Illinois Journal of Mathematics
Volume 45, Number 3, Fall 2001, Pages 981–998
S 0019-2082

ALEKSANDROV OPERATORS AS SMOOTHING OPERATORS

ALEC L. MATHESON

ABSTRACT. A holomorphic function b mapping the unit disk $\mathbb{D}$ into itself induces a family of measures τ_α, $|\alpha| = 1$, on the unit circle $\mathbb{T}$ by means of Herglotz's Theorem. This family of measures defines the Aleksandrov operator A_b by means of the formula $A_b f(\alpha) = \int_{\mathbb{T}} f(\zeta)\, d\tau_\alpha(\zeta)$, at least for continuous f. This operator preserves the smoothness classes determined by regular majorants, and is seen to be compact on these classes precisely when none of the measures τ_α has an atomic part. In the process, a duality theorem for smoothness classes is proved, improving a result of Shields and Williams, and various theorems about composition operators on weighted Bergman spaces are extended to spaces arising from regular weights.

1. Introduction

If K is a compact Hausdorff space and $\mathcal{X}$ is a Banach space, there is a one-to-one correspondence between bounded linear operators $T\colon \mathcal{X} \to C(K)$ and weak* continuous functions $\tau\colon K \to \mathcal{X}^*$ given by the formula

$$Tx(k) = \tau(k)(x),$$

or, equivalently,

$$\tau(k) = T^*(\delta_k),$$

where $\delta_k \in C(K)^*$ is the functional of evaluation at k, and $T^*\colon C(K)^* \to \mathcal{X}^*$ is the adjoint of T. By a classical theorem (see [6]) T is weakly compact if and only if τ is weakly continuous, and compact if and only if τ is norm continuous. According to the Riesz representation theorem, $C(K)^*$ can be identified with $M(K)$, the space of regular Borel measures on K, and then δ_k denotes the unit point mass at $k \in K$. Moreover, if $\mathcal{X} = C(L)$ for another compact Hausdorff space L, then $T^*(\mu)$ is a measure on L for each measure μ on K. In particular, the map τ assigns a measure $\tau(k)$ on L to each point $k \in K$.

Received July 27, 2000; received in final form January 19, 2001.
2000 *Mathematics Subject Classification.* Primary 47B38. Secondary 30D45, 30D50.

Specializing to the case $K = L = \mathbb{T}$, where $\mathbb{T}$ is the unit circle, there is a one-to-one correspondence between bounded operators T on $C = C(\mathbb{T})$ and weak* continuous functions τ from $\mathbb{T}$ to $M = M(\mathbb{T})$. An interesting class of operators arises from the family of Aleksandrov measures induced by a holomorphic map b of the unit disk $\mathbb{D}$ into itself. For each $\alpha \in \mathbb{T}$, the analytic function $\frac{\alpha+b(z)}{\alpha-b(z)}$ has positive real part $\frac{1-|b(z)|^2}{|\alpha-b(z)|^2}$, which, by Herglotz's Theorem, is the Poisson integral of a positive measure τ_α on $\mathbb{T}$. Thus,

$$\frac{1-|b(z)|^2}{|\alpha-b(z)|^2} = \int_{\mathbb{T}} P_z(\zeta)\, d\tau_\alpha(\zeta),$$

where $P_z(\zeta) = \frac{1-|z|^2}{|\zeta-z|^2}$ is the Poisson kernel. Evidently,

$$\|\tau_\alpha\| = \int_{\mathbb{T}} d\tau(\zeta) = \frac{1-|b(0)|^2}{|\alpha-b(0)|^2} \le \frac{1+|b(0)|}{1-|b(0)|},$$

with equality when α and $b(0)$ have the same argument. It follows that the Aleksandrov operator

$$A_b(f(\alpha)) = \int_{\mathbb{T}} f(\zeta)\, d\tau_\alpha(\zeta)$$

takes continuous functions to bounded functions. As Aleksandrov [1] remarks, the mapping $\alpha \to \tau_\alpha$ is weak* continuous, and consequently A_b takes continuous functions to continuous functions. Applying A_b to the constant function 1 shows that

$$\|A_b\| = \frac{1+|b(0)|}{1-|b(0)|}.$$

In a different direction Sarason [16] showed how to extend the composition operator $C_b f = f \circ b$ to act on M. Indeed, if μ is a positive measure on $\mathbb{T}$, its Poisson integral $u(z) = \int_{\mathbb{T}} P_z(\zeta)\, d\mu(\zeta)$ is a positive harmonic function. Thus $u \circ b$ is also a positive harmonic function, and hence, by Herglotz's Theorem, the Poisson integral of some positive measure ν. Sarason defines $C_b\mu = \nu$, and extends C_b to all of M by means of the Jordan decomposition of measures. Noting that $C_b\delta_\alpha = \tau_\alpha$ for each $\alpha \in \mathbb{T}$, it is easy to see that $\|C_b\|_M = \sup_K \|\tau_\alpha\| = \frac{1+|b(0)|}{1-|b(0)|}$. Among other results, Sarason showed that $C_b(L^1) \subset L^1$, and that C_b is compact on M if and only if $C_b(M) \subset L^1$, which means that each of the measures τ_α is absolutely continuous. He further showed [17] that if C_b is weakly compact on L^1, it is also compact. It is now known that this compactness is equivalent to compactness of C_b acting on any H^P space $(0 < p < \infty)$. Proofs of this can be found in [20] or [3], where the absolute continuity condition is shown to be equivalent to J. H. Shapiro's function theoretic characterization of compactness on H^2 (see [19]). Sarason also showed that the formal adjoint C_b^* takes continuous functions to continuous functions. This observation coincides with Aleksandrov's observation that $A_b(C) \subset C$, where C denotes the space of continuous functions on $\mathbb{T}$, since it is easy to see that $C_b \colon M \to M$ is the adjoint operator to $A_b \colon C \to C$ (see [4]). When restricted to analytic functions, the duality between A_b and C_b breaks down in a minor way unless $b(0) = 0$. Indeed, if $f \in A$, where A is the disk algebra, then $A_b f$ is meromorphic in $\mathbb{D}$ with a pole of order at most one at $b(0)$ and with residue $f(0)b(0)$. Hence $A_b f$ is analytic if and only if $f(0) = 0$ or $b(0) = 0$. It follows that the rank-one perturbation $\tilde{A}_b$ of A_b given by

$$\tilde{A}_b f(\alpha) = A_b f(\alpha) - f(0)\frac{b(0)}{\alpha - b(0)}$$

maps A into itself. On the other hand, Bourdon and Cima [2] showed that C_b maps the space K of Cauchy-Stieltjes transforms into itself, and it is well-known that K can be identified in a canonical way with the dual A^* of the disk algebra. It is shown in [4] that $C_b \colon K \to K$ is the adjoint of $\tilde{A}_b \colon A \to A$. In order to simplify the subsequent discussion, it will be assumed from now on that $b(0) = 0$, so that $\tilde{A}_b = A_b$.

It turns out that not only does A_b preserve continuity, but it also preserves smoothness in the following sense. A *majorant* is a nonnegative, continuous, increasing function ω on $[0, \infty)$ with $\omega(0) = 0$. The majorant ω is *Dini regular* if there is a positive constant α such that

$$\int_0^\delta \frac{\omega(t)}{t}\, dt \le \alpha\omega(\delta)$$

for all $\delta > 0$, and is said to satisfy the b_p condition if there is a positive constant β such that

$$\delta \int_\delta^\infty \frac{\omega(t)}{t^{1+p}}\, dt \le \beta\omega(\delta)$$

for all $\delta > 0$. A majorant is *regular* if it is both Dini regular and satisfies the b_1 condition.

Let Λ_ω be the collection of functions $f \in C$ whose modulus of continuity

$$\omega(f, \delta) = \sup\{\, |f(\zeta) - f(\zeta')| : |\zeta - \zeta'| \le \delta \,\}$$

is majorized by $\omega(\delta)$. With the norm

$$\|f\|_\omega = \|f\|_\infty + \sup_\delta \frac{\omega(f, \delta)}{\omega(\delta)},$$

it is easy to see (and well known) that Λ_ω is a Banach algebra.

THEOREM 1. *Let ω be a regular majorant. Then $A_b(\Lambda_\omega) \subset \Lambda_\omega$ and indeed A_b is a bounded operator on Λ_ω.*

The proof of this theorem will rely on the identification of $A_\omega = \Lambda_\omega \cap A$ with the dual space of a certain weighted Bergman space. The theorem will then be seen to be a consequence of Littlewood's Subordination Principle

[5]. Known arguments about the compactness of composition operators on Bergman spaces will then lead to the next theorem.

THEOREM 2. *The operator A_b is compact on Λ_ω if and only if τ_α has no atomic part for each $\alpha \in \mathbb{T}$.*

Specifically, A_ω is isomorphic to the dual of the weighted Bergman space $A^1(\psi)$, where $\psi(r) = \omega(1-r^2)/(1-r^2)$. In general $f \in A^p(\psi)$ if and only if the integral

$$\iint_{\mathbb{D}} |f(z)|^p \psi(|z|)\, dx\, dy$$

is finite. The following more general theorem will be obtained.

THEOREM 3. *Let $\psi(r) = \omega(1-r^2)/(1-r^2)$ where ω is a regular majorant. Then the composition operator C_b is bounded on the Bergman space $A^p(\psi)$ for $0 < p < \infty$, and is compact if and only if b has no angular derivatives.*

It was shown by R. Nevanlinna [14] that b has an angular derivative in the sense of Carathéodory at $\beta \in \mathbb{T}$ if and only if $\tau_\alpha(\{\beta\}) > 0$ for some $\alpha \in \mathbb{T}$. This should be contrasted with the compactness condition for C, which is equivalent to the absolute continuity of τ_α for each α.

2. Preliminaries

For a majorant ω, set

$$\omega_*(t) = \int_0^t \frac{\omega(s)}{s}\, ds,$$

and

$$\omega^*(t) = t\int_t^\infty \frac{\omega(s)}{s^2}\, ds.$$

Then ω is *Dini regular* if $\omega_*(t) \le \alpha\omega(t)$ for some $\alpha > 0$ and all $t \ge 0$, and satisfies the b_1 condition if $\omega^*(t) \le \beta\omega(t)$ for some $\beta > 0$ and all $t \ge 0$. The majorant ω is *regular* if it is Dini regular and satisfies the b_1 condition.

The regularity assumption derives its importance from the following two inequalities, whose proofs can be found in [11]. For a function f analytic in $\mathbb{D}$, let $M_\infty(f,r) = \sup\{\,|f(z)| : |z| = r\,\}$. If f is analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, with modulus of continuity $\omega(\delta) = \omega(f,\delta)$, then

$$M_\infty(f',r) = O(\omega^*(1-r)/(1-r)) \qquad \text{as } r \to 1^-.$$

On the other hand, if $M_\infty(f',r) = O(\omega(1-r)/(1-r))$, and if $\omega_*(t) \to 0$ as $t \to 0^+$, the function f extends to a continuous function on $\overline{\mathbb{D}}$ having modulus of continuity

$$\omega(f,\delta) = O(\omega(\delta) + \omega_*(\delta)) \qquad \text{as } \delta \to 0.$$

These two inequalities combine to yield the following known propositions.

PROPOSITION 1. *Let ω be a regular majorant, and let f be analytic in the unit disk $\mathbb{D}$. Then $f \in A_\omega$ if and only if*

$$M_\infty(f', r) = O(\omega(1-r)/(1-r)).$$

PROPOSITION 2. *Let ω be a regular majorant, and let u be a real valued function in Λ_ω. Then the harmonic conjugate $\tilde{u}$ also belongs to Λ_ω.*

The proof of the second proposition depends on the fact that the derivative of the analytic function $f = u + i\tilde{u}$ is given by the integral formula

$$f'(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{e^{it}}{(e^{it}-z)^2} u(e^{it})\, dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{e^{it}}{(e^{it}-z)^2}(u(e^{it}) - u(e^{is}))\, dt,$$

and it is this formula that is used to prove the first inequality above. The same methods produce the following theorem about Toeplitz operators (cf. [11]).

THEOREM 4. *Let s be a bounded analytic function on the unit disk $\mathbb{D}$, and let*

$$T_{\overline{s}} f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\overline{s(e^{it})} f(e^{it})}{e^{it} - z}\, dt.$$

Then $T_{\overline{s}}$ is a bounded operator on A_ω.

Corresponding results hold for the subspaces $\Lambda_{\omega,0}$ of Λ_ω and $A_{\omega,0}$ of A_ω consisting of those functions f which satisfy $\omega(f,\delta) = o(\omega(\delta))$.

Two regular majorants $\omega_1(t)$ and $\omega_2(t)$ are equivalent if there are positive constants c_1 and c_2 such that

$$c_1\omega_1(t) \le \omega_2(t) \le c_2\omega_2(t).$$

Clearly, equivalent majorants determine the same spaces of functions, with equivalent norms. The author would like to thank the referee for suggesting the following proposition and its proof. It leads to significant simplification of the subsequent arguments. The original approach followed the methods found in [12] and [7].

PROPOSITION 3. *If ω is a regular majorant, there exists an equivalent regular majorant ω_1 and exponents $0 < \nu < \mu < 1$ such that $t^{-\mu}\omega_1(t)$ increases to ∞ and $t^{-\nu}\omega_1(t)$ decreases to 0 as t decreases to 0.*

Proof. Choose A so that $\log A > \max(\alpha, \beta^2)$. Then

$$\omega_*(At) = \int_0^{At} \frac{\omega(s)}{s}\, ds \ge \omega(t)\int_t^{At} \frac{1}{s}\, ds = \omega(t)\log A.$$

Thus, $(1+\delta)\omega(t) \le \omega(At)$ for some positive δ. Next, there is an s, $t < s < At$, such that

$$\frac{\omega(s)}{s}\log A = \frac{\omega(s)}{s}\int_t^{At}\frac{1}{u}\,du \le \int_t^{At}\frac{\omega(u)}{u^2}\,du \le \frac{\omega^*(t)}{t} \le \beta\frac{\omega(t)}{t}.$$

Now, since $s < At$,

$$\frac{\omega(At)}{At} = \omega(At)\int_{At}^{\infty}\frac{1}{u^2}\,du \le \int_{At}^{\infty}\frac{\omega(u)}{u^2}\,du \le \frac{\omega^*(s)}{s}.$$

It follows that

$$\frac{\omega(At)}{At} \le \frac{\omega^*(s)}{s} \le \beta\frac{\omega(s)}{s} \le \frac{\beta^2}{\log A}\frac{\omega(t)}{t} = (1-\eta)\frac{\omega(t)}{t},$$

say, where $\eta > 0$. Thus $\omega(At) \le A(1-\eta)\omega(t)$.

Now for each integer n let $\omega_1(A^n) = \omega(A^n)$ and let $\log\omega_1$ be a linear function of $\log t$ between A^n and A^{n+1}. Clearly, ω and ω_1 are equivalent. Choosing μ, $0 < \mu < 1$, so that $A^\mu < 1+\delta$, shows that

$$A^{\mu(n+1)}\omega(A^{-(n+1)}) < A^{\mu n}(1+\delta)\omega(A^{-(n+1)}) \le A^{\mu n}\omega(A^{-n}),$$

so that the first assertion holds, at least at the special points A^n. Similarly, if $0 < \nu < 1$ and $A^{1-\nu}(1-\eta) < 1$,

$$A^{\nu n}\omega(A^{-n}) \le A^{1-\nu}(1-\eta)A^{\nu(n+1)}\omega(A^{-(n+1)} < A^{\nu(n+1)}\omega(A^{-(n+1)}).$$

A simple computation shows that the derivative of $t^{-a}\log\omega_1(t)$ with respect to $\log t$ on (A^n, A^{n+1}) is equal to $\log\left(\frac{\omega_1(A^{n+1})}{\omega_1(A^n)}\right)/\log A - a$. The inequality $(1+\delta)\omega(A^n) \le \omega(A^{n+1})$ shows that this derivative is greater than $\log(1+\delta)/\log A - a$ which is greater than 0 for $a = \mu$. Similarly, the inequality $\omega(A^{n+1}) \le A(1-\eta)\omega(A^n)$ shows that this derivative is smaller than $(\log A(1-\eta))/\log A - a$ which is less than 0 for $a = \nu$. This completes the proof. □

This approach can be used to construct regular majorants with various properties. For example if $\{\omega_n\}_{n\in\mathbb{Z}}$ is an increasing sequence such that

$$0 < a = \frac{1}{\log 2}\liminf_{n\to-\infty}\frac{\omega_{n+1}}{\omega_n} < \frac{1}{\log 2}\limsup_{n\to-\infty}\frac{\omega_{n+1}}{\omega_n} = b < 1,$$

the regular majorant given by $\omega(2^n) = \omega_n$, etc., satisfies the conclusion of Proposition 3 only if $0 < \mu < a$ and $b < \nu < 1$. This answers a question posed to the author by John R. Cannon.

It is easy to show that a majorant ω satisfying the conclusions of Proposition 3 must be regular. In the remainder of this paper it will be assumed that the regular majorant ω satisfies the conclusion of Proposition 3 with exponents μ and ν.

3. Duality

For a regular majorant ω, the results of the preceding section allow the identification of the space A_ω with a space of analytic functions of restricted growth. Specifically, if $\phi(r)$ is positive and continuous for $0 \le r < 1$ with $\lim_{r\to 1} \phi(r) = 0$, $A_\infty(\phi)$ will denote the space of functions f analytic in the unit disk and satisfying the growth condition

$$\sup_{0\le r<1} M_\infty(f,r)\phi(r) < \infty,$$

with the norm $\|f\|_\phi$ of f set equal to the quantity on the left-hand side of the above inequality. This space has a subspace $A_0(\phi)$ consisting of those functions f in $A_\infty(\phi)$ which satisfy

$$\lim_{r\to 1} M_\infty(f,r)\phi(r) = 0.$$

The codimension-one subspace of functions $f \in A_\omega$ which vanish at the origin is then identified with $A_\infty(\phi)$ under the correspondence of f with f', when $\phi(r) = (1-r^2)/\omega(1-r)$. Similarly, $A_0(\phi)$ corresponds to the subspace $A_{\omega,0}$ of functions vanishing at the origin.

Rubel and Shields [15] showed that under rather general conditions $A_\infty(\phi)$ is isomorphic to the second dual of $A_0(\phi)$, and subsequently Shields and Williams [21] identified the intervening dual space under more restrictive hypotheses on ϕ. To describe this, first consider a positive continuous function $\psi(r)$ for $0 < r \le 1$ with

$$\int_0^1 \psi(r)\, dr < \infty.$$

For f analytic in $\mathbb{D}$ define

$$\|f\|_\psi = \int_0^1 M_1(f,r)\psi(r)\, dr,$$

where

$$M_1(f,r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{it})|\, dt.$$

Then $A^1(\psi)$ will be the Banach space of analytic functions f with $\|f\|_\psi < \infty$.

A function ϕ as above will be called *normal* if there exist $k > \epsilon > 0$ and $r_0 < 1$ such that

$$\frac{\phi(r)}{(1-r)^\epsilon} \downarrow 0 \qquad \text{and} \qquad \frac{\phi(r)}{(1-r)^k} \uparrow \infty \qquad (r_0 \le r,\ r \to 1^-).$$

The functions (ϕ,ψ) then form a *normal pair* if there is $\alpha > k-1$ such that

$$\phi(r)\psi(r) = (1-r^2)^\alpha \qquad (0 \le r < 1).$$

If (ϕ, ψ) is a normal pair, there is a pairing between $A_\infty(\phi)$ and $A^1(\psi)$ given by

$$(f, g) = \frac{1}{\pi} \iint_{\mathbb{D}} f(z) g(\overline{z}) \phi(|z|) \psi(|z|)\, dx\, dy, \qquad f \in A_\infty(\phi), g \in A^1(\psi).$$

For $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A_\infty(\phi)$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n \in A^1(\psi)$, this pairing is also given by

$$(f, g) = \sum_{n=0}^{\infty} \frac{a_n b_n n!}{(\alpha+1)(\alpha+2)\cdots(\alpha+n+1)},$$

with the series converging in the sense of Abel. In particular, when $\alpha = 0$, the pairing is

$$(f, g) = \sum_{n=0}^{\infty} \frac{a_n b_n}{n+1}$$

Theorem 2 of [21] then asserts that the dual space $A_0(\phi)^*$ is isomorphic to $A^1(\psi)$, and that $A^1(\psi)^*$ is isomorphic to $A_\infty(\phi)$.

When ω is a regular majorant, the function $\phi(r) = (1-r^2)/\omega(1-r^2)$ is normal, and so the duality result of Shields and Williams holds between $A_\infty(\phi)$ and $A^1(\psi)$ with $\psi(r)\phi(r) = 1$. It will be more convenient to use the dual pairing

$$(f, g) = \frac{1}{\pi} \iint_{\mathbb{D}} f(z) \overline{z} g(\overline{z}) \phi(|z|) \psi(|z|)\, dx\, dy.$$

The analysis of Shields and Williams carries over. Moreover, in terms of coefficients the pairing is

$$(f, g) = \sum_{n=0}^{\infty} \frac{a_n b_{n+1}}{n+1},$$

where, as above, the series converges in the sense of Abel. If $h(z) = \sum_{n=0}^{\infty} c_n z^n \in A_\omega$, its derivative belongs to $A_\infty(\phi)$ and the dual pairing

$$\langle h, g\rangle = c_0 b_0 + (h', g) = c_0 b_0 + \sum_{n=0}^{\infty} \frac{(n+1)c_{n+1} b_{n+1}}{n+1} = \sum_{n=0}^{\infty} c_n b_n$$

leads to the following theorem.

THEOREM 5. *Let $\omega(r)$ be a regular majorant. If $\psi(r)$ is given by the formula $\psi(r) = (1-r^2)/\omega(1-r^2)$, the space $A^1(\psi)$ is isomorphic to the dual space of $A_{\omega,0}$, and in turn has dual space isomorphic to A_ω. For $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A_\omega$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n \in A^1(\psi)$, the dual pairing is given by*

$$\langle f, g\rangle = \sum_{n=0}^{\infty} a_n b_n,$$

where the series converges in the sense of Abel.

4. Aleksandrov operators

Throughout this section b will denote a holomorphic function mapping the unit disk into itself which fixes the origin. Theorem 1 now admits a simple proof based on duality and Littlewood's subordination principle in the form

$$M_1(f \circ b, r) \leq M_1(f, r).$$

The next lemma is immediate.

LEMMA 6. *Let $\psi(r)$ be a positive, increasing, integrable function on $0 \leq r < 1$. Then the composition operator C_b is bounded on $A^1(\psi)$.*

If ω is a regular majorant and $\psi(r) = \omega(1 - r^2)/(1 - r^2)$, then $\psi(r)$ satisfies the hypotheses of the lemma. Indeed,

$$\psi(r) = \frac{\omega(1 - r^2)}{(1 - r^2)} = \left(\frac{1}{1 - r^2}\right)^{1-\nu} \frac{\omega(1 - r^2)}{(1 - r^2)^{\nu}}$$

is increasing, since each factor on the right is. Thus C_b is a bounded operator on the dual space $A^1(\psi)$.

As noted in the introduction, it was shown in [4] that A_b is the formal adjoint of C_b, when the dual pairing is given by

$$(f, g) = \sum f_n \overline{g_n}.$$

With respect to the pairing used in this paper, C_{b^*} is the formal adjoint of A_b, where $b^*(z) = \overline{b(\overline{z})}$. In particular, A_b is bounded on polynomials in A_ω. In [16] Sarason observed that if $b(0) = 0$, the adjoint of C_b takes polynomials of degree n into polynomials of degree at most n. Since the polynomials are dense in $A_{\omega,0}$, it follows that A_b is a bounded operator on $A_{\omega,0}$, and hence also on A_ω since the latter space is isomorphic to the second dual of $A_{\omega,0}$.

If $f \in \Lambda_\omega$ and f is real valued, according to Proposition 2 its harmonic conjugate $\tilde{f}$ also belongs to Λ_ω. Hence $F = f + i\tilde{f}$ belongs to A_ω. Applying A_b to F and taking the real part shows that $A_b f \in \Lambda_\omega$. This completes the proof of Theorem 1.

5. Carleson measures and compactness

Because operators are compact if and only if their adjoints are, Theorem 2 will follow from the corresponding theorem for the composition operator C_b acting on $A^1(\psi)$, where $\psi(r) = \omega(1-r^2)/(1-r^2)$. Actually a somewhat more general result is true. For $0 < p < \infty$, let $A^p(\psi)$ denote the space of functions analytic in $\mathbb{D}$ for which

$$\|f\|_{p,\psi}^p = \frac{1}{\pi} \iint_{\mathbb{D}} |f(z)|^p \psi(|z|)\, dx\, dy < \infty.$$

For $1 \le p < \infty$, this is a Banach space and for all p it is an F-space. Littlewood's principle again guarantees that C_b is a bounded operator on each of these spaces.

THEOREM 7. *Let ω be a regular majorant, and let $\psi(r) = \omega(1-r^2)/(1-r^2)$. Then the composition operator C_b is compact on $A^p(\psi)$ if and only if b has no angular derivatives in the sense of Carathéodory.*

The proof of this theorem will use extensions of known arguments involving the Carleson measures for these spaces. The Carleson measure aspects will be treated in this section, and angular derivatives will be considered in the next.

A positive regular Borel measure μ on $\mathbb{D}$ is a *Carleson measure* for $A^p(\psi)$ if there is a constant C such that

$$\int_{\mathbb{D}} |f(z)|^p \, d\mu \le C \|f\|_{p,\psi}^p$$

for every $f \in A^p(\psi)$. As the following theorem indicates, Carleson measures admit a geometric characterization. This theorem, whose proof will be given below, is an extension of known results.

THEOREM 8. *Let I be an arc of the unit circle and let $R(I)$ denote the Carleson square on I. Thus $z \in R(I)$ if and only if $1-|z| < |I|$ and $z/|z| \in I$, where $|I|$ denotes the normalized length of I. If ω is a regular majorant and $\psi(r) = \omega(1-r^2)/(1-r^2)$, the positive regular Borel measure μ on $\mathbb{D}$ is a Carleson measure for $A^p(\psi)$ if and only if*

$$\mu(R(I)) \le C|I|\omega(|I|)$$

for all arcs I on the unit circle. When this happens, the constant C and the Carleson measure constant are proportional. Moreover, the embedding of $A^p(\psi)$ into $L^p(\mu)$ is compact if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\mu(R(I)) \le \epsilon|I|\omega(|I|)$$

whenever $|I| < \delta$.

For each holomorphic function b mapping the unit disk into itself there is an associated positive regular Borel measure μ_b determined by the change of variables formula

$$\int_{\mathbb{D}} u(\zeta) \, d\mu_b(\zeta) = \iint_{\mathbb{D}} u \circ b(z) \psi(|z|) \, dx \, dy.$$

Since

$$\int_{\mathbb{D}} |f(\zeta)|^p \, d\mu_b(\zeta) = \iint_{\mathbb{D}} |f \circ b(z)|^p \psi(|z|) \, dx \, dy,$$

for each $f \in A^p(\psi)$, it follows from the boundedness of C_b on $A^p(\psi)$ that μ_b is a Carleson measure for each space $A^p(\psi)$.

Now any operator T from $A^p(\psi)$ to a Banach space X is compact if and only if $\|Tf_n\| \to 0$ for every bounded sequence (f_n) in $A^p(\psi)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$. If μ_b satisfies the last condition of Theorem 8, the composition operator C_b is compact on $A^p(\psi)$. Indeed, given $\epsilon > 0$, choose $\delta > 0$ so that $\mu_b(R(I)) < \epsilon|I|\omega(|I|)$ whenever $|I| < \delta$. Let μ_1 be the restriction of μ_b to the disk $|z| \le 1 - \delta$ and let $\mu_2 = \mu_b - \mu_1$. If (f_n) is a bounded sequence in $A^p(\psi)$ which converges to 0 uniformly on compact subsets of the unit disk, then

$$\|C_b f_n\|_{p,\psi}^p = \int |f_n|^p \, d\mu_1 + \int |f_n|^p \, d\mu_2.$$

The first integral on the right tends to 0 because μ_1 is supported on a compact set, and the second does not exceed $c\epsilon^p$ for some constant c. It follows that C_b is compact.

The following norm estimate will be needed for the proof of the converse and again in the proof of Theorem 8. The argument comes from [21]. The expression $A \approx B$ means there are two positive constants c_1 and c_2 such that $c_1 \le A/B \le c_2$.

Proposition 4. *Let $k_a(z) = (1 - \overline{a}z)^{-2/p}$ for $a \in \mathbb{D}$. Then*

$$\|k_a\|_{p,\psi} \approx \frac{\omega(1-|a|^2)}{1-|a|^2} \qquad \text{as } |a| \to 1.$$

Proof. First note that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |k_a(re^{i\theta})|^p \, d\theta = \frac{1}{1-|a|^2 r^2}.$$

It follows that

$$\|k_a\|_{p,\psi}^p = \int_0^1 \frac{1}{1-|a|^2r^2}\frac{\omega(1-r^2)}{1-r^2}\, dr.$$

Since

$$\frac{1}{1-|a|^2r^2}\frac{\omega(1-r^2)}{1-r^2} \approx \frac{1}{1-|a|r}\frac{\omega(1-r)}{1-r},$$

it is enough to estimate the integral of the latter expression. Now, integrating by parts gives

$$\begin{aligned}
\int_0^1 \frac{1}{1-\rho r}\frac{\omega(1-r)}{1-r}\, dr &= \omega_*(1) + \rho\int_0^1 \frac{1}{(1-\rho r)^2}\int_0^r \frac{\omega(1-s)}{1-s}\, ds\, dr \\
&= \omega_*(1) + \rho\int_0^1 \frac{\omega_*(1-r)}{(1-\rho r)^2}\, dr \\
&\approx \omega_*(1) + \rho\int_0^1 \frac{\omega(1-r)}{(1-\rho r)^2}\, dr.
\end{aligned}$$

Since ω is increasing,

$$\begin{aligned}\rho\int_0^1 \frac{\omega(1-r)}{(1-\rho r)^2}\,dr &\le \rho\int_0^1 \frac{\omega(1-\rho r)}{(1-\rho r)^2}\,dr\\ &= \int_{1-\rho}^1 \frac{\omega(u)}{u^2}\,du\\ &\le \frac{\omega^*(1-\rho)}{1-\rho}\\ &\le \beta\frac{\omega(1-\rho)}{1-\rho},\end{aligned}$$

which is half of what is needed.

To obtain the lower estimate, note that

$$\begin{aligned}\rho\int_0^1 \frac{\omega(1-r)}{(1-\rho r)^2}\,dr &= \rho\int_0^1 \frac{(1-r)^\mu}{(1-\rho r)^2}\frac{\omega(1-r)}{(1-r)^\mu}\,dr\\ &\ge \frac{1}{2}\int_0^1 \frac{(1-r)^\mu\omega(1-\rho r)}{(1-\rho r)^{2+\mu}}\,dr\\ &\ge \frac{1}{2}\omega(1-\rho)\int_0^1 \frac{(1-r)^\mu}{(1-\rho r)^{2+\mu}}\,dr,\end{aligned}$$

when, say, $1/2 < \rho < 1$. The inequality then follows from the simple estimate

$$\int_0^1 (1-\rho r)^{-m}(1-r)^\gamma\,dr \ge c(1-\rho)^{1+\gamma-m},$$

which is valid for $\gamma < 1$ and $m > \gamma + 1$. $\square$

Suppose $p = 1$ (the argument for other values of p is similar). For each nonzero $a \in \mathbb{D}$, let I_a be the arc of the circle with length $|I_a| = 1 - |a|$ and center $a/|a|$. The functions $f_a = \frac{|I_a|}{\omega(|I_a|)}k_a$ are norm bounded, while the estimate $|1-\overline{a}z| \le c(1-|a|)$ for $z \in R(I_a)$ shows that $|f_a(z)| \ge C/(|I_a|\omega(|I_a|))$ for $z \in R(I_a)$. In particular, if μ is a Carleson measure for $A^1(\psi)$, then

$$\mu(R(I_a)) = \int_{R(I_a)} d\mu \le c|I_a|\omega(|I_a|)\int_{R(I_a)} |f_a|\,d\mu \le c'|I_a|\omega(|I_a|),$$

and the geometric condition of Theorem 8 holds for μ.

If the last condition of Proposition 8 fails for the measure μ, there is an $\eta > 0$ and a sequence of arcs I_n with $|I_n| \to 0$ and such that $\mu(R(I_n)) \ge \eta|I_n|\omega(|I_n|)$ for each n. If a_n is the point with $1-|a_n| = |I_n|$ and with $a_n/|a_n|$ the center of I_n, then the functions f_{a_n} tend to 0 uniformly on compact subsets of $\mathbb{D}$, but the above estimate shows that

$$\int |f_{a_n}|\,d\mu \ge C\eta \qquad \text{for all } n,$$

and so the embedding of $A^1(\psi)$ in $L^1(\mu)$ cannot be compact.

The easiest way to show that μ is a Carleson measure for $A^p(\psi)$ if $\mu(R(I)) \le C|I|\omega(|I|)$ is to use a method of Luecking [10]. This method uses a family of sets $E(a)$, $a \in \mathbb{D}$, with the following properties:

(i) $\overline{E(a)} \subset \mathbb{D}$;
(ii) $\operatorname{area}(E^*(a)) \le c_1 \operatorname{area}(E(a))$, with $E^*(a) = \cup\{E(b) : E(b) \cap E(a) \neq \emptyset\}$;
(iii) $\psi(b) \le c_2 \psi(a)$ for all $b \in E(a)$.

If the functions $f \in A^p(\psi)$ satisfy a mean value inequality

$$|f(z)|^p \le \frac{c_3}{\operatorname{area}(E(a))} \int_{E(a)} |f(z)|^p \, dx \, dy,$$

Luecking proves that μ is a Carleson measure for $A^p(\psi)$ provided $\mu(E(a)) \le c_4 \int_{E(a)} \psi(z) \, dx \, dy$ for each $a \in \mathbb{D}$.

In the current setting it is enough to let $E(a)$ be the hyperbolic disk with center a and fixed radius r. The mean value inequality and the properties (i) and (ii) are known to hold in this case. For the third property, if $E(a) = \{\, b : \left|\frac{a-b}{1-\overline{a}b}\right| < r \,\}$, then

$$\frac{|a|-r}{1-|a|r} \le |b| \le \frac{|a|+r}{1+|a|r}.$$

Since

$$1 - \left|\frac{|a|+r}{1+|a|r}\right|^2 = \frac{(1-|a|^2)(1-r^2)}{(1+|a|r)^2} \ge (1-|a|^2)\frac{1-r}{1+r},$$

it follows that $(1-|b|^2) \ge c(1-|a|^2)$ for all $b \in E(a)$. On the other hand,

$$1 - \left|\frac{|a|-r}{1-|a|r}\right|^2 = \frac{(1-|a|^2)(1-r^2)}{(1-|a|r)^2} \le (1-|a|^2)\frac{1+r}{1-r},$$

so $1-|b|^2 \le c(1-|a|^2)$ for all $b \in E(a)$, and thus $\omega(1-|b|^2) \le c'\omega(1-|a|^2)$ for some constant c' depending only on c and ω. This establishes (iii).

Finally, each $E(a)$ is contained in a Carleson square $R(I)$, where $|I|$ is proportional to $1-|a|$. Thus $\mu(E(a)) \le \mu(R(I)) \le C|I|\omega(|I|)$. On the other hand, $E(a)$ contains a rectangle

$$R(a) = \{\, z : |\arg z - \arg a| < \kappa(1-|a|), \quad \kappa(1-|a|) < 1-|z| < (1-|a|) \,\},$$

where $1 > \kappa > 0$ depends only on r. Hence

$$\begin{aligned}\int_{E(a)} \psi(z)\, dx\, dy &\ge \int_{R(a)} \psi(z)\, dx\, dy \\ &\ge c_5(1-|a|) \int_{\kappa(1-|a|)}^{1-|a|} \frac{\omega(t)}{t}\, dt \\ &\ge c_6(1-|a|)\omega(\kappa(1-|a|)).\end{aligned}$$

Finally, $\omega(\kappa(1-|a|)) \geq \kappa'\omega(|I|)$, where κ' depends only on κ. Thus the condition on μ follows.

Arguing as above, it is not difficult to see that when μ is a Carleson measure for $A^p(\psi)$, the embedding of $A^p(\psi)$ into $L^p(\mu)$ is compact if and only if

$$\mu(E(a)) = o((1-|a|)\omega(1-|a|))$$

as $|a| \to 1$. The compactness results can be summarized as follows.

Proposition 5. *The following are equivalent:*

(i) *C_b is compact on $A^p(\psi)$ for some $0<p<\infty$;*
(ii) *C_b is compact on $A^p(\psi)$ for all $0<p<\infty$;*
(iii) *$\mu_b(R(I)) = o(|I|\omega(|I|))$ as $|I| \to 0$.*

6. Angular derivatives

Since Proposition 5 shows that compactness of C_b on $A^p(\psi)$ does not depend on $p \in (0,\infty)$, it will be enough to consider $A^2(\psi)$. The proof of Theorem 2 will be completed by showing directly that compactness of C_b is equivalent to the nonexistence of angular derivatives. The argument is an adaptation to the current setting of an argument of J. H. Shapiro.

First suppose that b has an angular derivative. Then there is a sequence (a_n) in $\mathbb{D}$ with $a_n \to \alpha \in \mathbb{T}$ such that $\phi(a_n) \to \beta \in \mathbb{T}$ and $\frac{1-|\phi(a_n)|}{1-|a_n|} \to d \in (0,\infty)$. It follows that

$$\frac{d}{2}(1-|a_n|) \leq 1-|\phi(a_n)| \leq 2d(1-|a_n|)$$

for large n. Arguing as in [18, Chapter 3], let $f_n = k_{a_n}/\|k_{a_n}\|_{2,\psi}$. Note that $C_b^* k_a = k_{b(a)}$, and so

$$\begin{aligned}\|C_b^* f_n\|_{2,\psi}^2 &= \frac{\|k_{b(a_n)}\|_{2,\psi}^2}{\|k_{a_n}\|_{2,\psi}^2} \\ &\approx \frac{1-|a_n|^2}{1-|b(a_n)|^2}\,\frac{\omega(1-|b(a_n)|^2)}{\omega(1-|a_n|)},\end{aligned}$$

and this is bounded away from 0 for large n. Now suppose that C_b is compact, and, passing to a subsequence if necessary, that $C_b^* f_n \to g$. Let p be a polynomial. Then

$$\langle g, p\rangle = \lim_{n\to\infty} \langle C_b^* f_n, p\rangle = \lim_{n\to\infty} \langle f_n, C_b p\rangle = \lim_{n\to\infty} \frac{\overline{p(b(a_n))}}{\|k_{a_n}\|_{2,\psi}}.$$

Since p is bounded and $\|k_{a_n}\|_{2,\psi} \to \infty$, it follows that $\langle g, p\rangle = 0$ for all polynomials p. Since the polynomials are dense in $A^2(\psi)$ it follows that $g=0$, and so $\|C_b^* f_n\|_{2,\psi} \to 0$. This is a contradiction, so if C_b is compact, b has no angular derivatives.

Now suppose that b has no angular derivatives. To begin with, $A^2(\psi)$ admits an equivalent norm

$$\|f\|^2_{\mathcal{D},\psi} = |f(0)|^2 + \frac{1}{\pi}\iint_{\mathbb{D}} |f'(z)|^2 \Psi_0(|z|)\,dx\,dy,$$

where (cf. [5, p. 133])

$$\Psi_0(r) = \int_r^1 \int_s^1 \psi(t)\,dt\,ds.$$

A simple argument shows that $\Psi_0(r) \approx (1-r)\omega(1-r)$. Indeed,

$$\begin{aligned}
\Psi_0(r) &= \int_r^1 \int_s^1 \frac{\omega(1-t^2)}{1-t^2}\,dt\,ds \\
&\approx \int_r^1 \int_s^1 \frac{\omega(1-t)}{1-t}\,dt\,ds \\
&= \int_r^1 \int_s^1 \frac{\omega(1-t)}{(1-t)^{\nu}} \frac{1}{(1-t)^{1-\nu}}\,dt\,ds \\
&\le \int_r^1 \frac{\omega(1-s)}{(1-s)^{\nu}} \int_s^1 \frac{1}{(1-t)^{1-\nu}}\,dt\,ds \\
&= \frac{1}{\nu}\int_r^1 \omega(1-s)\,ds \\
&\le \frac{1}{\nu}(1-r)\omega(1-r).
\end{aligned}$$

An analogous argument produces the lower estimate. Thus $\Psi_0(r)$ can be replaced by $\Psi(r) = (1-r)\omega(1-r)$ in the equivalent norm.

Using $\Psi(r)$ in the equivalent norm, there is a change of variables formula

$$\iint_{\mathbb{D}} |(f\circ b)'(z)|^2 \Psi(|z|)\,dx\,dy = \iint_{b(\mathbb{D})} |f'(w)|^2 M_b(w)\,du\,dv,$$

where

$$M_b(w) = \sum_{b(z)=w} \Psi(|z|);$$

see [5, p. 36]. Since $b(0)=0$, the Schwarz Lemma shows that $|w| \le |z|$ whenever $w = b(z)$, and hence $1-|z| \le 1-|w|$. Suppose $w \ne 0$ and let z_0 be the value of z of least modulus with $b(z) = w$. Then, because ω is increasing,

$$\begin{aligned} M_b(w) &= \sum_{b(z)=w} (1-|z|)\omega(1-|z|) \\ &\le \omega(1-|z_0|) \sum_{b(z)=w} (1-|z|) \\ &\le \omega(1-|w|) \sum_{b(z)=w} (1-|z|) \\ &\le \omega(1-|w|) \sum_{b(z)=w} \log\frac{1}{|z|} \end{aligned}$$

since $1-|z| \le \log 1/|z|$. Now Littlewood's Inequality (see [18, p. 187]) asserts that

$$\sum_{b(z)=w} \log\frac{1}{|z|} \le \log\frac{1}{|w|},$$

so $M_b(w) \le \omega(1-|w|)\log 1/|w|$. Hence there is a constant C such that $M_b(w) \le C\Psi(|w|)$ if $|w| > 1/2$. On the other hand, $M_b(w)$ is clearly bounded if $|w| \le 1/2$. It follows that C_b is compact if $M_b(w) = o(\Psi(|w|))$, as $|w| \to 1$. To see this, it is enough to show that $\|C_b f_n\|_{\mathcal{D},\psi} \to 0$ if (f_n) is a bounded sequence in $A^2(\psi)$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$. But given $\epsilon > 0$ there is a $\delta > 0$ such that $M_b(w) < \epsilon$ whenever $1-|w| < \delta$. Then

$$\begin{aligned} \|C_b(f_n)\|^2_{\mathcal{D},\psi} &= |f_n(0)|^2 + \frac{1}{\pi}\iint_{b(\mathbb{D})} |f_n'(w)|^2 M_b(w)\,du\,dv \\ &\le |f_n(0)|^2 + \frac{C}{\pi}\iint_{|w|\le 1-\delta} |f_n'(w)|^2\,du\,dv \\ &\qquad + \frac{\epsilon}{\pi}\iint_{b(\mathbb{D})} |f_n'(w)|^2\Psi(w)\,du\,dv \\ &\le |f_n(0)|^2 + \frac{C}{\pi}\iint_{|w|\le 1-\delta} |f_n'(w)|^2\,du\,dv + \epsilon\|f_n\|^2_{\mathcal{D},\psi}. \end{aligned}$$

Since the first two terms clearly tend to 0, it follows that $\lim_{n\to\infty}\|C_b(f_n)\|_{2,\psi} = 0$ and so C_b is compact.

If b has no angular derivatives, then

$$\lim_{|z|\to 1} \frac{1-|b(z)|}{1-|z|} = \infty.$$

Otherwise there is a sequence $z_n \in \mathbb{D}$ with $|z_n| \to 1$ such that $\lim_{|z|\to 1} \frac{1-|b(z)|}{1-|z|} = a < \infty$, say. Passing to a subsequence if necessary, it may be supposed that $z_n \to \alpha \in \mathbb{T}$ and $b(z_n) \to \beta \in \mathbb{T}$. Then the Julia-Carathéodory Theorem shows that b has an angular derivative at α. Hence, given $\epsilon > 0$ there is a

$\delta > 0$ such that $1 - |z| < \epsilon(1 - |w|)$ whenever $|w| > 1 - \delta$ and $b(z) = w$. It follows that

$$\begin{aligned} M_b(w) &= \sum_{b(z)=w} (1 - |z|)\omega(1 - |z|) \\ &\le \omega(1 - |z_0|) \sum_{b(z)=w} 1 - |z| \\ &\le C\omega(\epsilon(1 - |w|))(1 - |w|) \\ &\le \frac{\omega(\epsilon(1 - |w|))}{\omega(1 - |w|)} \Psi(w), \end{aligned}$$

where z_0 is the value of z of least modulus such that $b(z) = w$ and the second inequality follows from Littlewood's Inequality. Finally, since $\omega(t)/t^\nu$ is increasing, it is easy to see that $\omega(\epsilon t)/\omega(t) \le \epsilon^\nu$. This completes the proof.

7. Concluding remarks

It would, of course, be of interest to see a proof of Theorem 1 that does not use duality as in the present paper. As remarked in the introduction, A_b is compact on C if and only if the function $\alpha \to \tau_\alpha$ is norm continuous. If this function has modulus of continuity $\omega(t)$, then $\omega(C_b f, t) \le C\omega(t)$ for every $f \in C$, and so $A_b(C) \subset \Lambda_\omega$. This suggests a different approach to smoothness. On the other hand, there are inner functions b without angular derivatives, and so the corresponding operator A_b is compact on the spaces Λ_ω, even though all of the measures τ_α are singular, and the function $\alpha \to \tau_\alpha$ has no point of continuity for the norm topology on M.

If $\alpha(z) = \frac{a-z}{1-\bar{a}z}$ is an automorphism of $\mathbb{D}$, then α is a homeomorphism of $\mathbb{T}$ onto itself. It is clear that α satisfies a Lipschitz condition

$$|\alpha(e^{i(t+\delta)}) - \alpha(e^{it})| \le \kappa\delta$$

for $\delta > 0$. Consequently, if $f \in \Lambda_\omega$,

$$\begin{aligned} |f(\alpha(e^{i(t+\delta)})) - f(\alpha(e^{it}))| &\le C_f \omega(|\alpha(e^{i(t+\delta)}) - \alpha(e^{it})|) \\ &\le C_f \omega(\kappa\delta) \le C_f C' \omega(\delta). \end{aligned}$$

Hence the restriction $b(0) = 0$ is inconsequential.

Finally, it is clear that the analysis here can be carried over to spaces of functions whose smoothness is determined by an integral modulus of continuity. The details will be left to others.

References

[1] A. B. Aleksandrov, *Multiplicity of boundary values of inner functions*, Izv. Akad. Nauk Armyan. SSR Ser. Mat. **22** (1987), 490–503, 515 (Russian).

[2] P. Bourdon and J. A. Cima, *On integrals of Cauchy-Stieltjes type*, Houston J. Math. **14** (1988), 465–474.

[3] J. A. Cima and A. Matheson, *Essential norms of composition operators and Aleksandrov measures*, Pacific J. Math. **179** (1997), 59–64.

[4] ———, *Cauchy transforms and composition operators*, Illinois J. Math. **42** (1998), 58–69.

[5] C. C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions*, CRC Press, Boca Raton, FL, 1995.

[6] N. Dunford and J. T. Schwartz, *Linear operators, Part I*, Interscience, New York, 1957.

[7] K. M. Dyakonov, *Equivalent norms on Lipschitz-type spaces of holomorphic functions*, Acta Math. **178** (1997), 143–167.

[8] J. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.

[9] K. H. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, NJ, 1962.

[10] D. Luecking, *A technique for characterizing Carleson measures on Bergman spaces* Proc. Amer. Math. Soc. **87** (1983), 656–660.

[11] A. Matheson, *Mean growth and smoothness of analytic functions*, Proc. Amer. Math. Soc. **85** (1982), 219–224.

[12] ——— *Cyclic vectors for invariant subspaces in some classes of analytic functions*, Illinois J. Math. **36** (1992), 136–144.

[13] B. D. MacCluer and J. H. Shapiro, *Angular derivatives and compact composition operators on the Hardy and Bergman spaces*, Canad. J. Math. **38** (1986), 878–906.

[14] R. Nevanlinna, *Remarques sur le lemme de Schwarz*, Comptes Rendues Acad. Sci. Paris **188** (1929), 1027–1029.

[15] L. A. Rubel and A. L. Shields, *The second duals of certain spaces of analytic functions*, J. Austral. Math. Soc. **11** (1970), 276–280.

[16] D. Sarason, *Composition operators as integral operators*, Analysis and partial differential equations (C. Sadosky, ed.), Marcel Dekker, New York, 1990; pp. 545–565.

[17] ———, *Weak compactness of holomorphic composition operators on* H^1, Functional analysis and operator theory (New Delhi, 1990), Springer-Verlag, Berlin (1992), pp. 75–79.

[18] J. H. Shapiro, *Composition operators and classical function theory*, Springer-Verlag, New York, 1993.

[19] ———, *The essential norm of a composition operator*, Ann. of Math. (2) **125** (1987), 375–404.

[20] J. H. Shapiro and C. Sundberg, *Compact composition operators on* L^1, Proc. Amer. Math. Soc. **108** (1990), 443–449.

[21] A. L. Shields and D. L. Williams, *Bounded projections, duality, and multipliers in spaces of analytic functions*, Trans. Amer. Math. Soc. **162** (1971), 287–302.

[22] A. Zygmund, *Trigonometric series*, Cambridge University Press, Cambridge, 1968.

Department of Mathematics, Lamar University, P.O. Box 10047, Beaumont, TX 10047, USA

E-mail address: matheson@math.lamar.edu