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INTERMEDIATE RINGS BETWEEN A LOCAL DOMAIN AND ITS COMPLETION, II

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Abstract. We present results connecting flatness of extension rings to the Noetherian property for certain intermediate rings between an excellent normal local domain and its completion. We consider conditions for these rings to have Cohen-Macaulay formal fibers. We also present several examples illustrating these results.

1. Introduction

In a series of papers [HRW1], [HRW2], [HRW3], the authors have been analyzing and further developing a technique for building new Noetherian (and non-Noetherian) commutative rings using completions, intersections, and homomorphic images. This technique goes back at least to Akizuki [A]. It is used by Nagata in his famous example of a normal Noetherian local domain for which the completion is not a domain [N1] or [N2, Example 7, pages 209-211]. The technique is also used by Rotthaus [R1], [R2], Ogoma [O], Brodmann-Rotthaus [BR1], [BR2], and Heitmann [H].

Let R be a commutative Noetherian integral domain with fraction field K, let y be a nonzero nonunit of R, and let R^* be the y-adic completion of R (the elements of R^* may be regarded as power series in y with coefficients in R). In [HRW1], we consider a form of the construction involving "intermediate" rings between R and its y-adic completion: Let $\tau_1, \ldots, \tau_s$ be elements of yR^* algebraically independent over K, let $S := R[\tau_1, \ldots, \tau_s]$ and define $C := L \cap R^*$, where L (contained in the total ring of fractions of R^*) is the field of fractions of S. It follows from [HRW2] that $C = L \cap R^*$ is both Noetherian and a localization of a subring of $S[1/y]$ if and only if the extension

$$S \hookrightarrow R^*[1/y] \tag{1.1}$$

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is flat [HRW2, Theorem 2.12].

In [HRW3], we use a generalization of the intermediate rings construction involving homomorphic images[1], namely $A := K \cap (R^*/I)$, for certain ideals I of R^*, where K is the field of fractions of R. For such ideals I, we show in [HRW3, Theorem 3.2] that $A = K \cap (R^*/I)$ is both Noetherian and a localization of a subring of $R[1/y]$ if and only if $R \hookrightarrow (R^*/I)_y$ is flat.

In this article, we present test criteria, involving the heights of certain prime ideals, for the extension in (1.1) to be flat[2]. We give in Theorem 2.1 a characterization of flatness in a more general related context under the condition that certain fibers are Cohen-Macaulay and other fibers are regular. We use this to show in Corollary 2.4 that, in the intermediate rings version of the construction, if $C = L \cap R^*$ is both Noetherian and a localization of a subring of $S[1/y]$, then the extension $C = L \cap R^* \hookrightarrow R^*[1/y]$ always has Cohen-Macaulay fibers.

In contrast with this, we present an example in Section 4, using the more general homomorphic image form of the construction, where $C = K \cap (R^*/I)$ is Noetherian and is a localization of a subring of $R[1/y]$, so the extension $C \hookrightarrow R^*[1/y]$ is flat, but fails to have Cohen-Macaulay fibers.

Rings with Cohen-Macaulay fibers are of interest in local cohomology (see, for example, [BS]). In some instances the condition that the base ring R is a homomorphic image of a regular ring can be weakened to the condition that R is universally catenary with Cohen-Macaulay formal fibers. Most notably Faltings' annihilator theorem holds for finitely generated modules over such rings. A related application to universally catenary local rings with Cohen-Macaulay formal fibers can be found in the theory of generalized Cohen-Macaulay modules. These are finitely generated modules M of dimension $n > 0$ for which the local cohomology modules $H^i_{\mathbf{m}}(M)$ are finitely generated in all possible cases, that is, for all $i < n$.

In Section 3 we observe that every Noetherian local ring containing an excellent local subring R and having the same completion as R has Cohen-Macaulay formal fibers; this applies to the intermediate rings construction, but not to the homomorphic images construction. In Remark 3.4, we discuss connections with a famous example of Ogoma.

We present in Section 4 non-Noetherian integral domains $B \subsetneq A$ arising from the intermediate rings construction. We also construct, using the homomorphic images construction, a two-dimensional Noetherian local domain C that is a homomorphic image of B and has the property that its generic formal fiber is not Cohen-Macaulay.

[1]As we point out in [HRW3], the intermediate rings construction (without homomophic images) is a special case of the homomorphic image form.

[2]We discovered these results while making corrections to Theorem 5.8 of our paper [HRW1]; the theorem there is stated incorrectly, as is explained in [HRW1c]. Theorem 2.2 of this paper is our restatement of that theorem.

Our conventions and terminology are as in [M2]. In particular, if $f : A \to B$ is a ring homomorphism and J is an ideal of B, then the ideal $f^{-1}(J)$ of A is denoted $A \cap J$.

2. Flatness criteria

We use a general result on flatness involving a trio of Noetherian local rings $(R, \mathbf{m}), (S, \mathbf{n})$ and (T, ℓ) and local maps

$$R \longrightarrow S \longrightarrow T.$$

THEOREM 2.1. *Assume that*

(i) $R \to T$ *is flat with Cohen-Macaulay fibers,*
(ii) $R \to S$ *is flat with regular fibers.*

Then the following statements are equivalent:

(1) $S \to T$ *is flat with Cohen-Macaulay fibers.*
(2) $S \to T$ *is flat.*
(3) *For each prime ideal* W *of* T*, we have* $\operatorname{ht}(W) \geq \operatorname{ht}(W \cap S)$*.*
(4) *For each prime ideal* W *of* T *such that* W *is minimal over* $\mathbf{n}T$*, we have* $\operatorname{ht}(W) \geq \operatorname{ht}(\mathbf{n})$*.*

Proof. The implications (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (4) are obvious, and the implication (2) $\Longrightarrow$ (3) is clear by [M2, Theorem 9.5]. For (4) $\Longrightarrow$ (1) we show:

(a) The map $S/\mathbf{m}S \longrightarrow T/\mathbf{m}T$ is faithfully flat.
(b) $\mathbf{m}S \otimes_S T \cong \mathbf{m}T$.

This will yield the flatness in item (1) by [M2, Theorem 22.3, (1) $\Longleftrightarrow$ (3)].

Proof of (b): Since $R \hookrightarrow S$ is flat, we have $\mathbf{m}S \cong \mathbf{m}R \otimes_R S$. Therefore

$$\mathbf{m}S \otimes_S T \cong (\mathbf{m} \otimes_R S) \otimes_S T \cong \mathbf{m} \otimes_R T \cong \mathbf{m}T.$$

Proof of (a): By assumption, $T/\mathbf{m}T$ is Cohen-Macaulay and $S/\mathbf{m}S$ is regular. Thus in view of [M2, Theorem 23.1], it suffices to show:

$$\dim(T/\mathbf{m}T) = \dim(S/\mathbf{m}S) + \dim(T/\mathbf{n}T).$$

Let W be a prime ideal of T such that $\mathbf{n}T \subseteq W$ and W is minimal over $\mathbf{n}T$. Since $\ell \cap S = \mathbf{n}$, we also have $W \cap S = \mathbf{n}$ and by [M2, Theorem 15.1], $\operatorname{ht}(W) \leq \operatorname{ht}(\mathbf{n})$. By assumption $\operatorname{ht}(W) \geq \operatorname{ht}(\mathbf{n})$, and therefore $\operatorname{ht}(W) = \operatorname{ht}(\mathbf{n})$. This equality holds for every minimal prime divisor of $\mathbf{n}T$, and therefore $\operatorname{ht}(\mathbf{n}) = \operatorname{ht}(\mathbf{n}T)$. Since the fibers of T over R are Cohen-Macaulay and hence catenary, we have

$$\begin{aligned}\dim(T/\mathbf{n}T) &= \dim(T) - \operatorname{ht}(\mathbf{n}T)\\ &= \dim(T) - \operatorname{ht}(W)\\ &= \dim(T) - \operatorname{ht}(\mathbf{n})\\ &= \dim(T) - \operatorname{ht}(\mathbf{m}T) - (\operatorname{ht}(\mathbf{n}) - \operatorname{ht}(\mathbf{m}S))\\ &= \dim(T/\mathbf{m}T) - \dim(S/\mathbf{m}S)).\end{aligned}$$

The next to last equality holds because T and S are flat over R, so $\operatorname{ht}(\mathbf{m}T) = \operatorname{ht}(\mathbf{m}S) = \operatorname{ht}(\mathbf{m})$ [M2, Theorem 15.1]. This completes the proof that $S \to T$ is flat.

To show Cohen-Macaulay fibers for $S \to T$, it suffices to show for each $Q \in \operatorname{Spec} T$ with $P := Q \cap S$ that T_Q/PT_Q is Cohen-Macaulay. Let $Q \cap R = q$. By passing to $R/q \subseteq S/qS \subseteq T/qT$, we may assume $Q \cap R = (0)$. Suppose that the height of $P = Q \cap S$ is $n \in \mathbb{N}$. Since $R \to S_P$ has regular fibers and $P \cap R = (0)$, the ideal PS_P is generated by n elements. Moreover, faithful flatness of the map $S_P \to T_Q$ implies that the ideal PT_Q has height n. Since T_Q is Cohen-Macaulay, a set of n generators of PS_P forms a regular sequence in T_Q. Hence T_Q/PT_Q is Cohen-Macaulay [M2, Theorems 17.4 and 17.3]. □

The following theorem is a restatement of [HRW1, Theorem 5.8]:

THEOREM 2.2. *Let $(R, \mathbf{m})$ be a normal excellent local domain and $y \in \mathbf{m}$. Suppose that R^* is the y-adic completion of R, that $\widehat{R}$ is the $\mathbf{m}$-adic completion of R, and that $\tau_1, \dots, \tau_s \in yR^*$ are algebraically independent over the fraction field of R. Then the following statements are equivalent:*

1. *$S := R[\tau_1, \dots, \tau_s]_{(m,\tau_1,\dots,\tau_s)} \hookrightarrow \widehat{R}[1/y]$ is flat.*
2. *Suppose P is a prime ideal of S and $\widehat{Q}$ is a prime ideal of $\widehat{R}$ minimal over $P\widehat{R}$. If $y \notin \widehat{Q}$, then $\operatorname{ht}(\widehat{Q}) = \operatorname{ht}(P)$.*
3. *If $\widehat{Q}$ is a prime ideal of $\widehat{R}$ with $y \notin \widehat{Q}$, then $\operatorname{ht}(\widehat{Q}) \geq \operatorname{ht}(\widehat{Q} \cap S)$.*

Moreover, if any of (1)–(3) hold, then $S \hookrightarrow \widehat{R}[1/y]$ has Cohen-Macaulay fibers. The corresponding statements with $\widehat{R}$ replaced by R^* also hold.

Proof. (1) $\implies$ (2): Let P be a prime ideal of S and let $\widehat{Q}$ be a prime ideal of $\widehat{R}$ that is minimal over $P\widehat{R}$ and is such that $y \notin \widehat{Q}$. The assumption of (1) implies flatness of the map:

$$\varphi_{\widehat{Q}} : S_{\widehat{Q} \cap S} \longrightarrow \widehat{R}_{\widehat{Q}}.$$

Furthermore the Going Down Theorem [M2, Theorem 9.5] implies that $\widehat{Q} \cap S = P$ and by [M2, Theorem 15.1], $\operatorname{ht}(\widehat{Q}) = \operatorname{ht}(P)$.

(2) $\implies$ (3): Let $\widehat{Q}$ be a prime ideal of $\widehat{R}$ with $y \notin \widehat{Q}$. Set $Q := \widehat{Q} \cap S$ and let $\widehat{W}$ be a prime ideal of $\widehat{R}$ which is minimal over $Q\widehat{R}$ and is contained in $\widehat{Q}$. Then $\operatorname{ht}(Q) = \operatorname{ht}(\widehat{W})$ by (2) since $y \notin \widehat{W}$ and therefore $\operatorname{ht}(\widehat{Q}) \geq \operatorname{ht}(Q)$.

(3) $\implies$ (1): Let $\widehat{Q}$ be a prime ideal of $\widehat{R}$ with $y \notin \widehat{Q}$. Then for every prime ideal $\widehat{W}$ contained in $\widehat{Q}$, we also have $y \notin \widehat{W}$, and by (3), $\operatorname{ht}(\widehat{W}) \geq \operatorname{ht}(\widehat{W} \cap S)$. Therefore, by Theorem 2.1, $\varphi_{\widehat{Q}} : S_{\widehat{Q} \cap S} \longrightarrow \widehat{R}_{\widehat{Q}}$ is flat with Cohen-Macaulay fibers.

The corresponding statements, with $\widehat{R}$ replaced by R^*, follow from [HRW1, Prop. 5.3] or [HRW2, Prop. 2.4]. □

As a corollary to Theorem 2.2, we have the following result.

Corollary 2.3. *With the notation of Theorem 2.2, assume that* $\widehat{R}[1/y]$ *is flat over* S. *Let* $P \in \operatorname{Spec} S$ *with* $\operatorname{ht}(P) \geq \dim(R)$. *Then:*

(1) *For every* $\widehat{Q} \in \operatorname{Spec} \widehat{R}$ *minimal over* $P\widehat{R}$ *we have* $y \in \widehat{Q}$.
(2) *Some power of* y *is in* $P\widehat{R}$.

Proof. Clearly items (1) and (2) are equivalent. To prove that these hold, suppose that $y \notin \widehat{Q}$. By Theorem 2.2 (2), $\operatorname{ht}(P) = \operatorname{ht}(\widehat{Q})$. Since $\dim(R) = \dim(\widehat{R})$, we have $\operatorname{ht}(\widehat{Q}) \geq \dim(\widehat{R})$. But then $\operatorname{ht}(\widehat{Q}) = \dim(\widehat{R})$ and $\widehat{Q}$ is the maximal ideal of $\widehat{R}$. This contradicts the assumption that $y \notin \widehat{Q}$. We conclude that $y \in \widehat{Q}$. □

Theorem 2.2, together with results from [HRW2] and [HRW3], gives the following corollary.

Corollary 2.4. *Let the notation be as Theorem 2.2 and let* L *denote the fraction field of* S. *Let* $C = L \cap R^*$. *Consider the following conditions:*

(1) C *is Noetherian and is a localization of a subring of* $S[1/y]$.
(2) $S \hookrightarrow R^*[1/y]$ *is flat.*
(3) $S \hookrightarrow R^*[1/y]$ *is flat with Cohen-Macaulay fibers.*
(4) *For every* $Q^* \in \operatorname{Spec}(R^*)$ *with* $y \notin Q^*$, *we have* $\operatorname{ht}(Q^*) \geq \operatorname{ht}(Q^* \cap S)$.
(5) C *is Noetherian.*
(6) $C \hookrightarrow R^*$ *is flat.*
(7) $C \hookrightarrow R^*[1/y]$ *is flat.*
(8) $C \hookrightarrow R^*[1/y]$ *is flat with Cohen-Macaulay fibers.*

Conditions (1)–(4) *are equivalent, conditions* (5)–(8) *are equivalent, and* (1)–(4) *imply* (5)–(8).

Proof. Item (1) is equivalent to (2) by [HRW3, (3.2)], (2) is equivalent to (3) and (7) is equivalent to (8) by Theorem 2.1, and (2) is equivalent to (4) by Theorem 2.2.

It is obvious that (1) implies (5). By [HRW2, (2.2)], R^* is the y-adic completion of C, and so (5) is equivalent to (6). By [HRW2, (2.2)] and [HRW3, part (1) of (3.1)], (6) is equivalent to (7). □

REMARK 2.5. (i) With the notation of Corollary 2.4, if $\dim C = 2$, it follows that condition (7) of Corollary 2.4 holds. For C is normal since R^* is normal. Thus if $Q^* \in \operatorname{Spec} R^*$ with $y \notin Q^*$, then $C_{Q^* \cap C}$ is either a DVR or a field. The map $C \to R^*_{Q^*}$ factors as $C \to C_{Q^* \cap C} \to R^*_{Q^*}$. Since $R^*_{Q^*}$ is a torsionfree and hence flat $C_{Q^* \cap C}$-module, it follows that $C \to R^*_{Q^*}$ is flat. Therefore $C \hookrightarrow R^*[1/y]$ is flat and C is Noetherian.

(ii) There exist examples where $\dim C = 2$ and conditions (5)–(8) of Corollary 2.4 hold, but yet conditions (1)–(4) fail to hold [HRW1, Theorem 2.4].

QUESTION 2.6. With the notation of Corollary 2.4, suppose that for every prime ideal Q^* of R^* with $y \notin Q^*$, $\operatorname{ht}(Q^*) \geq \operatorname{ht}(Q^* \cap C)$. Does it follow that R^* is flat over C or, equivalently, that C is Noetherian?

Theorem 2.2 also extends to give equivalences for the *locally flat in height* k property of [HRW1]. This property is defined as follows:

DEFINITION. Let $\phi : S \longrightarrow T$ be an injective morphism of commutative rings and let $k \in \mathbf{N}$ be an integer with $1 \leq k \leq d = \dim(T)$, where d is an integer or $d = \infty$. Then ϕ is called *locally flat in height* k, abbreviated as LF_k, if, for every prime ideal Q of T with $\operatorname{ht}(Q) \leq k$, the induced morphism on the localizations $\phi_Q : S_{Q \cap S} \longrightarrow T_Q$ is faithfully flat.

THEOREM 2.7. *Let $(R, \mathbf{m})$ be a normal excellent local domain and $y \in \mathbf{m}$. Suppose that R^* is the y-adic completion of R, that $\widehat{R}$ is the $\mathbf{m}$-adic completion of R, and that $\tau_1, \ldots, \tau_s \in yR^*$ are algebraically independent over the fraction field of R. Then the following statements are equivalent:*

1. $S := R[\tau_1, \ldots, \tau_s]_{(m, \tau_1, \ldots, \tau_s)} \hookrightarrow \widehat{R}[1/y]$ *is* LF_k.
2. *If P is a prime ideal of S and $\widehat{Q}$ is a prime ideal of $\widehat{R}$ minimal over $P\widehat{R}$ and if, moreover, $y \notin \widehat{Q}$ and* $\operatorname{ht}(\widehat{Q}) \leq k$, *then* $\operatorname{ht}(\widehat{Q}) = \operatorname{ht}(P)$.
3. *If $\widehat{Q}$ is a prime ideal of $\widehat{R}$ with $y \notin \widehat{Q}$ and* $\operatorname{ht}(\widehat{Q}) \leq k$, *then* $\operatorname{ht}(\widehat{Q}) \geq \operatorname{ht}(\widehat{Q} \cap S)$.

Proof. (1) $\Longrightarrow$ (2): Let P be a prime ideal of S and let $\widehat{Q}$ be a prime ideal of $\widehat{R}$ that is minimal over $P\widehat{R}$ with $y \notin \widehat{Q}$ and $\operatorname{ht}(\widehat{Q}) \leq k$. The assumption of (1) implies flatness for the map:

$$\varphi_{\widehat{Q}} : S_{\widehat{Q} \cap S} \longrightarrow \widehat{R}_{\widehat{Q}},$$

and we continue as in Theorem 2.2.

(2) $\Longrightarrow$ (3): Let $\widehat{Q}$ be a prime ideal of $\widehat{R}$ with $y \notin \widehat{Q}$ and $\operatorname{ht}(\widehat{Q}) \leq k$. Set $Q := \widehat{Q} \cap S$ and let $\widehat{W}$ be a prime ideal of $\widehat{R}$ which is minimal over $Q\widehat{R}$,

and so that $\widehat{W} \subseteq \widehat{Q}$. Then $\mathrm{ht}(Q) = \mathrm{ht}(\widehat{W})$ by (2) since $y \notin \widehat{W}$ and therefore $\mathrm{ht}(\widehat{Q}) \geq \mathrm{ht}(Q)$.

(3) $\implies$ (1): Let $\widehat{Q}$ be a prime ideal of $\widehat{R}$ with $y \notin \widehat{Q}$ and $\mathrm{ht}(\widehat{Q}) \leq k$. Then for every prime ideal $\widehat{W}$ contained in $\widehat{Q}$, we also have $y \notin \widehat{W}$ and by (3) $\mathrm{ht}(\widehat{W}) \geq \mathrm{ht}(\widehat{W} \cap S)$. To complete the proof it suffices to show that $\varphi_{\widehat{Q}} : S_{\widehat{Q} \cap S} \longrightarrow \widehat{R}_{\widehat{Q}}$ is flat, and this is a consequence of Theorem 2.1. $\square$

3. Cohen-Macaulay formal fibers and Ogoma's example

Our purpose in this section is to show that if R is excellent, then every Noetherian example C obtained via the intermediate rings construction has Cohen-Macaulay formal fibers. We observe in Remark 3.4 that this implies the non-Noetherian property of a certain intersection domain B that has Ogoma's example as a homomorphic image.

The following is an analogue of [M2, Theorem 32.1(ii)]. The difference is that we are considering regular fibers rather than geometrically regular fibers.

Proposition 3.1. *Suppose R, S, and T are Noetherian commutative rings and suppose we have maps $R \to S$ and $S \to T$ and the composite map $R \to T$. Assume that*

(i) *$R \to T$ is flat with regular fibers,*
(ii) *$S \to T$ is faithfully flat.*

Then $R \to S$ is flat with regular fibers.

As an immediate consequence of Theorem 2.1 and Proposition 3.1, we have the following implication concerning Cohen-Macaulay formal fibers.

Corollary 3.2. *Every Noetherian local ring B containing an excellent local subring R and having the same completion as R has Cohen-Macaulay formal fibers.*

Remark 3.3 (Cohen-Macaulay formal fibers). Corollary 3.2 implies that every Noetherian local ring S that has as its completion $\widehat{S}$ the formal power series ring $k[[x_1, \dots, x_d]]$ and that contains the polynomial ring $k[x_1, \dots, x_d]$ has Cohen-Macaulay formal fibers. In connection with Cohen-Macaulay formal fibers, Luchezar Avramov pointed out to us that every homomorphic image of a regular local ring has formal fibers that are complete intersections and therefore Cohen-Macaulay [Gr, (3.6.4), page 118]. Also every homomorphic image of a Cohen-Macaulay local ring has formal fibers that are Cohen-Macaulay [M2, page 181]. It is interesting that while regular local rings need not have regular formal fibers, they must have Cohen-Macaulay formal fibers.

Remark 3.4 (Ogoma's example). Corollary 3.2 sheds light on Ogoma's famous example [O] of a pseudo-geometric local domain of dimension 3 whose generic formal fiber is not locally equidimensional.

Ogoma's construction begins with a countable field k of infinite but countable transcendence degree over the field $\mathbb{Q}$ of rational numbers. Let x, y, z, w be variables over k, and let $S = k[x, y, z, w]_{(x,y,z,w)}$ be the localized polynomial ring. By a clever enumeration of the prime elements in S, Ogoma constructs three power series $g, h, \ell \in \widehat{S} = k[[x, y, z, w]]$ that satisfy the following conditions:

(a) g, h, ℓ are algebraically independent over $k(x, y, z, w) = Q(S)$.
(b) g, h, ℓ are part of a regular system of parameters for $\widehat{S} = k[[x, y, z, w]]$.
(c) If $\widehat{P} = (g, h, \ell)\widehat{S}$, then $\widehat{P} \cap S = (0)$, i.e., $\widehat{P}$ is in the generic formal fiber of S.
(d) If $I = (gh, g\ell)\widehat{S}$ and $R = Q(S) \cap (\widehat{S}/I)$, then R is a pseudo-geometric local domain[3] with completion $\widehat{R} = \widehat{S}/I$.
(e) It is then obvious that the completion $\widehat{R} = \widehat{S}/I$ of R has a minimal prime $g\widehat{S}/I$ of dimension 3 and a minimal prime $(h, \ell)\widehat{S}/I$ of dimension 2. Thus R fails to be formally equidimensional. Therefore R is not universally catenary [M2, Theorem 31.7] and provides a counterexample to the catenary chain condition.

Since R is not universally catenary, R is not a homomorphic image of a regular local ring. There exists a quasilocal integral domain B that dominates S, has completion $\widehat{S} = k[[x, y, z, w]]$, and contains an ideal J such that $R = B/J$. If B were Noetherian, then B would be a regular local ring and $R = B/J$ would be universally catenary. Thus B is necessarily non-Noetherian.

Theorem 2.1 provides a different way to understand that the ring B is non-Noetherian. To see this, we need more details about the construction of B. The ring B is defined as a nested union of rings:

Let $\lambda_1 = gh$ and $\lambda_2 = g\ell$ and define

$$B = \bigcup_{n=1}^{\infty} S[\lambda_{1n}, \lambda_{2n}]_{(x,y,z,w,\lambda_{1n},\lambda_{2n})} \subseteq k[[x, y, z, w]],$$

where the λ_{in} are end pieces of the λ_i. The construction is done in such a way that the λ's are in every completion of S with respect to a nonzero principal ideal. By the construction of the power series g, h, ℓ, for every nonzero element $f \in S$ the ring B/fB is essentially of finite type over the field k. This implies that the maximal ideal of B is generated by x, y, z, w and that the completion with respect to the maximal ideal of B is the formal power series

[3] Ogoma [O, page 158] actually constructs R as a directed union of birational extensions of S. He proves that R is Noetherian and that $\widehat{R} = \widehat{S}/I$. It follows that $R = Q(S) \cap (\widehat{S}/I)$. Heitmann observes in [H] that R is already normal.

ring $\widehat{S} = k[[x, y, z, w]]$. Let $K = k(x, y, z, w)$. Then $K \otimes_S B$ is a localization of the polynomial ring in two variables $K[\lambda_1, \lambda_2]$. Recall that $I = (\lambda_1, \lambda_2)\widehat{S}$ and $\widehat{P} = (g, h, \ell)\widehat{S}$. Let $J = I \cap B$. Since $\widehat{P} \cap S = (0)$ we see that $J = \widehat{P} \cap B$ is a prime ideal with $J(K \otimes_S B)$ a localization of the prime ideal $(\lambda_1, \lambda_2)K[\lambda_1, \lambda_2]$. Thus $J(K \otimes_S \widehat{S}) = (\lambda_1, \lambda_2)(K \otimes_S \widehat{S})$ and $\widehat{P}$ is in the formal fiber of B/J. Since $(\widehat{S}/I)_{\widehat{P}}$ is not Cohen-Macaulay, Corollary 3.2 implies that B is not Noetherian.

There is another intermediate ring between S and its completion $k[[x, y, z, w]]$ that carries information about R, namely the intersection ring

$$C = k(x, y, z, w, \lambda_1, \lambda_2) \cap k[[x, y, z, w]].$$

It is shown in [HRS, Claim 4.3] that the maximal ideal of C is generated by x, y, z, w, and it is shown in [HRS, Claim 4.4] that C is non-Noetherian.

4. Related examples not having Cohen-Macaulay fibers

In this section we adapt the two forms of the basic construction technique described in the introduction to obtain three rings A, B and C which we describe in detail. The setting is somewhat similar to that of Ogoma's example. It is simpler in the sense that it is fairly easy to see that the ring C, which corresponds to R in Ogoma's example, is Noetherian. Also C is a birational extension of a polynomial ring in 3 variables over a field. On the other hand, this setting seems more complicated, since for A and B (which are the two obvious choices of intermediate rings) the ring B maps surjectively onto C, while A does not.

4.1. Setting and notation. Let k be a field, and let x, y, z be variables over k. Let $\tau_1, \tau_2 \in xk[[x]]$ be formal power series in x which are algebraically independent over $k(x)$. Suppose that

$$\tau_i = \sum_{n=1}^{\infty} a_{in} x^n, \qquad \text{with } a_{in} \in k, \qquad \text{for } i = 1, 2.$$

The intersection ring $V := k(x, \tau_1, \tau_2) \cap k[[x]]$ is a discrete valuation domain which is a nested union of localized polynomial rings in 3 variables over k,

$$V = \bigcup_{n=1}^{\infty} k[x, \tau_{1n}, \tau_{2n}]_{(x, \tau_{1n}, \tau_{2n})},$$

where τ_{1n}, τ_{2n} are the endpieces

$$\tau_{in} = \sum_{j=n}^{\infty} a_{ij} x^{j-n+1}, \qquad \text{for all } n \in \mathbb{N} \quad \text{and } i = 1, 2.$$

We now define a 3-dimensional regular local ring D such that:

(i) D is a localization of a nested union of polynomial rings in 5 variables.

(ii) D has maximal ideal $(x, y, z)D$ and completion $\widehat{R} = k[[x, y, z]]$.

(iii) D dominates the localized polynomial ring $R := k[x,y,z]_{(x,y,z)}$:

$$D := V[y,z]_{(x,y,z)} = U_{(x,y,z)D\cap U}, \text{ where } U := \bigcup_{n=1}^{\infty} k[x,y,z,\tau_{1n},\tau_{2n}]. \tag{4.1.1}$$

Moreover, $D = k(x,y,z,\tau_1,\tau_2) \cap \widehat{R}$ (see [HRW2, (3.5)]).

We consider the following elements of $\widehat{R}$:

$$s := y+\tau_1, \quad t := z+\tau_2, \quad \rho := s^2 = (y+\tau_1)^2, \quad \sigma := st = (y+\tau_1)(z+\tau_2).$$

The elements s and t are algebraically independent over $k(x,y,z)$ as are also the elements ρ and σ. The endpieces of ρ and σ are given as

$$\rho_n := \frac{1}{x^n}((y+\tau_1)^2 - (y+\sum_{j=1}^{n} a_{1j}x^j)^2)$$

$$\sigma_n := \frac{1}{x^n}((y+\tau_1)(z+\tau_2) - (y+\sum_{j=1}^{n} a_{1j}x^j)(z+\sum_{j=1}^{n} a_{2j}x^j).$$

Obviously, the ideal $I := (\rho,\sigma)\widehat{R}$ has height 1 and is the product of two prime ideals $I = P_1P_2$ where $P_1 := s\widehat{R}$ and $P_2 := (s,t)\widehat{R}$. Observe that P_1 and P_2 are the associated prime ideals of I, and that P_1 and P_2 are in the generic formal fiber of R.

We now define rings A and C as follows:

$$A := Q(R)(\rho,\sigma) \cap k[[x,y,z]], \qquad C := Q(R) \cap (k[[x,y,z]]/I). \tag{4.1.2}$$

In analogy with the rings D and U of (4.1.1), we have rings $B \subseteq D$ and $W \subseteq U$ defined as follows:

$$B := \bigcup_{n=1}^{\infty} R[\rho_n,\sigma_n]_{(x,y,z,\rho_n,\sigma_n)} = W_{(x,y,z)B\cap W}, \tag{4.1.3}$$

$$\text{where } W := \bigcup_{n=1}^{\infty} k[x,y,z,\rho_n,\sigma_n].$$

It is clear that A, B and C are quasilocal domains and $B \subseteq A$ with A birationally dominating B. Moreover, $(x,y,z)B$ is the maximal ideal of B.

We show in Propositions 4.2 and 4.3 that A and B are non-Noetherian and that $B \subsetneq A$. In Proposition 4.5, we show that C is a Noetherian local domain with completion $\widehat{C} = \widehat{R}/I$ such that C has a non-Cohen-Macaulay formal fiber.

PROPOSITION 4.2. *With the notation of* (4.1)*, the quasilocal integral domains $B \subseteq A$ both have completion $\widehat{R}$ with respect to the powers of their maximal ideals. Also:*

(1) *We have* $P_1 \cap B = P_2 \cap B$.
(2) B *is factorial.*
(3) $\operatorname{ht}(P_1 \cap B) > \operatorname{ht}(P_1) = 1$.
(4) B *fails to have Cohen-Macaulay formal fibers.*
(5) B *is non-Noetherian.*

Proof. It follows from [HRW2, Propositon 2.2] that $\widehat{R}$ is the completion of both A and B.

For (1), it suffices to show $P_1 \cap W = P_2 \cap W$. It is clear that $P_1 \cap W \subseteq P_2 \cap W$. Let $v \in P_2 \cap W$. Then there is an integer $n \in \mathbb{N}$ such that $x^n v \in k[x, y, z, \rho, \sigma]$. Thus

$$x^n v = \sum b_{ij} \rho^i \sigma^j, \qquad \text{where } b_{ij} \in k[x, y, z], \qquad \text{for all } i, j \in \mathbb{N}.$$

Since $P_2 \cap k[x, y, z] = (0)$ and since $\rho, \sigma \in P_2$ we have that $b_{00} = 0$. This implies that $v \in P_1$. Thus (1) holds.

For (2), since $B/xB = R/xR$, the ideal $xB = \mathbf{q}$ is a principal prime ideal in B. Since B is dominated by $\widehat{R}$, we have $\cap_{n=1}^{\infty} \mathbf{q}^n = (0)$. Hence $B_{\mathbf{q}}$ is a DVR. Moreover, by construction, B_x is a localization of $(B_0)_x$, where $B_0 := R[\rho, \sigma]_{(x,y,z,\rho,\sigma)}$, and $(B_0)_x$ is factorial. Therefore $B = B_x \cap B_{\mathbf{q}}$ is factorial [S, (6.3), page 21].

For (3), we note that B_x is a localization of the ring $(B_0)_x$ and the ideal $J = (\rho, \sigma)B_0$ is a prime ideal of height 2. Let $Q = P_1 \cap B$; then $x \notin Q$ and $B_Q = (B_0)_J$. Therefore $\operatorname{ht} Q = 2$. Since $P_1 = s\widehat{R}$ has height one, this proves (3).

Item (3) implies (5), since $\operatorname{ht}(P_1 \cap B) > \operatorname{ht} P_1$ implies that $B \to \widehat{R}$ fails to satisfy the going down property, so $\widehat{R}$ is not flat over B and B is not Noetherian.

For (4), as we saw above, $Qk[[x, y, z]]_{P_2} = (\rho, \sigma)_{P_2} = I_{P_2}$. Thus $\widehat{R}_{P_2}/I\widehat{R}_{P_2}$ is a formal fiber of B. Since $k[[x, y, z]]/I = k[[x, s, t]]/(s^2, st)$, we see that $P_2/I = (s, t)\widehat{R}/(s^2, st)\widehat{R}$ is an embedded associated prime of the ring $k[[x, y, z]]/I$. Hence $(k[[x, y, z]]/I)_{P_2}$ is not Cohen-Macaulay and the embedding $B \longrightarrow k[[x, y, z]]$ fails to have Cohen-Macaulay formal fibers. This also implies that B is non-Noetherian by Corollary 2.4. □

Proposition 4.3. *With the notation of* (4.1) *we have:*

(1) A *is a quasilocal Krull domain with maximal ideal* $(x, y, z)A$ *and completion* $\widehat{R}$.
(2) $P_1 \cap A \subsetneq P_2 \cap A$, *so* $B \subsetneq A$.
(3) A *is non-Noetherian.*

Proof. For item (1), it follows from [HRW2, Propositon 2.2] that $(x, y, z)A$ is the maximal ideal of A. It remains to observe that A is a Krull domain, and this is clear since by definition A is the intersection of a field with the Krull domain $\widehat{R}$.

For item (2), let $Q_i := P_i \cap A$, for $i = 1, 2$. Observe that

$$\sigma^2/\rho = (z + \tau_2)^2 = t^2 \in (Q_2 - B) - Q_1.$$

For item (3), assume A is Noetherian. Then A is a regular local ring and the embedding $A \longrightarrow \widehat{R} = k[[x, y, z]]$ is flat. In particular, A is factorial and the ideal $P := s\widehat{R} \cap A = P_1 \cap A$ is a prime ideal of height one in A. Thus P is principal. We have that $\rho = s^2 \in P$ and $\sigma^2 = \rho(\sigma^2/\rho)$, and therefore $st = \sigma \in P$. Let v be a generator of P. Then $v = sa$ where a is a unit in $D \subseteq k[[x, y, z]]$. We write

(4.3.1) $\quad v = sa = h(\rho, \sigma)/g(\rho, \sigma), \quad$ where $\quad h(\rho, \sigma), g(\rho, \sigma) \in k[x, y, z][\rho, \sigma].$

Now $a \in D = U_{(x,y,z)D\cap U}$, so $a = g_1/g_2$, where $g_1, g_2 \in k[x, y, z, \tau_{1n}, \tau_{2n}]$, for some $n \in \mathbb{N}$, and g_2 as a power series in $k[[x, y, z]]$ has nonzero constant term. There exists $m \in \mathbb{N}$ such that $x^m g_1 := f_1$ and $x^m g_2 := f_2$ are in the polynomial ring $k[x, y, z, \tau_1, \tau_2] = k[x, y, z][s, t]$. We regard $f_2(s, t)$ as a polynomial in s and t with coefficients in $k[x, y, z]$. We have $f_2 k[[x, y, z]] = x^m k[[x, y, z]] = x^m k[[x, s, t]]$. Therefore $f_2 \notin (s, t)k[[x, s, t]]$. It follows that the constant term of $f_2(s, t) \in k[x, y, z][s, t]$ is a nonzero element of $k[x, y, z]$. Since we have

$$a = \frac{x^m g_1}{x^m g_2} = \frac{f_1}{f_2}, \tag{4.3.2}$$

and a is a unit of D, the constant term of $f_1(s, t) \in k[x, y, z][s, t]$ is also nonzero. The equations (4.3.1) and (4.3.2) together yield

$$s f_1(s, t) h(s^2, st) = f_2(s, t) g(s^2, st). \tag{4.3.3}$$

The term of lowest total degree in s and t on the left hand side of (4.3.3) has odd degree, while the term of lowest total degree in s and t on the right hand side has even degree, a contradiction. Therefore the assumption that A is Noetherian leads to a contradiction. We conclude that A is not Noetherian. $\square$

Remark 4.4. (i) Although A is not Noetherian, the proof of Proposition 4.3 does not rule out the possibility that A is factorial. The proof does show that if A is factorial, then $\mathrm{ht}(P_1 \cap A) > \mathrm{ht}(P_1)$. It would be interesting to know whether the non-flat morphism $A \to \widehat{A} = \widehat{R}$ has the property that $\mathrm{ht}(\widehat{Q} \cap A) \le \mathrm{ht}(\widehat{Q})$, for each $\widehat{Q} \in \operatorname{Spec} \widehat{R}$. It would also be interesting to know the dimension of A.

(ii) We observe the close connection of the integral domains $A \subseteq D$ of (4.1). The extension of fraction fields $Q(A) \subseteq Q(D)$ has degree two and $A = Q(A) \cap D$, yet A is non-Noetherian while D is Noetherian.

Proposition 4.5. *With the notation of* (4.1), *C is a two-dimensional Noetherian local domain having completion $\widehat{R}/I$ and the generic formal fiber of C is not Cohen-Macaulay.*

Proof. By [HRW3, (2.4)], the completion of C is $\widehat{R}/I$. Hence if C is Noetherian, then $\dim(C) = \dim(\widehat{R}/I) = 2$.

To show that C is Noetherian, according to [HRW3, Theorem 3.2], it suffices to show that the canonical morphism

$$\begin{aligned} R = k[x,y,z]_{(x,y,z)} \xrightarrow{\varphi} (\widehat{R}/I)_x &= (k[[x,y,z]]/I)_x \\ &= (k[[x,s,t]]/(s^2,st)k[[x,s,t]])_x \end{aligned}$$

is flat. It suffices to show for every prime ideal $\widehat{Q}$ of $\widehat{R}$ with $x \notin \widehat{Q}$ that the morphism

$$\varphi_{\widehat{Q}} : R \longrightarrow \widehat{R}_{\widehat{Q}}/I\widehat{R}_{\widehat{Q}} = (\widehat{R}/I)_{\widehat{Q}}$$

is flat. We may assume $I = P_1P_2 \subseteq \widehat{Q}$.

If $\widehat{Q} = P_2 = (s,t)\widehat{R}$, then $\varphi_{\widehat{Q}}$ is flat since $P_2 \cap R = (0)$.

If $\widehat{Q} \neq P_2$, then $P_2\widehat{R}_{\widehat{Q}} = \widehat{R}_{\widehat{Q}}$, because $\operatorname{ht} P_2 = 2$. Hence $I\widehat{R}_{\widehat{Q}} = P_1\widehat{R}_{\widehat{Q}} = s\widehat{R}_{\widehat{Q}}$. Thus we need to show that

$$\varphi_{\widehat{Q}} : R \longrightarrow \widehat{R}_{\widehat{Q}}/s\widehat{R}_{\widehat{Q}} = (\widehat{R}/s\widehat{R})_{\widehat{Q}}$$

is flat. To see that $\varphi_{\widehat{Q}}$ is flat, we observe that, since $R \subseteq D_{\widehat{Q}\cap D} \subseteq \widehat{R}_{\widehat{Q}}$ and $s\widehat{R} \cap R = (0)$, the morphism $\varphi_{\widehat{Q}}$ factors through a homomorphic image of $D = V[y,z]_{(x,y,z)}$. That is, $\varphi_{\widehat{Q}}$ is the composition of the following maps

$$R \xrightarrow{\gamma} (D/sD)_{D\cap\widehat{Q}} \xrightarrow{\psi_{\widehat{Q}}} (\widehat{R}/s\widehat{R})_{\widehat{Q}}.$$

Since D is Noetherian, the map $\psi_{\widehat{Q}}$ is faithfully flat. Thus it remains to show that γ is flat. Since $x \notin \widehat{Q}$, the ring $(D/sD)_{D\cap\widehat{Q}}$ is a localization of $(D/sD)[1/x]$. Thus it is a localization of the polynomial ring

$$k[x,y,z,\tau_1,\tau_2]/sk[x,y,z,\tau_1,\tau_2] = k[x,y,z,s,t]/sk[x,y,z,s,t],$$

which is clearly flat over R. Thus C is Noetherian.

Now $P_2/I = \mathbf{p}$ is an embedded associated prime of (0) of $\widehat{C}$, so $\widehat{C}_{\mathbf{p}}$ is not Cohen-Macaulay. Since $\mathbf{p} \cap C = (0)$, the generic formal fiber of C is not Cohen-Macaulay. $\square$

Proposition 4.6. *The canonical morphism $B \longrightarrow \widehat{R}/I$ factors through C. We have $B/Q \cong C$, where $Q = I \cap B = s\widehat{R} \cap B$. On the other hand, the canonical morphism $A \longrightarrow \widehat{R}/I$ fails to factor through C.*

Proof. We have canonical maps $B \to \widehat{R}/I$ and $C \to \widehat{R}/I$. We define a map $\phi : B \twoheadrightarrow C$ such that the following diagram commutes:

$$
\begin{array}{ccc}
B & \longrightarrow & \widehat{R} \\
{\scriptstyle\phi}\big\downarrow & & \big\downarrow \\
C & \longrightarrow & \widehat{R}/I.
\end{array}
\tag{4.6.1}
$$

We write C as a nested union [HRW3, (3.2)]:

$$C = \bigcup_{n=1}^{\infty} R[\bar{\rho_n}, \bar{\sigma_n}]_{(x,y,z,\bar{\rho_n},\bar{\sigma_n})}.$$

Here $\bar{\rho_n}, \bar{\sigma_n}$ are the nth frontpieces of ρ and σ:

$$\bar{\rho_n} = \frac{1}{x^n}(y + \sum_{j=1}^{n} a_{1j}x^j)^2, \quad \bar{\sigma_n} = \frac{1}{x^n}(y + \sum_{j=1}^{n} a_{1j}x^j)(z + \sum_{j=1}^{n} a_{2j}x^j).$$

Then

$$B = \bigcup_{n=1}^{\infty} R[\rho_n, \sigma_n]_{(x,y,z,\rho_n,\sigma_n)}, \quad C = \bigcup_{n=1}^{\infty} R[\bar{\rho_n}, \bar{\sigma_n}]_{(x,y,z,\bar{\rho_n},\bar{\sigma_n})}.$$

It is clear that for each $n \in \mathbb{N}$ there is a surjection

$$R[\rho_n, \sigma_n]_{(x,y,z,\rho_n,\sigma_n)} \longrightarrow R[\bar{\rho_n}, \bar{\sigma_n}]_{(x,y,z,\bar{\rho_n},\bar{\sigma_n})},$$

which maps $\rho_n \mapsto \bar{\rho_n}$ and $\sigma_n \mapsto \bar{\sigma_n}$ and which extends to a surjective homomorphism ϕ on the directed unions such that diagram (4.6.1) commutes. This shows that $C \cong B/Q$ is a homomorphic image of B.

In order to see that the canonical map $\zeta : A \longrightarrow \widehat{R}/I$ fails to factor through C, we note that $I \cap D = (\rho, \sigma)D$ and so ζ factors through D:

$$A \xrightarrow{\gamma} D/(\rho,\sigma)D \xrightarrow{\delta} \widehat{R}/I$$

where δ is injective. The map γ sends the element $\sigma^2/\rho = t^2$ to the residue class of $t^2 = (z+\tau_2)^2$ in $D/(\rho,\sigma)D$. This element is algebraically independent over R, which shows that the ring $A/\ker(\gamma)$ is transcendental over R. Since C is a birational extension of R, the morphism $A \longrightarrow \widehat{R}/I$ fails to factor through C. □

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