b'$. If $k > 0$, we call s a Type 1 square ($a \times b$ is mapped to another square), and if $k < 0$, we call it a Type 2 square (another square maps to $a \times b$).

There is a correspondence between Type 1 and Type 2 squares. Given a Type 1 square, it corresponds to the Type 2 square to which it first returns under some iterate of $T \times T$. Note that this is a bijection, and thus the Type 1 squares have the same total measure as the Type 2 squares. Furthermore, the measure of the union of these is at least $\alpha^2\left(\lambda \times \lambda(I \times J)\right)$. Hence the Type 1 squares have total measure at least $\frac{1}{2}\alpha^2\left(\lambda \times \lambda(I \times J)\right)$, as do the Type 2 squares. Since $I \times J$ is at least $\frac{7}{8}$-full of A, we can see that more than half of the Type 1 squares must be $\frac{1}{2}$-full of A, and more than half of the Type 2 squares must be $\frac{1}{2}$-full of A. By the pigeonhole principle, there exists a pair related through the correspondence, say C and $(T \times T)^k C$, with $k \neq 0$, such that both are at least $\frac{1}{2}$-full. Then $\lambda \times \lambda((T \times T)^k (A \cap C) \cap (A \cap C)) > 0$. Hence $\lambda \times \lambda((T \times T)^k (A) \cap A) > 0$, so $T \times T$ is conservative. $\square$

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