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THE STRONG TEST IDEAL

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Let R be a Noetherian commutative ring containing a field. The *test ideal,* introduced by Hochster and Huneke in [HH1], has emerged as an important object associated to R. The test ideal can be defined as the largest ideal J of R such that $JI^* \subset I$ for all ideals I of R where I^* denotes the tight closure of I. Although it is not obvious that a ring R admits a non-zero test ideal, Hochster and Huneke showed nearly every ring of interest possesses a non-zero test ideal. (The definition of tight closure and basic features of the test ideal are recalled in Section 0.)

In [Hu], Craig Huneke introduced the related concept of a *strong test ideal:* an ideal J of R such that $JI^* = JI$ for all ideals I of the ring, where I^* denotes the tight closure of I. Huneke showed that non-trivial strong test ideals exist for a reasonably large class of rings, and put them to interesting use bounding the degrees of the equations of integral dependence for certain elements in the integral closure of an ideal. He also asked whether the blowup of the maximal strong test ideal might be a variety with only rational singularities, or some other good properties.

The purpose of this paper is to show that in many cases the test ideal is itself a strong test ideal. Since a sum of strong test ideals is a strong test ideal, there exists a unique maximal strong test ideal for R, and it is natural to call it *the* strong test ideal. From the definitions, we see that every strong test ideal is contained in the test ideal, but there is no *a priori* reason to expect them to coincide. Our paper shows that in many cases, the strong test ideal and the test ideal coincide. This provides numerous non-trivial examples in which the strong test ideal, proven to exist but not constructed explicitly by Huneke, can be explicitly described. This allows us to answer questions posed in [Hu].

An outline of the main results of the paper follows. Section 0 reviews basic definitions and properties of tight closure and test ideals.

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In Section 1, we give a simple proof that in any ring in which the test ideal is maximal, the test ideal coincides with the strong test ideal. This allows us to produce easy examples in which the blowup of the maximal strong test ideal has a non-rational singularity — indeed, examples in which its blowup does not appear to be in any way nicer than the original ring. This provides a negative answer to Question 3.3 of [Hu] and lays to rest any speculations about good properties of blowups of strong test ideals.

In Section 2, we study further situations in which test ideals are strong test ideals. We begin from the point of view that Huneke's existence proof for strong test ideals suggests that, if a ring has a filtration of ideals with certain nice properties, then the ideals appearing in this filtration are strong test ideals. Accordingly, if the test ideal appears in such a nice filtration, then it is a strong test ideal. We apply this method to the graded and canonical filtrations, and show that the test ideal is a strong test ideal under some additional conditions.

Based on our work here, we have come to believe that the test ideal may equal the strong test ideal in any excellent local reduced ring of prime characteristic. Such a result would provide an affirmative answer to a strengthened form of Huneke's Question 3.1, which asks whether the test ideal and strong test ideal might have the same radical in general. Our work also provides evidence for an affirmative answer to a strengthened form of Question 3.4 of [Hu], in which Huneke asks whether the strong test ideal is tightly closed. In fact, at least in the geometric situation explored here in Section 2, the strong test ideal is actually integrally closed (so, in particular, it is tight closed). The point is that in this setting the test ideal is integrally closed, as proved in [H2] and [S2].

We should mention that as this paper was under preparation, Adela Vraciu, working on her PhD thesis under the direction of Mel Hochster, found a proof that the test ideal is a strong test ideal in complete local rings of characteristic $p > 0$. Her proof is quite simple and it gives further confirmation of our conjecture that the test ideal is always a strong test ideal; see [V]. Although our method presented in Section 2 is considerably less elementary than this argument, we feel that it is of interest and significance from the viewpoint of, say, the study of F-rationality and F-regularity of Rees algebras.

0. Definitions and preliminaries

Tight closure, and hence the test ideal and strong test ideals, are primarily characteristic p notions defined via iteration of the Frobenius map. However, they can be defined also in characteristic zero, by reduction to characteristic p. First let us begin with the definition in characteristic $p > 0$. The reader is referred to [HH1], [HH2] for general properties of tight closure and the test ideal in characteristic p.

0.1. Definition ([HH1]). Let R be a Noetherian commutative ring of prime characteristic $p > 0$. For an ideal I of R and a power $q = p^e$ of p, we denote by $I^{[q]}$ the ideal generated by the qth powers of the elements of I. The *tight closure* I^* of I is defined to be the ideal consisting of all elements $z \in R$ for which there is an element c not in any minimal prime of R such that

$$cz^q \in I^{[q]}$$

for all large $q = p^e$. The *test ideal* of R, denoted by $\tau(R)$ or simply by τ when R is clear from the context, is the unique largest ideal τ such that

$$\tau I^* \subset I$$

for all ideals I. An element of τ which is not in any minimal prime of R is called a *test element*. A test element $c \in R$ is characterized by the property that for any ideal I and any $z \in R$, one has $z \in I^*$ if and only if $cz^q \in I^{[q]}$ for all powers $q = p^e$.

0.2. Definition ([Hu]). Let R be a Noetherian commutative ring of characteristic $p > 0$. An ideal J of R is said to be a *strong test ideal* if

$$JI^* \subset JI,$$

or equivalently, if $JI^* = JI$. The unique ideal J maximal with respect to this property is called *the* strong test ideal. It is clear that the strong test ideal is contained in the test ideal τ.

A ring is said to be *weakly F-regular* if all of its ideals are tightly closed. Clearly, a ring is weakly F-regular if and only if its test ideal is the unit ideal. In this case, the strong test ideal is obviously also the unit ideal.

The test ideal is much better understood in general than the strong test ideal. For example, we have the following useful characterization of the test ideal of a local ring of prime characteristic p.

0.3. Let (R, m) be a Noetherian local ring of characteristic $p > 0$ and let $E = E_R(R/m)$ be the injective hull of the residue field R/m as an R-module. An element $\xi \in E$ is in 0^*_E if and only if there exists $c \in R$ not in any minimal prime of R such that $c \otimes \xi = 0$ in $R^{(e)} \otimes_R E$ for all large $e \in \mathbb{N}$, where $R^{(e)}$ denotes the R-bimodule which is R with the usual R-module structure on the left, but where the right R-module structure is given by the e-times iterated Frobenius. It is proved in [AM] and [LS] that, if R is $\mathbb{Q}$-Gorenstein or an $\mathbb{N}$-graded ring over a field R_0, then 0^*_E is equal to the finitistic tight closure 0^{*fg}_E of zero in E (see [HH1] for the definition of 0^{*fg}_E). Hence by [HH1, 8.23], the test ideal τ of R is equal to $\mathrm{Ann}_R(0^*_E)$, the annihilator of 0^*_E in R, if R is $\mathbb{Q}$-Gorenstein or $\mathbb{N}$-graded over a field.

0.4. Reduction to prime characteristic. Given a ring containing a field of characteristic zero, one can define tight closure using the method of "reduction to characteristic p." This is easiest to describe when R is an algebra essentially of finite type over a field k of characteristic zero. One can choose a finitely generated $\mathbb{Z}$-algebra A contained in k and a subalgebra R_A of R essentially of finite type over A such that the natural map $R_A \otimes_A k \to R$ is an isomorphism. For a maximal ideal μ of A, we consider the base change $\operatorname{Spec} A/\mu \to \operatorname{Spec} A$ to get a prime characteristic ring $R_\mu = R_A \otimes_A A/\mu$. This is considered to be a "prime characteristic model" of the original ring R, and we refer to such R_μ for maximal ideals μ in a suitable dense open subset of $\operatorname{Spec} A$ as "reduction to characteristic $p \gg 0$" of R. Furthermore, given a resolution of singularities $f : X \to \operatorname{Spec} R$, we can reduce this entire setup to characteristic $p \gg 0$. (See [H1, 2], [S1, 2] for details.) We use the phrase "in characteristic $p \gg 0$" when we speak of such a setup reduced from characteristic zero to characteristic $p \gg 0$.

0.5. Characteristic zero. If R has characteristic zero, one needs to reduce to prime characteristic in order to define the tight closure of I^* of an ideal $I \subset R$, and there is more than one way to proceed with this reduction; see [Ho], [HH4]. However, we must be cautious speaking about the test ideal (resp. the strong test ideal) in characteristic zero. It still makes sense to consider the largest ideal τ' (resp. J') such that $\tau' I^* \subset I$ (resp. $J' I^* \subset J' I$) for all ideals I of R, but this may not be quite what we want to study. Rather, we want to consider the ideal τ (resp. J) of R which reduces to the test ideal (resp. the strong test ideal) in "almost all prime characteristic reductions" of R.

Let R be a ring essentially of finite type over a field of characteristic zero. An ideal τ (resp. J) of R will be called a *universal test ideal* (resp. a *universal strong test ideal*) if choosing A and R_A as in 0.4 such that R_A contains the generators for τ (resp. J), then setting $\tau_A = \tau \cap R_A$ (resp. $J_A = J \cap R_A$), the ideal $\tau_A \bmod \mu$ (resp. $J_A \bmod \mu$) is the test ideal (resp. a strong test ideal) for the ring $R_A \bmod \mu$ for all μ in a dense open subset of $\operatorname{Spec} A$. If R admits a universal test ideal, it is unique and contains the unique largest universal strong test ideal of R. In this case, we will simply call them the test ideal and the strong test ideal of R, respectively.

In characteristic zero, as in the prime characteristic case, the test ideal is better understood than the strong test ideal.

0.6. THEOREM ([H2], [S2]). *Let R be a normal $\mathbb{Q}$-Gorenstein ring essentially of finite type over a field of characteristic zero. Then R admits a (universal) test ideal, denoted $\tau(R)$, and it is equal to the multiplier ideal. Namely, if $f : X \to \operatorname{Spec} R$ is a resolution of singularities with simple normal crossing exceptional divisor and if we write $K_X = f^* K_R + \Delta$ for an*

f-exceptional $\mathbb{Q}$-divisor Δ, then

$$\tau(R) = H^0(X, \mathcal{O}_X(\lceil\Delta\rceil)).$$

Although the above theorem is stated in characteristic zero, it is essentially a statement in prime characteristic. It says that, if $f : X \to \operatorname{Spec} R$ is a resolution of singularities in characteristic $p \gg 0$ reduced from the original setting in characteristic zero as in 0.4, then one has $\tau(R) = H^0(X, \mathcal{O}_X(\lceil\Delta\rceil))$.

1. Counterexamples to Huneke's question

In [Hu], Huneke introduced the notion of strong test ideals and asked whether the blowup with respect to the strong test ideal might have only rational singularities. In this section we give simple examples showing that the answer to this question is negative. The main point of our argument is the following simple characterization of the strong test ideal in a ring whose test ideal is maximal.

1.1. Theorem. *Let (R, m) be a local ring containing a field, and assume that m is the test ideal of R. Then m is the strong test ideal of R.*

Many rings have the property that the test ideal is maximal. For example, any F-pure local ring with an isolated non-strongly-F-regular point has this property.

Proof. If m is not the strong test ideal of R, then there exists an ideal I of R such that mI^* is not contained in mI. Among all such counterexample ideals I, choose one which is generated by the minimal possible number of elements.

Since mI^* is contained in I but not in mI, there must be a $c \in m$ and a $z \in I^*$ such that cz is in I but not in mI. This means that cz is a minimal generator for I. Let $cz, x_1, x_2, \ldots, x_r$ be a minimal generating set for I.

Since $z \in I^*$ and c is a test element, we have equations

$$cz^q = a_q(cz)^q + b_{1q}x_1^q + \cdots + b_{rq}x_r^q$$

with $a_q, b_{1q}, \ldots, b_{rq} \in R$ for all $q = p^e$, whence

$$cz^q(1 - a_q c^{q-1}) \in (x_1^q, \ldots, x_r^q)$$

for all $q = p^e$. Since $1 - a_q c^{q-1}$ is a unit in R, we see that

$$z \in (x_1, \ldots, x_r)^*.$$

But now $I^* = (x_1, \ldots, x_r)^*$, because $I = (cz, x_1, \ldots, x_r) \subset (z, x_1, \ldots, x_r)$, so that $I^* \subset (z, x_1, \ldots, x_r)^* = (x_1, \ldots, x_r)^*$, while the reverse inclusion is trivial. Because $(x_1, \ldots, x_r)$ is generated by strictly fewer elements than I, we have that $m(x_1, \ldots, x_r)^* \subset m(x_1, \ldots, x_r)$.

We conclude that $mI^* = m(x_1, \ldots, x_r)^* \subset m(x_1, \ldots, x_r) \subset mI$, contrary to our choice of I. This contradiction completes the proof. $\square$

We now put to rest Huneke's speculations about the nice properties the blow up of the strong test ideal might have.

1.2. Example. Let R be the localization at the maximal ideal (x, y, z) of the two dimensional normal graded ring $k[x, y, z]/(x^2 + y^3 + z^6)$, where k is a field of characteristic $p \geq 5$. The test ideal of this ring is the unique homogeneous maximal ideal, by Theorems 4.3 and 5.4 of [HS] (see also [H1, H2] for more similar results that hold in any dimension and $p \gg 0$, but without the specific bound on p). However, the blowup of the maximal ideal produces a normal scheme with an isolated non-rational singularity. Explicitly, the unique singular point corresponds to the maximal ideal (x', y', z) in the affine scheme $\operatorname{Spec} k[x', y', z]/((x')^2 + (y')^3 z + z^4)$, where $x' = \frac{x}{z}$ and $y' = \frac{y}{z}$. Because this can be graded with degree $x' = 2$ and degree $y' =$ degree $z = 1$, we see that the a-invariant is again zero, so we have a non-rational singularity. The test ideal, and hence the strong test ideal, at this point (at least in large characteristic) is the maximal ideal (x', y', z). Blowing this up again, we achieve a scheme that is not even regular in codimension one; for example, the affine patch where z does not vanish is isomorphic to $\operatorname{Spec} k[x'', y'', z]/((x'')^2 + (y'')^3 z^2 + z^2)$, which is singular along the curve where x'' and z are zero. This shows that the blowup of the strong test ideal need not even been normal in general, nor is an iterated blow-up of strong test ideals likely to have good properties.

As far as we know, it is possible that the test ideal is always the strong test ideal. The next result gives an elementary argument that suggests that for graded rings, the test ideal always coincides with the strong test ideal. We will later strengthen this result; see Theorem 2.6.

1.3. Theorem. *Let R be an $\mathbb{N}$-graded ring. Assume that the test ideal τ is generated by all elements of degrees exceeding a certain fixed number a. Then the test ideal is a strong test ideal for homogeneous ideals. That is, $\tau I^* = \tau I$ for all homogeneous ideals I, where τ is the test ideal of R.*

For an example of a ring satisfying the above condition on τ, let R be any $\mathbb{Q}$-Gorenstein $\mathbb{N}$-graded ring with an isolated non-F-regular point, in characteristic $p \gg 0$ (including characteristic zero); see Remark 2.7.

Proof. Suppose that the claim is false, so there exists a homogeneous I such that τI^* is not contained in τI. Choose such a counterexample ideal I generated by the fewest possible homogeneous elements $y_1, y_2, \ldots, y_r$. Let δ be the unique integer such that some minimal homogeneous generator of I has degree δ, but no minimal generator of I has degree exceeding δ. Since I^* and τ are homogeneous, it suffices to show that for every homogeneous $z \in I^*$ and homogeneous c in τ, the product cz is in τI. But since $cz \in I$, we can write

$$cz = b_1 y_1 + \cdots + b_r y_r$$

and since $\tau = R_{>a}$ we only need to check that the coefficients b_i have degrees greater than a. For this is sufficient that the degree of z is at least the degree of each y_i.

Assume, on the contrary, that the degree of z is strictly less than the degree of some y_i. In particular, the degree of z is less than δ. Let $I = I_1 + I_2$ where I_1 is homogeneous and generated by elements of degree less than δ, and I_2 is generated by homogeneous elements of degree exactly δ. But since the degree of z is less than δ, the equations

$$cz^q \in I_1^{[q]} + I_2^{[q]}$$

for large q can hold only if the coefficients on the generators of $I_2^{[q]}$ are zero, since the degrees of these generators are much larger than the degree of cz^q. This implies that $z \in I_1^*$. By our assumption on the minimality of our counterexample, we see that $\tau z \subset \tau I_1^* \subset \tau I_1 = \tau I$. Thus $\tau I^* \subset \tau I$ in either case, and the proof is complete. □

Theorem 1.3 raises the following natural question: If $\tau I^* \subset \tau I$ for all homogeneous ideals I in a graded ring, is it true that $\tau I^* \subset \tau I$ for all ideals I of R? It seems quite plausible that there is an elementary proof that the answer is yes, especially because τ itself is homogeneous. If so, then Theorem 1.3 would imply that the test ideal in a graded ring satisfying the hypothesis above coincides with the strong test ideal. In any case, in Theorem 2.6, we establish that the test ideal is the strong test ideal in a graded ring satisfying the hypothesis of Theorem 1.2, although our argument is somewhat less elementary.

In every example in which the strong test ideal is known, it is equal to the test ideal. Thus we are led to the problem: Is the test ideal always equal to the strong test ideal?

2. The identification of the test and strong test ideals

In this section, all rings of characteristic $p > 0$ are assumed to be F-finite (which, in particular, implies excellence [Ku]). We will also change notation slightly, letting (A, m) denote the local ring we study, rather than (R, m).

The main results here are Theorem 2.6 and Theorems 2.12, each proving that the test ideal is the strong test ideal in situations where we understand the test ideal well enough.

Our strategy to attack the problem is based on the theory of Rees algebras and "filtered blowings-up." A filtration on a Noetherian domain A is a decreasing sequence of ideals $\{J_n\}_{n\geq 0}$ of A satisfying the conditions: (i) $J_0 = A$ and $J_n \neq 0$ for every $n \geq 0$; and (ii) $J_m \cdot J_n \subset J_{m+n}$ for every $m, n \geq 0$. A filtration $\{J_n\}_{n\geq 0}$ is called a Noetherian (resp. normal) filtration if the Rees algebra $\mathcal{R} = \bigoplus_{n\geq 0} J_n T^n \subset A[T]$ is a finitely generated (resp. normal) A-algebra, where T is an indeterminate of degree 1.

The following key lemma is observed by Huneke in [Hu].

2.1. Lemma (cf. [Hu]). *Let A be a Noetherian domain of characteristic $p > 0$ and let $\mathcal{R} = \bigoplus_{n\geq 0} J_nT^n$ be the Rees algebra of a Noetherian filtration $\{J_n\}$ of ideals on A. Assume that for an integer $N \geq 0$, the degree N piece $\mathcal{R}_N = J_NT^N$ of $\mathcal{R}$ is contained in the test ideal $\tau(\mathcal{R})$. Then $J_N \subset A$ is a strong test ideal of A.*

Proof. Let $I \subset A$ be any ideal of A. Then $I^* \subset (I\mathcal{R})^*$, so that

$$J_NI^* \cdot T^N = I^*\mathcal{R}_N \subset I\mathcal{R}$$

by our hypothesis. Looking at the degree N part of this containment, we see that $J_NI^* \cdot T^N \subset (I\mathcal{R})_N = IJ_N \cdot T^N$. Thus $J_NI^* \subset J_NI$ for all $I \subset A$, whence the result. □

2.2. Setup. Let (A, m) be a d-dimensional Noetherian local or $\mathbb{N}$-graded ring over a field of characteristic $p > 0$ and $\{J_n\}$ be a Noetherian normal filtration of ideals on A. Let $\mathcal{R} = \bigoplus_{n\geq 0} J_nT^n$ be the Rees algebra associated with $\{J_n\}$ and $\mathfrak{M} = m\mathcal{R} + \mathcal{R}_+$ be the homogeneous maximal ideal of $\mathcal{R}$. We set $X = \operatorname{Proj}\mathcal{R}$ and $\mathcal{O}_X(n) := \mathcal{R}(n)^\sim$ on X for $n \in \mathbb{Z}$. The birational projective morphism $\psi : X \to \operatorname{Spec} A$ is called the *filtered blowup* of $\{J_n\}_{n\geq 0}$. We have the following fundamental commutative diagram [TW]:

$$\begin{array}{ccccc}
 & Y' = \operatorname{Spec}_X\left(\bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(n)T^n\right) & \xrightarrow{\sim} & \operatorname{Spec}\mathcal{R} \setminus V(\mathcal{R}_+) \\
 & \downarrow & & \downarrow \\
S \hookrightarrow & Y = \operatorname{Spec}_X\left(\bigoplus_{n\geq 0} \mathcal{O}_X(n)T^n\right) & \xrightarrow{\varphi} & \operatorname{Spec}\mathcal{R} \\
 & \downarrow \varpi & & \downarrow \\
 & E \hookrightarrow X = \operatorname{Proj}\mathcal{R} & \xrightarrow{\psi} & \operatorname{Spec} A
\end{array}$$

Here E denotes the closed fiber of ψ, and S is the zero-section of ϖ, i.e., the Weil divisor on Y defined by the ideal sheaf $\bigoplus_{n>0} \mathcal{O}_X(n)T^n$ of $\mathcal{O}_Y = \bigoplus_{n\geq 0} \mathcal{O}_X(n)T^n$.

By Demazure [D], there exists a ψ-ample $\mathbb{Q}$-Cartier divisor Δ on X such that $\mathcal{O}_X(n) = \mathcal{O}_X(n\Delta) := \mathcal{O}_X(\lfloor n\Delta \rfloor)$ for every $n \in \mathbb{Z}$. For simplicity, we assume that Δ is an integral Weil divisor. (When Δ has non-integer coefficients, we have to make a modification by the "fractional part" Δ' in Lemma 2.3 below, as in Lemma 2.8.)

In the following, the canonical sheaf (resp. canonical divisor) of a normal variety V is denoted by ω_V (resp. K_V), and the divisorial sheaf corresponding to the divisor nK_V ($n \in \mathbb{Z}$) is denoted by $\omega_V^{(n)}$ or $\mathcal{O}_V(nK_V)$.

To apply our ideas, we require a few easy lemmas.

2.3. LEMMA. *Let the situation be as in* 2.2. *Then we have the following isomorphism of graded $\mathcal{R}$-modules for each $i \in \mathbb{Z}$:*

$$H^{d+1}_{\mathfrak{M}}(\omega^{(i)}_{\mathcal{R}}) \cong \bigoplus_{n<i} H^d_E(\omega^{(i)}_X(n))T^n.$$

Proof. Since $K_Y = \pi^* K_X - S$ in the divisor class group of Y (cf. [TW, Proposition 3.11]), one has $\omega^{(i)}_Y \cong \bigoplus_{n\geq i} \omega^{(i)}_X(n)T^n$. This implies that

$$\omega^{(i)}_{\mathcal{R}} \cong H^0(Y, \omega^{(i)}_Y) \cong \bigoplus_{n\geq i} H^0(X, \omega^{(i)}_X(n))T^n,$$

since $\omega^{(i)}_{\mathcal{R}}$ and $\omega^{(i)}_Y$ are reflexive and $\varphi : Y \to \operatorname{Spec}\mathcal{R}$ is isomorphic in codimension one (cf. [TW, Remark 1.3 (iii)]). Hence by the duality for graded $\mathcal{R}$-modules [TW, Proposition A.3.13],

$$\begin{aligned} H^{d+1}_{\mathfrak{M}}(\omega^{(i)}_{\mathcal{R}}) &\cong \operatorname{Hom}_A(\operatorname{Hom}_{\mathcal{R}}(\omega^{(i)}_{\mathcal{R}}, \omega_{\mathcal{R}}), E_A(A/m)) \\ &\cong \operatorname{Hom}_A(\omega^{(1-i)}_{\mathcal{R}}, E_A(A/m)) \\ &\cong \bigoplus_{n\geq 1-i} \operatorname{Hom}_A(H^0(X, \omega^{(1-i)}_X(n))T^n, E_A(A/m)), \end{aligned}$$

where $E_A(A/m)$ is the injective envelope of the A-module A/m, sitting in degree 0. The right-hand side of this equality is identified with

$$\bigoplus_{n\geq 1-i} H^d_E(\omega^{(i)}_X(-n))T^{-n} = \bigoplus_{n<i} H^d_E(\omega^{(i)}_X(n))T^n$$

by the Grothendieck duality, as required. □

2.4. LEMMA. *Let $c \in R$ be a nonzero element of an integral domain R of characteristic $p > 0$ satisfying the following property: There exists a test element $d \in R$, a power $q = p^e$ and $\theta \in \operatorname{Hom}_R(R^{1/q}, R)$ such that $\theta(d^{1/q}) = c$. Then c is a test element of R.*

Proof. Let $I \subset R$ be an ideal and let $x \in I^*$. Then one has $dx^q \in I^{[q]}$, so that

$$cx = \theta((dx^q)^{1/q}) \in \theta((I^{[q]})^{1/q}) = \theta(I \cdot R^{1/q}) \subset I.$$

Hence $cI^* \subset I$ for all I, which means that c is a test element. □

We can now prove a key proposition.

2.5. PROPOSITION. *Assume notation as in* 2.2. *Then there exists an element $c \in A$ that is a test element for $\mathcal{R}$. For such a $c \in A$ and a given integer $N \geq 0$, assume further that the following condition holds: There exists an $e_0 \in \mathbb{N}$ such that the map*

$$(c, cF, \ldots, cF^{e_0}) : H^d_E(\omega_X(-N)) \longrightarrow \bigoplus_{e=0}^{e_0} H^d_E(F^e_*(\omega^{(q)}_X(-qN)))$$

is injective, where $cF^e : H^d_E(\omega_X(-N)) \to H^d_E(F^e_*(\omega_X^{(q)}(-qN)))$ *is the induced e-times iterated Frobenius map followed by multiplication by c. Then* J_N *is a strong test ideal of A.*

Proof. We can choose a nonzero element $c \in J_1$ such that A_c is regular. Then $\mathcal{R}_c = A_c[T]$ is also regular, so that some power of c, which we relabel as c, is a test element of $\mathcal{R}$ by [HH3].

By Lemmas 2.1 and 2.4, we see that J_N is a strong test ideal if $\mathcal{R}_N = J_N T^N$ lies in the image of the map

$$\bigoplus_{e=0}^{e_0} \operatorname{Hom}_{\mathcal{R}}(\mathcal{R}^{1/q}, \mathcal{R}) \xrightarrow{\delta} \bigoplus_{e=0}^{e_0} \operatorname{Hom}_{\mathcal{R}}(\mathcal{R}^{1/q}, \mathcal{R}) \to \operatorname{Hom}_{\mathcal{R}}(\mathcal{R}, \mathcal{R}) = \mathcal{R},$$

where δ denotes the diagonal map $\operatorname{diag}(c, c^{1/p}, \ldots, c^{1/p^{e_0}})$. Here we note that $\mathcal{R}^{1/q}$ and so $\operatorname{Hom}_{\mathcal{R}}(\mathcal{R}^{1/q}, \mathcal{R})$ have natural $\frac{1}{q}\mathbb{Z}$-grading, and the above map is viewed as a graded homomorphism via the $p^{-e_0}\mathbb{Z}$-grading, since so is $\mathcal{R} \hookrightarrow \mathcal{R}^{1/q} \xrightarrow{c^{1/q}} \mathcal{R}^{1/q}$ for $q = 1, p, \ldots, p^{e_0}$. Therefore it is sufficient to prove that the above map is surjective in degree N.

Now the Matlis dual of $\operatorname{Hom}_{\mathcal{R}}(\mathcal{R}^{1/q}, \mathcal{R}) \cong \operatorname{Hom}_{\mathcal{R}}((\omega_{\mathcal{R}}^{(q)})^{1/q}, \omega_{\mathcal{R}})$ in the category of graded $\mathcal{R}$-modules is $H^{d+1}_{\mathfrak{M}}((\omega_{\mathcal{R}}^{(q)})^{1/q})$. Hence it suffices to show that the map

$$H^{d+1}_{\mathfrak{M}}(\omega_{\mathcal{R}}) \to \bigoplus_{e=0}^{e_0} H^{d+1}_{\mathfrak{M}}((\omega_{\mathcal{R}}^{(q)})^{1/q}) \xrightarrow{\delta} \bigoplus_{e=0}^{e_0} H^{d+1}_{\mathfrak{M}}((\omega_{\mathcal{R}}^{(q)})^{1/q})$$

is injective in degree $-N$. This map is identified with the map

$$(c, cF, \ldots, cF^{e_0}) : H^{d+1}_{\mathfrak{M}}(\omega_{\mathcal{R}}) \to \bigoplus_{e=0}^{e_0} H^{d+1}_{\mathfrak{M}}(\omega_{\mathcal{R}}^{(q)})$$

by [W2]. Hence the required injectivity follows from our assumption, because we have the graded decomposition

$$H^{d+1}_{\mathfrak{M}}((\omega_{\mathcal{R}}^{(q)})^{1/q}) \cong \bigoplus_{n<q} H^d_E(F^e_*(\omega_X^{(q)}(n)))T^{n/q}$$

by Lemma 2.3, and its degree $-N$ component is $H^d_E(F^e_*(\omega_X^{(q)}(-qN)))$. The proposition is proved. □

2.6. Theorem. *Let* $A = \bigoplus_{k \geq 0} A_k$ *be a Noetherian normal graded ring over a field* A_0*. Assume that the test ideal* τ *of A is described as the ideal of all elements of A whose degrees are more than a fixed integer,* $A_{\geq N}$*. Then* $\tau = A_{\geq N}$ *is the strong test ideal of A.*

2.7. Remark. If A is a normal graded ring with an isolated non-F-regular point and if A is $\mathbb{Q}$-Gorenstein, i.e., $\omega_A^{(r)} \cong A(b)$ for some integers $r > 0$ and b, then in characteristic $p \gg 0$ (including characteristic zero), one has $\tau = R_{>b/r}$ ([H2], see also [HS] for the Gorenstein case). So in this case, the assumption of the theorem is satisfied.

Proof. We make use of the graded filtration as the filtration $\{J_n\}_{n\geq 0}$ in Proposition 2.5, namely, $J_n = A_{\geq n} = \bigoplus_{k\geq n} A_k$. In this case, the divisor Δ in Setup 2.2 is $-E$ (integral!), and we can choose a *homogeneous* test element c of A which is also a test element of $\mathcal{R} = \bigoplus_{n\geq 0} J_n T^n$.

The graded ring A is represented by $A = R(E, D) = \bigoplus_{k\geq 0} H(E, \mathcal{O}_E(kD))t^k$ for an ample $\mathbb{Q}$-Cartier divisor D on $E = \operatorname{Proj} A$ and a homogeneous element t of degree 1 in the fraction field of A (see [D]). Then $X = \operatorname{Proj}\mathcal{R} \cong \operatorname{Spec}_E(\bigoplus_{k\geq 0} \mathcal{O}_E(kD)t^k)$, and we have not only the fundamental diagram for the Rees algebra $\mathcal{R}$ in Setup 2.2, but also that for the graded ring A:

$$\begin{array}{ccccc} & & X - E & \xrightarrow{\sim} & \operatorname{Spec} A \setminus \{m\} \\ & & \downarrow & & \downarrow \\ E & \hookrightarrow & X & \xrightarrow{\psi} & \operatorname{Spec} A \\ & =\searrow & \downarrow \pi & & \\ & & E = \operatorname{Proj} A & & \end{array}$$

We define the "fractional part" D' of the $\mathbb{Q}$-divisor D as follows [W1]: Let D be expressed as $D = \sum (e_i/d_i)D_i$, where each D_i is a prime divisor on E and d_i and e_i are coprime integers with $d_i > 0$. Then set $D' = \sum((d_i - 1)/d_i)D_i$.

We will describe the map

$$cF^e : H^d_E(\omega_X(-N)) \longrightarrow H^d_E(F^e_*(\omega_X^{(q)}(-qN)))$$

in Proposition 2.5 in terms of the Demazure representation.

2.8. Lemma. *With the notation as above, one has the following isomorphisms of graded A-modules for each integers i and n:*

(i) $H^d_m(\omega_A^{(i)}) \cong \bigoplus_{k\in\mathbb{Z}} H^{d-1}(E, \mathcal{O}_E(i(K_E + D') + kD))t^k$.

(ii) $H^d_E(\omega_X^{(i)}(n)) \cong \bigoplus_{k<i+n} H^{d-1}(E, \mathcal{O}_E(i(K_E + D') + kD))t^k$.

In particular, $H^d_E(\omega_X^{(i)}(n))$ is the quotient A-module of $H^d_m(\omega_A^{(i)})$ consisting of the graded part of degree $< i + n$.

Proof. (i) is shown in [W2]. Since $\psi : X \to \operatorname{Spec} A$ is isomorphic outside of E, we have an exact sequence

$$H^{d-1}(X, \omega_X^{(i)}(n)) \xrightarrow{\rho} H^d_m(\omega_A^{(i)}) \to H^d_E(\omega_X^{(i)}(n)) \to 0.$$

Now, $K_X = \pi^*(K_E + D') - E$ and $\mathcal{O}_X(i) = \mathcal{O}_X(-iE) = \bigoplus_{k \geq i} \mathcal{O}_E(kD)t^k$ by [W1], so that

$$\omega_X^{(i)}(n) = \mathcal{O}_X(i\pi^*(K_E + D') - (i+n)E) = \bigoplus_{k \geq i+n} \mathcal{O}_E(i(K_E + D') + kD)t^k.$$

Hence the map ρ in the above exact sequence is isomorphic in degree $\geq i + n$. This proves (ii). □

We continue the proof of Theorem 2.6. By Proposition 2.5 and Lemma 2.8, it is sufficient to show that there exists an $e_0 \in \mathbb{N}$ such that the map

$$(c, cF, \ldots, cF^{e_0}) : H_m^d(\omega_A) \longrightarrow \bigoplus_{e=0}^{e_0} H_m^d(\omega_A^{(q)})$$

is injective in degree $\leq -N$. However, by [W2], this map is identified with the map

$$\Phi(e_0) = (c, cF, \ldots, cF^{e_0}) : E(A/m) \longrightarrow \bigoplus_{e=0}^{e_0} A^{(e)} \otimes_A E(A/m)$$

induced for the injective envelope $E(A/m)$ of A/m in the category of graded A-modules. Since $E(A/m)$ is an Artinian A-module, we can choose an e_0 from which the kernel of $\Phi(e_0)$ becomes stationary: $\mathrm{Ker}(\Phi(e_0)) = \mathrm{Ker}(\Phi(e_0 + 1)) = \cdots$.

Now by [LS], the test ideal $\tau(A) = A_{\geq N}$ is equal to $\mathrm{Ann}_A(0^*_{E(A/m)})$, so that $0^*_{E(A/m)} = E(A/m)_{>-N}$ by the duality. Therefore, if ξ is a nonzero element of $E(A/m)$ of degree $\leq -N$, then $\xi \notin 0^*_{E(A/m)}$, whence $cF^e(\xi) \neq 0$ in $A^{(e)} \otimes_A E(A/m)$ for some $q = p^e$. This means that the stationary kernel $\mathrm{Ker}(\Phi(e_0))$ has no graded component of degree $\leq -N$. This completes the proof of the theorem. □

We now turn to the local case. A normal variety V is said to be $\mathbb{Q}$-Gorenstein (resp. quasi-Gorenstein) if $\omega_V^{(r)}$ is invertible for some integer $r > 0$ (resp. ω_V is invertible). The following proposition allows us to reduce the problem for $\mathbb{Q}$-Gorenstein rings to the quasi-Gorenstein case.

2.9. PROPOSITION. *Let (A, m) be a normal $\mathbb{Q}$-Gorenstein local ring such that the order r of its canonical class is not divisible by the characteristic p. Given an isomorphism $\omega_A^{(r)} \cong A$, we consider the canonical covering $B = \bigoplus_{i=0}^{r-1} \omega_A^{(i)}$ with the corresponding ring structure. Suppose that the test ideal $\tau(B)$ of B is the strong test ideal of B. Then the test ideal $\tau(A)$ of A is also the strong test ideal of A.*

Proof. Since r is not divisible by p, $\tau(B)$ is a homogeneous ideal under the $\mathbb{Z}/r\mathbb{Z}$-grading of B, and its degree 0 part $\tau(B)_0$ is $\tau(B) \cap A = \tau(A)$ (see, e.g., [H2], [S2]).

Now let $I \subset A$ be any ideal of A. Then

$$\tau(A)I^* \subset \tau(B) \cdot (IB)^* \subset \tau(B) \cdot IB = \tau(B)I,$$

since $\tau(B)$ is a strong test ideal. It is easy to see that the degree 0 part of $\tau(B)I$ is $\tau(B)_0 I = \tau(A)I$, whence $\tau(A)I^* \subset \tau(A)I$. $\square$

2.10. The canonical model. Let (A, m) be a normal local ring of $\dim A = d \geq 2$, essentially of finite type over a field. A *canonical model* $\psi : X \to \operatorname{Spec} A$ is a birational projective morphism from a normal variety X satisfying the following conditions (cf. [KMM]):

(i) X has only canonical singularities, i.e., X is $\mathbb{Q}$-Gorenstein and, for a resolution of singularities $g : \tilde{X} \to X$, one has $g_*\mathcal{O}_{\tilde{X}}(nK_{\tilde{X}}) = \mathcal{O}_X(nK_X)$ for all $n \geq 0$.

(ii) K_X is ψ-ample.

Then the canonical ring

$$\mathcal{R} := \bigoplus_{n\geq 0} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(nK_{\tilde{X}}))T^n = \bigoplus_{n\geq 0} H^0(X, \mathcal{O}_X(nK_X))T^n$$

is a finitely generated A-algebra with $X = \operatorname{Proj} \mathcal{R}$, and $\mathcal{O}_X(n) := \mathcal{R}(n)^\sim = \mathcal{O}_X(nK_X)$ for $n \in \mathbb{Z}$.

2.11. Remark. In characteristic zero, the existence of the canonical model is equivalent to the finite generation of the A-algebra $\bigoplus_{n\geq 0} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(nK_{\tilde{X}}))T^n$, where $\tilde{X} \to \operatorname{Spec} A$ is a resolution of the singularity. In general it is difficult to prove the existence of canonical models. But it is known that canonical models exist for surface singularities in arbitrary characteristic; 3-dimensional singularities in characteristic zero [M]; "nondegenerate" hypersurface singularities of any dimension [I]; and so on.

2.12. Theorem. *Let (A, m) be a $\mathbb{Q}$-Gorenstein normal local ring essentially of finite type over a field whose canonical covering admits a canonical model. Then in characteristic $p \gg 0$ (including characteristic zero: see* 0.4 *and* 0.5*), the test ideal $\tau(A)$ is the strong test ideal of A.*

Proof. By Proposition 2.9 we may assume that A is quasi-Gorenstein and has a canonical model $\psi : X \to \operatorname{Spec} A$. Then the discrepancy $\Delta = K_X - \psi^*K_A$ of ψ is an integral divisor. We define a filtration $\{J_n\}_{n\geq 0}$ by $J_n := H^0(X, \mathcal{O}_X(n\Delta)) \cong H^0(X, \mathcal{O}_X(nK_X))$. The Rees algebra $\mathcal{R} = \bigoplus_{n\geq 0} J_n T^n$ then is Noetherian and normal, and its filtered blowup is the canonical model $\psi : X = \operatorname{Proj} \mathcal{R} \to \operatorname{Spec} A$.

Now let $\tilde{f} : \tilde{X} \to \operatorname{Spec} A$ be a resolution of singularities reduced from characteristic zero to characteristic p, and let $\tilde{\Delta} = K_{\tilde{X}} - \tilde{f}^* K_A$ be the discrepancy of $\tilde{f}$. Then in characteristic $p \gg 0$, the test ideal of A is described as $\tau(A) = H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\tilde{\Delta}))$, and this is identified with $H^0(\tilde{X}, \omega_{\tilde{X}}) = H^0(X, \omega_X)$ via the identification $A \cong \omega_A$ (Theorem 0.6). Hence we have that $J_1 = \tau(A)$.

We will apply Proposition 2.5 to this situation. Let $0 \neq c \in A$ be a test element for $\mathcal{R}$ as in Proposition 2.5. Then, since $\mathcal{O}_X(n) \cong \mathcal{O}_X(nK_X)$, it is sufficient to prove the following

Claim. *There exists $e \in \mathbb{N}$ such that the map $cF^e : H^d_E(\mathcal{O}_X) \to H^d_E(F^e_* \mathcal{O}_X)$ is injective.*

Proof of Claim. Since $H^d_E(\mathcal{O}_X)$ is an Artinian A-module, we only have to show that, for any nonzero element $\xi \in H^d_E(\mathcal{O}_X)$, $c\xi^q \neq 0$ in $H^d_E(F^e_* \mathcal{O}_X)$ for all $q = p^e \gg 0$.

Let ξ be a nonzero element of $H^d_E(\mathcal{O}_X)$. We note that the dual form of the formula $\tau(A) \cong H^0(X, \omega_X)$ is

$$0^*_{H^d_m(A)} = \operatorname{Ker}\left(H^d_m(A) \xrightarrow{\sigma} H^d_E(\mathcal{O}_X)\right),$$

where the map σ is the Matlis dual of the natural inclusion $H^0(X, \omega_X) \hookrightarrow \omega_A$, whence surjective (cf. [H2], [MS]).

Pick a $\tilde{\xi} \in H^d_m(A)$ which maps to $\xi \in H^d_E(\mathcal{O}_X)$. Then $\tilde{\xi} \notin 0^*_{H^d_m(A)}$, so that $c\tilde{\xi}^{q'} \notin 0^*_{H^d_m(A)}$ for some power q' of p, because otherwise, given a test element $d \in A$, $cd\tilde{\xi}^q = 0$ for all $q = p^e$, which is a contradiction.

This implies that $(c\tilde{\xi}^{q'})^q \notin 0^*_{H^d_m(A)}$ for all $q = p^e$. To see this, assume to the contrary that $(c\tilde{\xi}^{q'})^q \in 0^*_{H^d_m(A)}$ for some power q of p. Then we have that $H^d_m(A) = 0^*_{H^d_m(A)} + A(c\tilde{\xi}^{q'}, (c\tilde{\xi}^{q'})^p, \ldots, (c\tilde{\xi}^{q'})^{q/p})$, since $0^*_{H^d_m(A)}$ is the maximal proper submodule of $H^d_m(A)$ which is stable under the Frobenius action [S1]. But this implies that $H^d_E(\mathcal{O}_X) \cong H^d_m(A)/0^*_{H^d_m(A)}$ has finite length, which is absurd.

Consequently, we have $c\tilde{\xi}^q \notin 0^*_{H^d_m(A)}$ for all $q \geq q'$, so that $c\xi^q \neq 0$ for all $q \geq q'$, as required. This completes the proof of the theorem. □

As we mentioned in Remark 2.11, Theorem 2.12 is applicable in several cases. In particular, in the local ring of any $\mathbb{Q}$-Gorenstein normal surface singularity, the test ideal is equal to the strong test ideal, if the characteristic is zero or $p \gg 0$. What's more, given the dual graph of the resolution of such a singularity, we can determine an explicit lower bound of p for which the identification of the test and strong test ideals occurs (cf. [H3]).

However, this result is not yet satisfactory, because it is likely that the test ideal is a strong test ideal even in small characteristic $p > 0$. Also, for our purpose it seems too much to ask for the existence of canonical models. We ask if Proposition 2.5 is applicable to easier filtrations $\{J_n\}_{n\geq 0}$, e.g., $J_n = \tau^n$, or $J_n = \overline{\tau^n}$.

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