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AN ERGODIC AND TOPOLOGICAL APPROACH TO DISCONJUGATE LINEAR HAMILTONIAN SYSTEMS

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ABSTRACT. This paper is devoted to the qualitative study of disconjugate random linear Hamiltonian systems. We relate the principal solutions at $\pm\infty$ with the ergodic structure of the flow, the presence of exponential dichotomy, and the description of the Sacker-Sell spectrum. A continuity theorem for the principal solutions is also provided.

1. Introduction

In this paper, we study some problems relating to time-varying linear Hamiltonian differential equations using techniques of the theory of random differential systems. We will see that, using these techniques, it is possible to gain new information about such classical objects as the principal solutions and the Lyapunov type numbers of linear Hamiltonian equations. It is also possible to explore such themes of more recent interest as the recurrence properties of the solutions of such equations.

Before giving an outline of the results presented here, we try to put them in perspective. In the last 40 years or so, an approach to the study of non-autonomous linear differential equations

$$\mathbf{x}' = A(t)\,\mathbf{x}\,, \qquad \mathbf{x} \in \mathbb{R}^n,\ t \in \mathbb{R}\,, \tag{1.1}$$

has been developed using methods of ergodic theory and topological dynamics. One begins by introducing the so-called *hull* H_A of the (bounded measurable) function A; this is the closure of the set of translates $A_s(\cdot) = A(s+\cdot)$ in an appropriate function space. One then observes that the solutions of (1.1) define a flow in the product space $H_A \times \mathbb{R}^n$. One can then study Lyapunov numbers and exponential dichotomies with the help of this flow structure; one obtains a theory of wide applicability to nonautonomous differential equations,

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nonlinear as well as linear. Notable results were obtained using these ideas in the 1960s and 1970s by Artstein, Bronstein, Millionščikov, Sell, and other authors.

In the 1980s another technique was added to those already existing for the study of (1.1), namely that of rotation numbers for linear Hamiltonian systems (Johnson and Moser [9], Johnson [8]). It turns out that there is a close connection between Lyapunov exponents, rotation numbers, exponential dichotomies, and the classical Weyl m-functions. These themes were all explored and applications were worked out in, among others papers, Johnson and Nerurkar [10], Johnson et al. [11], Novo et al. [16]. We note that there are substantial applications of all these concepts to the study of problems as diverse as the spectral theory of the quasi-periodic Schrödinger operator, the existence of chaotic orbits in singularly perturbed ODEs (Batelli and Palmer [3]), non-autonomous control theory, and the theory of orthogonal polynomials.

The conglomeration of methods and applications which we have just described is the field of random differential systems. Some authors prefer to reserve this term for stochastically driven differential equations, so our terminology is perhaps not entirely standard.

Let us explain in a bit more detail what we mean by a *random* Hamiltonian differential system. Let Ω be a compact metric space which supports a continuous real flow $\sigma : \Omega \times \mathbb{R} \to \Omega$, $(\xi, t) \mapsto \xi\cdot t$. Consider the family of equations

$$\mathbf{z}' = \begin{bmatrix} H_1(\xi\cdot t) & H_2(\xi\cdot t) \\ H_3(\xi\cdot t) & -H_1^T(\xi\cdot t) \end{bmatrix} \mathbf{z} = H(\xi\cdot t)\,\mathbf{z}\,, \qquad \xi \in \Omega\,, \tag{1.2}$$

where H_1, H_2, H_3 are continuous, $n \times n$ matrix-valued functions on Ω, and H_2, H_3 are symmetric. The flow (Ω, σ) may be (but need not be) that defined by translation on the hull of a single function $H(\cdot)$. We will assume that H_2 is positive semidefinite, and that the system (1.2) is disconjugate on $\mathbb{R}$ for each $\xi \in \Omega$. This implies the existence of principal solutions of (1.2) both as $t \to \infty$ and as $t \to -\infty$.

We now outline the results to be discussed in this paper. Let $L_{\mathbb{R}}$ be the manifold of real Lagrange planes in $\mathbb{R}^{2n}$, and let $K_{\mathbb{R}} = \Omega \times L_{\mathbb{R}}$. The equations (1.2) induce in a natural way a flow on $K_{\mathbb{R}}$. We will discuss the supports of the invariant measures on the Lagrange bundle, then give a formula for the sum γ of the non-negative Lyapunov exponents of (1.2) whose proof makes use of the ergodic structure of the flow on $K_{\mathbb{R}}$. Next we will give a necessary and sufficient condition for the presence of exponential dichotomy in (1.2), using the principal solutions. Then we turn to the recurrence properties of solutions of (1.2), or more precisely of their images in the Lagrange bundle $K_{\mathbb{R}}$. We show that, if the base flow (Ω, σ) is minimal, then the principal solutions define almost automorphic minimal subsets of $K_{\mathbb{R}}$. When (Ω, σ) is minimal, we also prove a property of the Sacker-Sell spectrum of (1.2) when exponential dichotomy is not present. Finally, we will discuss an L^2-convergence result of the principal solutions when (1.2) depends in a certain way on a parameter E and E tends to a value E_0 at which the dichotomy property does not hold.

This paper is part of what is now a long-standing effort to study linear non-autonomous Hamiltonian systems from the point of view of random systems; see, e.g., [8], [10], [16], and especially [11] (where, however, it is assumed that $H_2 > 0$).

2. Preliminaries

Let Ω be a compact metric space, $\sigma : \mathbb{R} \times \Omega \to \Omega,\ (t, \xi) \mapsto \xi{\cdot}t$ a continuous flow and m_0 a fixed σ-ergodic measure on Ω. We consider the family of linear Hamiltonian systems (1.2) where, as we have said before, H is a continuous real $2n \times 2n$ matrix-valued function and H_2 and H_3 are $n \times n$ symmetric matrices. This family of systems induces in a natural way a skew-product flow on $\Omega \times \mathbb{C}^{2n}$. If $U(t, \xi)$ represents the fundamental matrix solution of equation (1.2) for $\xi \in \Omega$ with $U(0, \xi) = I_{2n}$, the trajectory of $(\xi, \mathbf{z})$ is $\{(\xi{\cdot}t, U(t, \xi)\,\mathbf{z}) |\, t \in \mathbb{R}\}$.

It is known that for each $t \in \mathbb{R}$ and $\xi \in \Omega$, $U(t, \xi)$ lies in the symplectic group $\mathrm{Sp}(n, \mathbb{R}) = \{G \in M_{\mathbb{R}}(2n) |\, G^T J G = J\}$, where $J = \left[\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right]$. We recall that an n-dimensional vector subspace $F \subset \mathbb{C}^{2n}$ is called a *complex Lagrange plane* if $\mathbf{x}^T J \mathbf{y} = 0$ for all $\mathbf{x}$, $\mathbf{y} \in F$. The space $L_{\mathbb{C}}$ of all complex Lagrange planes of $\mathbb{C}^{2n}$ is a compact orientable manifold of dimension $n(n + 1)/2$. Since $U(t, \xi)\, F$ lies in $L_{\mathbb{C}}$ whenever $F \in L_{\mathbb{C}}$, the map $\tau : \mathbb{R} \times \Omega \times L_{\mathbb{C}} \to \Omega \times L_{\mathbb{C}},\ (t, \xi, F) \mapsto (\xi{\cdot}t, U(t, \xi)\, F)$ defines a continuous skew-product flow on $K_{\mathbb{C}} = \Omega \times L_{\mathbb{C}}$.

An element F of $L_{\mathbb{C}}$ can be represented by a $2n \times n$ matrix $\left[\begin{smallmatrix} F_1 \\ F_2 \end{smallmatrix}\right]$ of range n with $F_1^T F_2 = F_2^T F_1$. The column vectors form the basis of the Lagrange subspace; so two matrices $\left[\begin{smallmatrix} F_1 \\ F_2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix}\right]$ represent the same complex Lagrange plane if and only if there is a non-singular $n \times n$ complex matrix P such that $F_1 = G_1 P$ and $F_2 = G_2 P$. The set $S_{\mathbb{C}}(n)$ of symmetric $n \times n$ complex matrices parametrizes an open dense subset of $L_{\mathbb{C}}$, $\widetilde{\mathcal{D}} = \left\{ \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right] |\, M \in S_{\mathbb{C}}(n) \right\}$. Taking these complex coordinates in (1.2), we obtain the Riccati equations

$$M' = -M H_2(\xi{\cdot}t) M - M H_1(\xi{\cdot}t) - H_1^T(\xi{\cdot}t) M + H_3(\xi{\cdot}t)\,, \qquad \xi \in \Omega\,. \tag{2.1}$$

The flow on $\Omega \times \widetilde{\mathcal{D}}$ is then given by $(\xi, M) \cdot t = (\xi \cdot t, M(t, \xi, M))$, where $M(t, \xi, M)$ is the solution of (2.1) with initial data $M(0, \xi, M) = M$.

Analogously, we consider the space $L_{\mathbb{R}}$ of real Lagrange planes of $\mathbb{R}^{2n}$, a compact manifold of dimension $n(n + 1)/2$. As in the complex case, we can represent the elements of $L_{\mathbb{R}}$ by $2n \times n$ real matrices of range n of the form $\left[\begin{smallmatrix} F_1 \\ F_2 \end{smallmatrix}\right]$ with $F_1^T F_2 = F_2^T F_1$. By taking an orthonormal basis of the subspace,

$L_{\mathbb{R}}$ can be identified with the homogeneous space of left cosets $\mathcal{G}/\mathcal{H}$, where

$$\begin{aligned}\mathcal{G} &= \mathrm{Sp}(n,\mathbb{R})\cap \mathrm{SO}(2n,\mathbb{R})\\ &= \left\{ \begin{bmatrix}\Phi & -\Psi\\ \Psi & \Phi\end{bmatrix} \mid \Phi^T\Phi+\Psi^T\Psi = I_n\,,\ \Phi^T\Psi=\Psi^T\Phi\right\},\\ \mathcal{H} &= O(n,\mathbb{R}) = \left\{ \begin{bmatrix}R & 0\\ 0 & R\end{bmatrix} \mid R^TR = I_n\right\}.\end{aligned}$$

Then $L_{\mathbb{R}}$ is a symmetric Riemannian space with a $\mathcal{G}$-invariant metric. It can be shown that $\mathcal{G}/\mathcal{H}$ is orientable when n is odd and non-orientable when n is even (see Matsushima [13] and Mishchenko et al. [15]). In any case, we can consider the set of real Lagrange planes with an assigned orientation, $\mathcal{L}_{\mathbb{R}}$, which is an orientable compact manifold, a two-covering of $L_{\mathbb{R}}$, and which can be identified with the homogeneous space $\mathcal{G}/\mathcal{H}_1$, where

$$\mathcal{H}_1 = \mathrm{SO}(n,\mathbb{R}) = \left\{ \begin{bmatrix}R & 0\\ 0 & R\end{bmatrix} \mid R^TR = I_n\,,\ \det R = 1\right\}.$$

The next theorem explains the transformation of the systems (1.2) when *generalized polar coordinates* are used (see Reid [20]). The application of the polar transformation to the study of matrix differential equations was first presented by Barret [2] and was subsequently refined by Reid [18] for differential systems.

THEOREM 2.1. *Let* $\left[\begin{smallmatrix}F_1\\ F_2\end{smallmatrix}\right]$ *be a real Lagrange plane and* Φ, Ψ *and* R $n\times n$ *real matrices such that* $\left[\begin{smallmatrix}F_1\\ F_2\end{smallmatrix}\right] = \left[\begin{smallmatrix}\Phi R\\ \Psi R\end{smallmatrix}\right]$, *with* $\left[\begin{smallmatrix}\Phi & -\Psi\\ \Psi & \Phi\end{smallmatrix}\right]\in\mathcal{G}$ *and* R *non-singular. Then the* $2n\times n$ *solution of* (1.2) *corresponding to the initial data* $\left[\begin{smallmatrix}F_1\\ F_2\end{smallmatrix}\right]$ *is*

$$\begin{bmatrix}F_1(t,\xi,F_1,F_2)\\ F_2(t,\xi,F_1,F_2)\end{bmatrix} = \begin{bmatrix}\Phi(t,\xi,\Phi,\Psi)\,R(t,\xi,\Phi,\Psi,R)\\ \Psi(t,\xi,\Phi,\Psi)\,R(t,\xi,\Phi,\Psi,R)\end{bmatrix},$$

where $\Phi(t,\xi,\Phi,\Psi)$, $\Psi(t,\xi,\Phi,\Psi)$ *and* $R(t,\xi,\Phi,\Psi,R)$ *are the solutions of*

$$\begin{aligned}\Phi' &= \Psi\,Q(\xi{\cdot}t,\Phi,\Psi)\,,\\ \Psi' &= -\Phi\,Q(\xi{\cdot}t,\Phi,\Psi)\,,\end{aligned}\tag{2.2}$$

$$R' = S(\xi{\cdot}t,\Phi,\Psi)\,R\tag{2.3}$$

given by the initial data Φ, Ψ, *and* R, *respectively, with*

$$Q(\xi,\Phi,\Psi) = \left[\Phi^T\,\Psi^T\right] JH(\xi)\begin{bmatrix}\Phi\\ \Psi\end{bmatrix},\quad S(\xi,\Phi,\Psi) = \left[\Phi^T\,\Psi^T\right] H(\xi)\begin{bmatrix}\Phi\\ \Psi\end{bmatrix}.$$

Furthermore,

$$\begin{aligned}R^T(t,\xi,\Phi,\Psi,R)\,R(t,\xi,\Phi,\Psi,R) &= F_1^T(t,\xi,F_1,F_2)\,F_1(t,\xi,F_1,F_2)\\ &\quad + F_2^T(t,\xi,F_1,F_2)\,F_2(t,\xi,F_1,F_2)\end{aligned}$$

and

$$\begin{bmatrix} \Phi(t,\xi,\Phi,\Psi) & -\Psi(t,\xi,\Phi,\Psi) \\ \Psi(t,\xi,\Phi,\Psi) & \Phi(t,\xi,\Phi,\Psi) \end{bmatrix} \in \mathcal{G}$$

for all $t \in \mathbb{R}$.

Therefore, with these coordinates, the skew-product flow τ induced by equations (1.2) on the compact metric space $K_{\mathbb{R}} = \Omega \times L_{\mathbb{R}}$ can be expressed in the following way: if $\left[\begin{smallmatrix} \Phi \\ \Psi \end{smallmatrix}\right]$ is a real Lagrange plane with $\Phi^T\Phi + \Psi^T\Psi = I_n$ and $\Phi(t,\xi,\Phi,\Psi)$ and $\Psi(t,\xi,\Phi,\Psi)$ are the matrix solutions of the equations (2.2) with initial data Φ and Ψ, then

$$\tau(t,\xi,\Phi,\Psi) = (\xi{\cdot}t, \Phi(t,\xi,\Phi,\Psi), \Psi(t,\xi,\Phi,\Psi))$$

defines the equation of the flow on $K_{\mathbb{R}}$. The relation $M = \Psi\Phi^{-1}$ gives us the change between the systems of coordinates that we have introduced.

The concept of *rotation number* for systems (1.2) was discussed in Novo et al. [16] in terms of the argument of a symplectic fundamental matrix solution. Let $V(t,\xi) = \left[\begin{smallmatrix} V_1(t,\xi) & V_3(t,\xi) \\ V_2(t,\xi) & V_4(t,\xi) \end{smallmatrix}\right]$ be a fundamental symplectic matrix solution of (1.2). The rotation number is defined as

$$\alpha = \lim_{t\to\infty} \frac{1}{t}\operatorname{Arg} V(t,\xi)\,,$$

where Arg is any argument equivalent to that defined by $\operatorname{Arg}_1 V = \arg\det(V_1 - iV_2)$ and a continuous branch of the argument is taken. The rotation number is well-defined, i.e., the limit exists and takes the same value for almost every $\xi \in \Omega$ with respect to m_0, and is independent of the choices of the equivalent argument and the fundamental matrix. A geometric introduction of the rotation number and its relation with the Arnold-Maslov index [1] is given in Johnson [8] and Johnson and Nerurkar [10].

To end this section we recall the definition of the *Lyapunov exponent* of the systems (1.2) with respect to m_0. Letting $\wedge^n$ denote the n-th wedge product, we define

$$\gamma = \lim_{t\to\infty} \frac{1}{t}\ln \|\wedge^n U(t,\xi)\|\,,$$

which exists for almost every $\xi \in \Omega$ with respect to m_0. As a matter of fact, $\gamma = \sum_{j=1}^n \gamma_j$ where $\gamma_1 \geq \cdots \geq \gamma_n \geq 0$ are the non-negative Lyapunov exponents of the systems (1.2) with respect to m_0 and, since $U(t,\xi)$ is a symplectic matrix, the remaining Lyapunov exponents are $-\gamma_1 \leq \cdots \leq -\gamma_n \leq 0$.

Ergodic representations for the rotation number and the Lyapunov exponent, in terms of the generalized polar coordinates, were obtained in [16].

3. Ergodic properties of disconjugate linear Hamiltonian systems

Throughout this section, we will study ergodic properties of linear Hamiltonian systems (1.2) satisfying the following conditions.

Assumption 3.1.

(1) The $n \times n$ matrix-valued function $H_2(\xi) \geq 0$ is positive semidefinite on Ω.
(2) The systems (1.2) are *identically normal* for each $\xi \in \Omega$, i.e., for any nontrivial solution $\mathbf{z}(t,\xi) = (\mathbf{z}_1(t,\xi), \mathbf{z}_2(t,\xi))^T$, the vector $\mathbf{z}_1(t,\xi)$ does not vanish throughout any interval of $\mathbb{R}$.
(3) There exists a $2n \times n$ matrix solution of (1.2), $\left[\begin{smallmatrix} F_1(t,\xi) \\ F_2(t,\xi) \end{smallmatrix}\right] \in L_{\mathbb{R}}$, such that $\det F_1(t,\xi) \neq 0$ for each $t \in \mathbb{R}$ and $\xi \in \Omega$.

It is known that under these hypotheses the systems (1.2) are *disconjugate* on $\mathbb{R}$ for each $\xi \in \Omega$, that is, for every non-zero solution $\mathbf{z}(t,\xi) = (\mathbf{z}_1(t,\xi), \mathbf{z}_2(t,\xi))^T$, the vector $\mathbf{z}_1(t,\xi)$ vanishes at most once on $\mathbb{R}$. In fact, if $H_2(\xi) \geq 0$ and systems (1.2) are identically normal, the disconjugacy is equivalent to property (3) of the Assumption 3.1 (see Chapter 2.1 of Coppel [4]). A detailed study of the properties of disconjugate linear Hamiltonian systems can be found in Hartman [5], Coppel [4] and Reid [20].

We also recall that a $2n \times n$ matrix solution of (1.2), $\left[\begin{smallmatrix} F_1(t,\xi) \\ F_2(t,\xi) \end{smallmatrix}\right] \in L_{\mathbb{R}}$, is called *principal at* $+\infty$ (resp. *at* $-\infty$) if $\det F_1(t,\xi) \neq 0$ for each $t \in \mathbb{R}$, and

$$\left(\int_{t_0}^{t} F_1^{-1}(s,\xi)\, H_2(\xi{\cdot}s)\, (F_1^{-1})^T(s,\xi)\, ds \right)^{-1} \to 0$$

as $t \to +\infty$ (resp. $t \to -\infty$). Note that, as shown in Proposition 2.2 of [4], when (1) and (2) of Assumption 3.1 hold the symmetric matrix $\int_{t_0}^{t} F_1^{-1}(s,\xi)$ $H_2(\xi{\cdot}s)\,(F_1^{-1})^T(s,\xi)\,ds$ is invertible for $t > t_0$.

From Assumption 3.1, the systems (1.2) possess principal solutions at $\pm\infty$, which will be denoted by $\left[\begin{smallmatrix} F_1^{\pm}(t,\xi) \\ F_2^{\pm}(t,\xi) \end{smallmatrix}\right]$, and they are unique as elements of $L_{\mathbb{R}}$. In addition, since $\det F_1^{\pm}(0,\xi) \neq 0$, we can define

$$N^{\pm}(\xi) = F_2^{\pm}(0,\xi)(F_1^{\pm})^{-1}(0,\xi).$$

Without loss of generality, in the rest of the paper the initial data of the principal solutions at $\pm\infty$ are assumed to be $\left[\begin{smallmatrix} F_1^{\pm}(0,\xi) \\ F_2^{\pm}(0,\xi) \end{smallmatrix}\right] = \left[\begin{smallmatrix} I_n \\ N^{\pm}(\xi) \end{smallmatrix}\right]$. It was shown in Johnson et al. [11] that $N^{\pm}(\xi)$ are pointwise limits of continuous matrix-valued functions

$$N^{\pm}(\xi) = \lim_{r \to \pm\infty} M_r(\xi), \tag{3.1}$$

where $M_r(\xi) = F_{2,r}(0,\xi)F_{1,r}^{-1}(0,\xi)$ and $\left[\begin{smallmatrix} F_{1,r}(t,\xi) \\ F_{2,r}(t,\xi) \end{smallmatrix}\right]$ is a non-trivial solution of (1.2) with $F_{1,r}(r,\xi) = 0$, irrespective of the value of $F_{2,r}(r,\xi)$. They are also bounded solutions along the flow of the Riccati equation (2.1) and define τ-invariant subsets $\{(\xi, N^{\pm}(\xi)) \,|\, \xi \in \Omega\} \subset K_{\mathbb{R}}$ which concentrate two singular τ-invariant measures (which may coincide). The next theorem was proved in [11] for $H_2(\xi) > 0$. It remains valid with the same proof in the present case, and characterizes the set where all the τ-invariant measures are concentrated.

THEOREM 3.2. *Let* $\mathcal{J} = \{(\xi, M) \in \Omega \times S_{\mathbb{R}}(n) \mid N^+(\xi) \le M \le N^-(\xi)\}$.

(i) $(\xi, M) \in \mathcal{J}$ *if and only if the solution* $F(t,\xi) = \left[\begin{smallmatrix} F_1(t,\xi) \\ F_2(t,\xi) \end{smallmatrix}\right]$ *of* (1.2) *with initial condition* $F(0,\xi) = \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right]$ *satisfies* $\det F_1(t,\xi) \ne 0$ *for every* $t \in \mathbb{R}$.

(ii) $\mathcal{J}$ *is a compact invariant set.*

(iii) *Every* τ*-invariant measure* ν *on* $K_{\mathbb{R}}$ *is concentrated on* $\mathcal{J}$, *i.e.,* $\nu(\mathcal{J}) = 1$. *In particular, if* m_0 *is an ergodic measure on* Ω *and* $\left\{\left(\xi, \left[\begin{smallmatrix} F_1(\xi) \\ F_2(\xi) \end{smallmatrix}\right]\right) \in K_{\mathbb{R}} \mid \xi \in \Omega_0\right\}$ *is a* τ*-invariant subset of* $K_{\mathbb{R}}$ *with* $m_0(\Omega_0) = 1$, *then the set*

$$A = \{\xi \in \Omega_0 \mid \det F_1(\xi) \ne 0 \text{ and } N^+(\xi) \le F_2(\xi) F_1^{-1}(\xi) \le N^-(\xi)\}$$

is σ*-invariant and* $m_0(A) = 1$.

(iv) $\mathcal{J}$ *is the maximal invariant subset of* $\mathcal{D} = \{(\xi, M) \mid \xi \in \Omega,\ M \in S_{\mathbb{R}}(n)\}$. *Moreover, each minimal subset of* $K_{\mathbb{R}}$ *is contained in* $\mathcal{J}$.

Let $\gamma = \sum_{j=1}^n \gamma_j$ be the Lyapunov exponent of the systems (1.2) with respect to an ergodic measure m_0, where $\gamma_1 \ge \cdots \ge \gamma_n \ge 0$ are the non-negative Lyapunov exponents of (1.2) with respect to m_0. The remaining Lyapunov exponents are $-\gamma_1 \le \cdots \le -\gamma_n \le 0$. From Oseledets' multiplicative ergodic theorem (see Oseledets [17] and Johnson et al. [12]), as in Lemma 2.3 of [16], we can construct for almost every $\xi \in \Omega$ a basis $\{\mathbf{x}_{\xi,1}, \dots, \mathbf{x}_{\xi,n}, \mathbf{y}_{\xi,1}, \dots, \mathbf{y}_{\xi,n}\}$ of $\mathbb{R}^{2n}$ satisfying

(i) $\lim_{|t|\to\infty} (1/t) \ln \|U(t,\xi)\, \mathbf{x}_{\xi,j}\| = \gamma_j$ for $j = 1, \dots, n$,

(ii) $\lim_{|t|\to\infty} (1/t) \ln \|U(t,\xi)\, \mathbf{y}_{\xi,j}\| = -\gamma_j$ for $j = 1, \dots, n$,

(iii) the subspaces $F_{\xi,1} = \langle \mathbf{x}_{\xi,1}, \dots, \mathbf{x}_{\xi,n} \rangle$ and $F_{\xi,2} = \langle \mathbf{y}_{\xi,1}, \dots, \mathbf{y}_{\xi,n} \rangle$ are real Lagrange planes.

(Recall that $U(t,\xi)$ is the fundamental matrix solution of (1.2) with $U(0,\xi) = I_{2n}$.)

In addition, we will represent by $V_+(\xi)$, $V_0(\xi)$ and $V_-(\xi)$ the sum of the subspaces corresponding to strictly negative, null and strictly positive Lyapunov exponents, respectively. That is,

$$\begin{aligned} V_+(\xi) &= \langle \mathbf{y}_{\xi,1}, \dots, \mathbf{y}_{\xi,n-s} \rangle\,, \\ V_0(\xi) &= \langle \mathbf{x}_{\xi,n-s+1}, \dots, \mathbf{x}_{\xi,n}, \mathbf{y}_{\xi,n-s+1}, \dots \mathbf{y}_{\xi,n} \rangle\,, \\ V_-(\xi) &= \langle \mathbf{x}_{\xi,1}, \dots, \mathbf{x}_{\xi,n-s} \rangle\,, \end{aligned}$$

where $0 \le s \le n$ and $2s$ is the number of null Lyapunov exponents.

The next two propositions explain the behaviour of some solutions of the family of systems (1.2) which are important for the ergodic characterization given in the last theorem of the section.

PROPOSITION 3.3. *For almost every $\xi \in \Omega$, there exists a $2n \times n$ matrix solution $\begin{bmatrix} F_1(t,\xi) \\ F_2(t,\xi) \end{bmatrix} \in L_{\mathbb{R}}$ of* (1.2) *such that* $\det F_1(t,\xi) \neq 0$ *for each* $t \in \mathbb{R}$ *and*

$$\lim_{t\to\infty} \frac{1}{2t} \ln\det\left(F_1^T(t,\xi)F_1(t,\xi)\right) = \gamma\,.$$

Proof. As shown in Proposition 2.5 of [16], there is a τ-ergodic measure ν_0 on $K_{\mathbb{R}}$ such that

$$\gamma = \int_{K_{\mathbb{R}}} \operatorname{tr} S(\xi,\Phi,\Psi)\,d\nu_0\,,$$

where the $n \times n$ matrix-valued function $S(\xi,\Phi,\Psi)$ was introduced in Theorem 2.1. Therefore, Birkhoff's ergodic theorem assures that, for almost every (ξ,Φ,Ψ) with respect to ν_0,

$$\gamma = \lim_{t\to\infty} \frac{1}{t}\int_0^t \operatorname{tr} S(\tau(s,\xi,\Phi,\Psi))\,ds\,.$$

Moreover, denoting by $\begin{bmatrix} F_1(t,\xi) \\ F_2(t,\xi) \end{bmatrix}$ the $2n \times n$ matrix solution of (1.2) with initial data $\begin{bmatrix} F_1(0,\xi) \\ F_2(0,\xi) \end{bmatrix} = \left[\begin{smallmatrix} \Phi \\ \Psi \end{smallmatrix}\right]$, and using the equation (2.3) satisfied by the generalized polar coordinate $R(t,\xi)$ with $R^T(t,\xi)R(t,\xi) = F_1^T(t,\xi)F_1(t,\xi) + F_2^T(t,\xi)F_2(t,\xi)$, we conclude that

$$\lim_{t\to\infty} \frac{1}{2t} \ln\det\left(F_1^T(t,\xi)F_1(t,\xi) + F_2^T(t,\xi)F_2(t,\xi)\right) = \gamma\,.$$

In addition, from Theorem 3.2 we deduce that the τ-ergodic measure ν_0 is concentrated on $\mathcal{J}$, which implies that for almost every $\xi \in \Omega$ and each $t \in \mathbb{R}$, $\det F_1(t,\xi) \neq 0$ and $N^+(\xi{\cdot}t) \leq F_2(t,\xi)F_1^{-1}(t,\xi) \leq N^-(\xi{\cdot}t)$. Therefore,

$$\begin{aligned}\gamma &= \lim_{t\to\infty} \frac{1}{2t} \ln\det\left(F_1^T(t,\xi)F_1(t,\xi) + F_2^T(t,\xi)F_2(t,\xi)\right)\\ &= \lim_{t\to\infty} \frac{1}{2t} \ln\det\left(F_1^T(t,\xi)\left[I_n + M^2(t,\xi)\right]F_1(t,\xi)\right),\end{aligned}$$

and the boundedness of the symmetric matrix $M(t,\xi) = F_2(t,\xi)F_1^{-1}(t,\xi)$ yield

$$\lim_{t\to\infty} \frac{1}{2t} \ln\det\left(F_1^T(t,\xi)F_1(t,\xi)\right) = \gamma$$

for almost every $\xi \in \Omega$, as asserted. $\square$

PROPOSITION 3.4. *Let $G(t,\xi) = \begin{bmatrix} G_1(t,\xi) \\ G_2(t,\xi) \end{bmatrix}$ be a $2n \times n$ matrix solution of* (1.2) *linearly independent with the principal solution $F^+(t,\xi) = \begin{bmatrix} F_1^+(t,\xi) \\ F_2^+(t,\xi) \end{bmatrix}$ almost everywhere, i.e.,* $\det\begin{bmatrix} G_1(t,\xi) & F_1^+(t,\xi) \\ G_2(t,\xi) & F_2^+(t,\xi) \end{bmatrix} \neq 0$. *Then for almost every* $\xi \in \Omega$

$$\limsup_{t\to\infty} \frac{1}{2t} \ln\det\left(G_1^T(t,\xi)G_1(t,\xi)\right) = \gamma\,.$$

Proof. Let $\left[\begin{smallmatrix} F_1(t,\xi) \\ F_2(t,\xi) \end{smallmatrix}\right]$ be any $2n \times n$ matrix solution of (1.2). Since $G(t,\xi)$ and $F^+(t,\xi)$ provide a fundamental matrix solution, there are constant $n \times n$ matrices $C_1(\xi)$ and $C_2(\xi)$ such that $F_1(t,\xi) = G_1(t,\xi)C_1(\xi) + F_1^+(t,\xi)C_2(\xi)$. Therefore, setting $P(t,\xi) = G_1^{-1}(t,\xi)F_1^+(t,\xi)$, we have

$$\begin{aligned} F_1^T(t,\xi)F_1(t,\xi) = {} & [C_1(\xi) + P(t,\xi)C_2(\xi)]^T \times \\ & \times G_1^T(t,\xi)G_1(t,\xi)\,[C_1(\xi) + P(t,\xi)C_2(\xi)]\,, \end{aligned}$$

and since $P(t,\xi) \to 0$ as $t \to \infty$, as shown in Proposition 2.4 of [4], we deduce that

$$\limsup_{t\to\infty} \frac{1}{2t} \ln\det\left(F_1^T(t,\xi)F_1(t,\xi)\right) \le \limsup_{t\to\infty} \frac{1}{2t} \ln\det\left(G_1^T(t,\xi)G_1(t,\xi)\right)\,.$$

Finally, since these limits are always $\le \gamma$, and by Proposition 3.3 there exists a solution $\left[\begin{smallmatrix} F_1(t,\xi) \\ F_2(t,\xi) \end{smallmatrix}\right]$ for which this limit is exactly γ almost everywhere, we obtain the result. □

Remark 3.5. Similarly, a $2n \times n$ matrix solution $\left[\begin{smallmatrix} G_1(t,\xi) \\ G_2(t,\xi) \end{smallmatrix}\right]$ of (1.2) that is linearly independent with $\left[\begin{smallmatrix} F_1^-(t,\xi) \\ F_2^-(t,\xi) \end{smallmatrix}\right]$ satisfies

$$\limsup_{t\to-\infty} \frac{1}{2t} \ln\det\left(G_1^T(t,\xi)G_1(t,\xi)\right) = -\gamma\,,$$

for almost every $\xi \in \Omega$. Note that in both cases we have not assumed the initial data to be a Lagrange plane.

We will denote by $F^\pm(\xi)$ both the real Lagrange planes $\left[\begin{smallmatrix} I_n \\ N^\pm(\xi) \end{smallmatrix}\right]$ obtained from the principal solutions at $\pm\infty$ and the $2n \times n$ real matrix $\left[\begin{smallmatrix} I_n \\ N^\pm(\xi) \end{smallmatrix}\right]$. The context will give in each case the exact meaning of the symbol.

The next result yields an ergodic representation of the Lyapunov exponent with respect to an ergodic measure m_0 on Ω, and characterizes the Lagrange planes $F^\pm(\xi)$ in terms of the Lyapunov exponents of the system.

Theorem 3.6.

(i) *For almost every $\xi \in \Omega$, the Lagrange plane $F^+(\xi)$ coincides with $V_+(\xi) \oplus L_0(\xi)$, where $L_0(\xi) \subset V_0(\xi)$.*
(ii) *For almost every $\xi \in \Omega$, the Lagrange plane $F^-(\xi)$ coincides with $V_-(\xi) \oplus \widetilde{L}_0(\xi)$, where $\widetilde{L}_0(\xi) \subset V_0(\xi)$.*
(iii) $\gamma = \mp \int_\Omega \operatorname{tr}[H_1(\xi) + H_2(\xi)\,N^\pm(\xi)]\,dm_0$.
(iv) *For almost every $\xi \in \Omega$, $L_0(\xi) = \widetilde{L}_0(\xi)$ and* $\dim(F^+(\xi) \cap F^-(\xi)) =$ *$s = \dim V_0(\xi)/2$. In particular, we have:*

- *$\gamma = 0$ if and only if $N^+(\xi) = N^-(\xi)$ almost everwhere.*
- *$\gamma_1 \geq \ldots \geq \gamma_n > 0$ if and only if $F^+(\xi)$ and $F^-(\xi)$ are supplementary subspaces for almost every $\xi \in \Omega$.*

Proof. (i) Let us assume that there is $j \in \{1, 2, \ldots, n-s\}$ such that $\mathbf{y}_{\xi,j} \notin F^+(\xi)$. We consider a supplementary subspace of $F^+(\xi)$ generated by the vectors $\{\mathbf{v}_{\xi,1}, \mathbf{v}_{\xi,2}, \ldots, \mathbf{v}_{\xi,n}\} \subset \{\mathbf{x}_{\xi,1}, \ldots, \mathbf{x}_{\xi,n}, \mathbf{y}_{\xi,1}, \ldots, \mathbf{y}_{\xi,n}\}$, where $\mathbf{v}_{\xi,1} = \mathbf{y}_{\xi,j}$, and which is not necessarily a Lagrange plane. We denote by $G(\xi)$ the $2n \times n$ matrix $G(\xi) = [\mathbf{v}_{\xi,1}\mathbf{v}_{\xi,2} \cdots \mathbf{v}_{\xi,n}]$ and let $G(t,\xi) = \left[\begin{smallmatrix} G_1(t,\xi) \\ G_2(t,\xi) \end{smallmatrix}\right]$ be the $2n \times n$ matrix solution of (1.2) with initial data $G(0,\xi) = G(\xi)$. Notice that

$$\sum_{k=1}^{n} \lim_{t\to\infty} \frac{1}{2t} \ln \|U(t,\xi)\, \mathbf{v}_{\xi,k}\| < \gamma$$

because

$$\lim_{t\to\infty} \frac{1}{2t} \ln \|U(t,\xi)\, \mathbf{v}_{\xi,1}\| = -\gamma_j < 0\,.$$

It is easy to prove, as in Proposition 2.5 of [16], that

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{2t} \ln\det\left(G_1^T(t,\xi)G_1(t,\xi)\right) \\
&\qquad\leq \limsup_{t\to\infty} \frac{1}{2t} \ln\det\left(G_1^T(t,\xi)G_1(t,\xi) + G_2^T(t,\xi)G_2(t,\xi)\right) \\
&\qquad\leq \sum_{k=1}^{n} \lim_{t\to\infty} \frac{1}{2t} \ln \|U(t,\xi)\, \mathbf{v}_{\xi,k}\| < \gamma\,,
\end{aligned}$$

which contradicts Proposition 3.4. Thus, necessarily $V_+(\xi) \subset F^+(\xi)$. In addition, let us suppose that $\mathbf{v} = \mathbf{v}_- + \mathbf{v}_+ + \mathbf{v}_0 \in F^+(\xi)$ with $\mathbf{v}_\pm \in V_\pm(\xi)$, $\mathbf{v}_0 \in V_0(\xi)$ and $\mathbf{v}_- \neq 0$. Since $V_+(\xi) \subset F^+(\xi)$, we deduce that $\mathbf{v}_- + \mathbf{v}_0 \in F^+(\xi)$ and $\mathbf{y}_{\xi,j}^T\, J\, (\mathbf{v}_- + \mathbf{v}_0) = 0$ for every $j = 1, 2, \ldots, n-s$. Moreover, since $U(t,\xi)$ is a sympletic fundamental matrix solution, $\mathbf{y}^T U^T(t,\xi) J U(t,\xi)\, \mathbf{x}$ is independent of t for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{2n}$. Together with the behavior of the solutions at $\pm\infty$ this implies that $\mathbf{y}_{\xi,j}^T\, J\, \mathbf{v}_0 = 0$ for every $j = 1, 2, \ldots, n-s$. Therefore, $\mathbf{y}_{\xi,j}^T\, J\, \mathbf{v}_- = 0$ for every $j = 1, 2, \ldots, n$ and we will obtain $n+1$ isotropic independent vectors, which is impossible. Consequently, $F^+(\xi) = V_+(\xi) \oplus L_0(\xi)$, as asserted.

The proof of (ii) is analogous, in view of Remark 3.5, and (iii) can be proved using the same argument as in the proof of Theorem 4.6(iii) of [11].

(iv) By (i) and (ii), $F^+(\xi) = V_+(\xi) \oplus L_0(\xi)$ and $F^-(\xi) = V_-(\xi) \oplus \widetilde{L}_0(\xi)$. We consider the invariant Lagrange plane $F(\xi) = V_+(\xi) \oplus \widetilde{L}_0(\xi)$ which admits a representation of the form $\left[\begin{smallmatrix} I_n \\ M(\xi) \end{smallmatrix}\right]$ because $\left\{(\xi, F(\xi)) = \left(\xi, \left[\begin{smallmatrix} F_1(\xi) \\ F_2(\xi) \end{smallmatrix}\right]\right) \mid \xi \in \Omega\right\}$ is a τ-invariant subset of $K_{\mathbb{R}}$ and therefore, by Theorem 3.2, $\det F_1(\xi) \neq 0$ almost everywhere. As in (iii), we have $-\gamma = \int_\Omega \operatorname{tr}\left[H_1(\xi) + H_2(\xi)\, M(\xi)\right]\, dm_0$, which together with $-\gamma = \int_\Omega \operatorname{tr}[H_1(\xi) + H_2(\xi)\, N^+(\xi)]\; dm_0$ yields

$$\int_\Omega \operatorname{tr}\big[H_2(\xi)(M(\xi) - N^+(\xi))\big]\; dm_0 = 0\,.$$

Moreover, by Theorem 3.2 we have $N^+(\xi) \le M(\xi)$ for almost every $\xi \in \Omega$, and since $H_2(\xi) \ge 0$ we conclude that $H_2(\xi)N^+(\xi) = H_2(\xi)M(\xi)$ almost everywhere on Ω. Therefore, if we consider the $2n \times n$ matrix solution of (1.2) $F(t,\xi) = \left[\begin{smallmatrix} F_1(t,\xi) \\ F_2(t,\xi)\end{smallmatrix}\right] \in L_{\mathbb{R}}$, with initial data $\left[\begin{smallmatrix} F_1(0,\xi) \\ F_2(0,\xi)\end{smallmatrix}\right] = \left[\begin{smallmatrix} I_n \\ M(\xi)\end{smallmatrix}\right]$, we obtain

$$\begin{aligned} \frac{d}{dt} F_1^+(t,\xi) &= H_1(\xi{\cdot}t)\, F_1^+(t,\xi) + H_2(\xi{\cdot}t)\, F_2^+(t,\xi) \\ &= \big[H_1(\xi{\cdot}t) + H_2(\xi{\cdot}t)\, N^+(\xi{\cdot}t)\big]\, F_1^+(t,\xi)\,, \\ \frac{d}{dt} F_1(t,\xi) &= H_1(\xi{\cdot}t)\, F_1(t,\xi) + H_2(\xi{\cdot}t)\, F_2(t,\xi) \\ &= [H_1(\xi{\cdot}t) + H_2(\xi{\cdot}t)\, M(\xi{\cdot}t)]\, F_1(t,\xi)\,. \end{aligned}$$

Since $F_1(0,\xi) = F_1^+(0,\xi) = I_n$ and $H_2(\xi)N^+(\xi) = H_2(\xi)M(\xi)$ almost everywhere on Ω, this yields $F_1(t,\xi) = F_1^+(t,\xi)$ for almost every $\xi \in \Omega$. Finally, since the systems (1.2) are identically normal, the same holds for the other components of the solutions, i.e., we have $F_2(t,\xi) = F_2^+(t,\xi)$. Hence we conclude that $F(t,\xi) = F^+(t,\xi)$ for almost every $\xi \in \Omega$. This implies that $N^+(\xi) = M(\xi)$, $L_0(\xi) = \widetilde{L}_0(\xi)$ and $\dim(F^+(\xi) \cap F^-(\xi)) = \dim L_0(\xi) = s = \dim V_0(\xi)/2$ almost everywhere, as asserted. □

4. Topological properties of disconjugate linear Hamiltonian systems

In this section we consider some important topological properties satisfied by the family of systems (1.2) (under Assumption 3.1) and a perturbation of this family. We begin with a characterization of exponential dichotomy in terms of the principal solutions.

Proposition 4.1. *The family of linear Hamiltonian systems* (1.2) *admits an exponential dichotomy on Ω if and only if $\mathbb{R}^{2n} = F^+(\xi) \oplus F^-(\xi)$ for every $\xi \in \Omega$.*

Proof. If (1.2) admits an exponential dichotomy (E.D.), there are closed invariant Lagrange planes $W^+(\xi)$ and $W^-(\xi)$ with $\mathbb{R}^{2n} = W^+(\xi) \oplus W^-(\xi)$ for every $\xi \in \Omega$, and positive constants C, β such that

$$\begin{aligned} &\|U(t,\xi)\,\mathbf{z}\| \le Ce^{-\beta t}\|\mathbf{z}\| \qquad \text{for every } t \ge 0 \text{ and } \mathbf{z} \in W^+(\xi)\,, \\ &\|U(t,\xi)\,\mathbf{z}\| \le Ce^{\beta t}\|\mathbf{z}\| \qquad \text{for every } t \le 0 \text{ and } \mathbf{z} \in W^-(\xi). \end{aligned}$$

As in the proof of Theorem 3.6, we then see that $F^+(\xi) = W^+(\xi)$ and $F^-(\xi) = W^-(\xi)$ for every $\xi \in \Omega$. Note that we have equality for *every* $\xi \in \Omega$ since, in this case, the $2n \times n$ matrix solution $\left[\begin{smallmatrix} F_1(t,\xi) \\ F_2(t,\xi) \end{smallmatrix}\right] \in L_{\mathbb{R}}$ of (1.2) with initial data $W^-(\xi)$ satisfies Proposition 3.3 for every $\xi \in \Omega$.

Conversely, let us assume that $\mathbb{R}^{2n} = F^+(\xi) \oplus F^-(\xi)$ for every $\xi \in \Omega$. We denote, as usual, by $F^\pm(t,\xi) = \left[\begin{smallmatrix} F_1^\pm(t,\xi) \\ F_2^\pm(t,\xi) \end{smallmatrix}\right]$ the principal solutions at $\pm\infty$ with initial data $\left[\begin{smallmatrix} I_n \\ N^\pm(\xi) \end{smallmatrix}\right]$. Since $F^+(\xi)$ and $F^-(\xi)$ are supplementary subspaces, by Proposition 2.4 of [4] we have, for each $\xi \in \Omega$,

$$\lim_{t\to\infty} (F_1^-)^{-1}(t,\xi)\, F_1^+(t,\xi) = 0. \tag{4.1}$$

In addition, the fact that the expression

$$(F_2^+)^T(t,\xi)\, F_1^-(t,\xi) - (F_1^+)^T(t,\xi)\, F_2^-(t,\xi) = N^+(\xi) - N^-(\xi) \tag{4.2}$$

is independent of t, for each $\xi \in \Omega$, together with (4.1) yields

$$\begin{aligned} 0 &= \lim_{t\to\infty} \left[(F_2^+)^T(t,\xi)\, F_1^-(t,\xi) - (F_1^+)^T(t,\xi)\, F_2^-(t,\xi)\right] (F_1^-)^{-1}(t,\xi)\, F_1^+(t,\xi) \\ &= \lim_{t\to\infty} (F_1^+)^T(t,\xi) \left[N^+(\xi{\cdot}t) - N^-(\xi{\cdot}t)\right] F_1^+(t,\xi)\,. \end{aligned}$$

Since, in this case, $N^-(\xi) - N^+(\xi) > 0$, we conclude that

$$\lim_{t\to\infty} \left[N^-(\xi{\cdot}t) - N^+(\xi{\cdot}t)\right]^{1/2} F_1^+(t,\xi) = 0\,. \tag{4.3}$$

Moreover, from (4.2) we obtain

$$N^+(\xi) - N^-(\xi) = (F_1^+)^T(t,\xi) \left[N^+(\xi{\cdot}t) - N^-(\xi{\cdot}t)\right] F_1^-(t,\xi)\,,$$

that is,

$$(F_1^-)^{-1}(t,\xi) = \left[N^+(\xi) - N^-(\xi)\right]^{-1} (F_1^+)^T(t,\xi)\, \left[N^+(\xi{\cdot}t) - N^-(\xi{\cdot}t)\right],$$

and (4.3) yields

$$\lim_{t\to\infty} (F_1^-)^{-1}(t,\xi) = 0\,. \tag{4.4}$$

Let $\mathbf{z}(t,\xi) = \left[\begin{smallmatrix} \mathbf{z}_1(t,\xi) \\ \mathbf{z}_2(t,\xi) \end{smallmatrix}\right]$ be a bounded solution of the system (1.2). Since $F^+(t,\xi)$ and $F^-(t,\xi)$ form a fundamental matrix solution, we can express the first n components of the solution as

$$\mathbf{z}_1(t,\xi) = F_1^+(t,\xi)\, \mathbf{c}_1(\xi) + F_1^-(t,\xi)\, \mathbf{c}_2(\xi)$$

for some constants $\mathbf{c}_1(\xi)$ and $\mathbf{c}_2(\xi)$, i.e.,

$$(F_1^-)^{-1}(t,\xi)\, \mathbf{z}_1(t,\xi) = (F_1^-)^{-1}(t,\xi)\, F_1^+(t,\xi)\mathbf{c}_1(\xi) + \mathbf{c}_2(\xi)\,.$$

From (4.1) and (4.4) we deduce that $\mathbf{c}_2(\xi) = 0$ for each $\xi \in \Omega$.

Analogously, the behaviour of the principal solution at $-\infty$ shows that $\mathbf{c}_1(\xi) = 0$ for each $\xi \in \Omega$, so we conclude that $\mathbf{z}_1(t,\xi) = 0$. Therefore, since the systems (1.2) are identically normal, they do not admit a non-zero bounded solution. This implies E.D. on Ω (see Selgrade [23] and Sacker and Sell [21]) in case the flow on the base Ω is minimal. In the general case, the above argument provides E.D. with subbundles of constant dimension on any minimal subset of Ω, and from this we obtain E.D. over the entire space Ω (see Sacker and Sell [22]). □

Let us assume that the flow (Ω,σ) is minimal and let $\pi : K_{\mathbb{R}} \to \Omega$ be the projection on the base Ω. Recall that a minimal subset of $K_{\mathbb{R}}$, K, is called an *almost automorphic extension* of the base Ω if there exists $\xi \in \Omega$ such that $\mathrm{card}(\pi^{-1}(\xi) \cap K) = 1$. A point $(\xi, F) \in K$ is called an *almost automorphic point* for the flow if, whenever $\lim_{n\to\infty} \tau(t_n, \xi, F) = (\xi_0, F_0)$, we have $\lim_{n\to\infty} \tau(-t_n, \xi_0, F_0) = (\xi, F)$, and (K, τ) is an *almost automorphic flow* if there exists an almost automorphic point. (See Veech [24] for these definitions.) The following proposition shows that the principal solutions provide minimal almost automorphic extensions of the base Ω. For the case $H_2(\xi) > 0$ this was shown in [11], and the same proof applies here.

Proposition 4.2. *Let us assume that the flow (Ω,σ) is minimal and that the linear Hamiltonian systems* (1.2) *do not admit an exponential dichotomy. Then there is a residual invariant subset $\Omega_0 \subset \Omega$ of continuity points of $N^{\pm}$, and an integer k with $1 \le k \le n$ such that*

(i) $\dim(F^+(\xi) \cap F^-(\xi)) = k$ *for every $\xi \in \Omega_0$, and* $\dim(F^+(\xi) \cap F^-(\xi)) \le k$ *for every $\xi \in \Omega$;*

(ii) *the compact invariant sets $K_{\pm} = \mathrm{cls}\{(\xi, N^{\pm}(\xi)) \,|\, \xi \in \Omega_0\}$ are minimal almost automorphic extensions of the base Ω.*

Remark 4.3. In some cases, the invariant subset Ω_0 of continuity points of $N^{\pm}$ has null measure $m_0(\Omega_0) = 0$; see, for instance, the examples of disconjugate linear bidimensional systems given by Millionščikov [14] and Vinograd [25].

Let us consider the perturbed family of linear Hamiltonian systems

$$\mathbf{z}' = (H(\xi{\cdot}t) + EJ^{-1}\Gamma(\xi{\cdot}t))\,\mathbf{z}\,, \quad \xi \in \Omega\,, \tag{4.5$_E$}$$

where the unperturbed systems (1.2) satisfy Assumption 3.1, $E \in \mathbb{C}$ is a complex parameter and $\Gamma = \left[\begin{smallmatrix}\Gamma_1 & 0\\ 0 & 0\end{smallmatrix}\right] \ge 0$ is a symmetric positive semidefinite continuous $2n \times 2n$ matrix-valued function on Ω with $\Gamma_1(\xi) > 0$. Since the unperturbed systems (1.2) are identically normal, it is easy to show that Γ satisfies the following Atkinson type condition: each minimal subset of Ω contains at least one point ξ such that

$$\int_{-\infty}^{\infty} \mathbf{v}^T U^T(t,\xi)\,\Gamma^2(\xi{\cdot}t)\,U(t,\xi)\,\mathbf{v}\,dt > 0 \quad \text{for all } \mathbf{v} \in \mathbb{C}^{2n} - \{0\}\,. \tag{4.6}$$

Johnson and Nerurkar [10] showed that this property guarantees the exponential dichotomy of the systems $(4.5)_E$ for $\Im E > 0$, that is, the existence of a splitting of the complex bundle into two τ_E-invariant n-dimensional closed subbundles, $\Omega \times \mathbb{C}^{2n} = F^+_{\Gamma,E} \oplus F^-_{\Gamma,E}$, and positive constants C, β such that

(i) $\|U_E(t,\xi)\,\mathbf{z}\| \le Ce^{-\beta t}\|\mathbf{z}\|$ for every $t \ge 0$ and $(\xi,\mathbf{z}) \in F^+_{\Gamma,E}$,
(ii) $\|U_E(t,\xi)\,\mathbf{z}\| \le Ce^{\beta t}\|\mathbf{z}\|$ for every $t \le 0$ and $(\xi,\mathbf{z}) \in F^-_{\Gamma,E}$.

Moreover, for every $\xi \in \Omega$, the sections $F^{\pm}_{\Gamma,E}(\xi) = \{\mathbf{z} \in \mathbb{C}^{2n} \,|\, (\xi,\mathbf{z}) \in F^{\pm}_{\Gamma,E}\}$ are complex Lagrange planes and can be represented in terms of the Weyl M-functions by $\begin{bmatrix} I_n \\ M^{\pm}_{\Gamma}(\xi,E)\end{bmatrix}$; see also Hinton and Shaw [6, 7] and Johnson [8]. The functions $M^{\pm}_{\Gamma}(\xi,E)$, defined for $\Im E \neq 0$ and $\xi \in \Omega$, are symmetric complex $n\times n$ matrix functions, that are continuous in both variables, and analytic outside the real axis for each fixed $\xi \in \Omega$. Moreover, $\pm\Im E\,\Im M^{\pm}_{\Gamma}(\xi,E) > 0$, and $M^{\pm}_{\Gamma}(\xi,\overline{E}) = (M^{\pm}_{\Gamma})^*(\xi,E)$.

We assume that the flow on Ω is minimal, which implies that the spectrum of the corresponding self-adjoint operator $L_\xi = J\,(d/dt - H(\xi{\cdot}t))$ is independent of ξ. In fact, E belongs to the resolvent if and only if $(4.5)_E$ admits an exponential dichotomy on Ω.

Theorem 4.4.

(i) *For each $E<0$ the system $(4.5)_E$ is disconjugate on $\mathbb{R}$ and admits an exponential dichotomy on Ω, i.e., $(-\infty,0)$ is contained in the resolvent set.*
(ii) *We have*
$$\lim_{E\to 0^-} N^{\pm}(\xi,E) = N^{\pm}(\xi)\,,$$
where $N^{\pm}(\xi,E)$ are given by the principal solutions of $(4.5)_E$ at $\pm\infty$.

Proof. (i) It is easily seen that the perturbed systems $(4.5)_E$ also satisfy conditions (1) and (2) of Assumption 3.1. Under these conditions, and since for $E<0$ we have $JH(\xi\cdot t) + E\Gamma(\xi\cdot t) \le JH(\xi\cdot t)$ for every $t \in \mathbb{R}$, the comparison theorem given in Proposition 2.10 of [4] yields the disconjugacy of the systems $(4.5)_E$ on $\mathbb{R}$.

This implies that the rotation number $\alpha_\Gamma(E)$ is 0 for each $E \le 0$. Thus, the characterization of the exponential dichotomy in terms of the rotation number, given by Johnson and Nerurkar in Theorem 2.10 of [10], establishes the exponential dichotomy of system $(4.5)_E$ on Ω for $E<0$. Obviously, the closed n-dimensional invariant subbundles provided by the exponential dichotomy admit the representation $\left\{\left(\xi, \begin{bmatrix} I_n \\ N^{\pm}(\xi,E)\end{bmatrix}\right) \,|\, \xi \in \Omega\right\}$ in terms of the principal solutions.

(ii) As explained at the beginning of Section 3, we have $N^{\pm}(\xi,E) = \lim\limits_{r\to\pm\infty} M_r(\xi,E)$, where $M_r(\xi,E) = F_{2,r}(0,\xi,E)F^{-1}_{1,r}(0,\xi,E)$ and $\begin{bmatrix} F_{1,r}(t,\xi,E) \\ F_{2,r}(t,\xi,E)\end{bmatrix}$ is the solution of $(4.5)_E$ with initial conditions $F_{1,r}(r,\xi,E)=0$, $F_{2,r}(r,\xi,E) = I_n$. As in Proposition 5.1 and Theorem 5.2 of [11] we deduce that for $E \le E'$, $M_r(\xi,E) \le M_r(\xi,E')$ if $r>0$, $M_r(\xi,E) \ge M_r(\xi,E')$ if $r<0$ and
$$N^+(\xi,E) \le N^+(\xi,E') \le N^-(\xi,E') \le N^-(\xi,E)\,.$$

Therefore, the limits

$$\lim_{E\to 0^-} N^{\pm}(\xi,E) = N_0^{\pm}(\xi)$$

exist and are finite. To show that $N_0^{\pm}(\xi)$ and $N^{\pm}(\xi)$ are equal, we note that, for every $\xi\in\Omega$,

$$\begin{aligned}
&N^+(\xi) = \lim_{r\to\infty} M_r(\xi)\,,\\
&N^+(\xi) \le N_0^+(\xi)\,,\\
&N^+(\xi) \ge M_r(\xi) \ge M_r(\xi,E) \quad (E\le 0,\ r>0)\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
0 &\le N_0^+(\xi) - N^+(\xi) \le N_0^+(\xi) - M_r(\xi)\\
&\le N_0^+(\xi) - N^+(\xi,E) + N^+(\xi,E) - M_r(\xi,E)\,,
\end{aligned}$$

and since $N^+(\xi,E)\to N_0^+(\xi)$ as $E\to 0^-$ and $M_r(\xi,E)\to N^+(\xi,E)$ as $r\to\infty$, we conclude that $N^+(\xi)=N_0^+(\xi)$. The case $N^-(\xi)$ is analogous. $\square$

REMARK 4.5. Note that the functions $N^{\pm}(\xi,E)$ are the analytic continuations to the real axis of the Weyl M-functions $M_\Gamma^{\pm}(\xi,E)$ with $\Im E>0$. Therefore, the Herglotz properties of these functions also yield the existence of the limits of $N^{\pm}(\xi,E)$ as $E\to 0^-$. The result is obvious if $E=0$ is also in the resolvent set, so the main interest of the above theorem lies in the case when the unperturbed system does not admit an exponential dichotomy and thus $E=0$ is the first point of the spectrum.

The next result gives information on the Sacker-Sell spectrum of $(4.5)_E$ when the dichotomy property does not hold at $E=0$. Recall that, for a fixed value of the parameter E, the Sacker-Sell spectrum Σ_E of $(4.5)_E$ consists of those real numbers λ for which the translated systems

$$\mathbf{z}' = [-\lambda I_{2n} + H(\xi{\cdot}t) + EJ^{-1}\Gamma(\xi{\cdot}t)]\,\mathbf{z}\,,\quad \xi\in\Omega$$

do not admit an E.D. See Sacker and Sell [21, 22] and also Selgrade [23]. We note that, if $(4.5)_E$ does not have the dichotomy property at $E=0$, then 0 is an element of $\Sigma_0=\Sigma_{E=0}$. On the other hand, $0\notin\Sigma_E$ if $E<0$.

THEOREM 4.6. *Suppose that $N^+(\xi)=N^-(\xi)$ for all $\xi\in\Omega_0$. Then the dynamical spectrum Σ_0 of $(4.5)_E$ at $E=0$ is a single interval containing $\lambda=0$ (which may reduce to $\{0\}$).*

Proof. Suppose, to the contrary, that Σ_0 is not a single interval. Then symmetry considerations show that $\Sigma_0=J_+\cup J_0\cup J_-$, where $J_+\subset(-\infty,0)$, $J_-\subset(0,\infty)$, $0\in J_0$ and the sets J_+, J_0, J_- are compact and pairwise disjoint. Here we have used basic properties of the Sacker-Sell spectrum given in [22]. By [22], there exists a decomposition

$$\Omega\times\mathbb{R}^{2n} = W_+\oplus W_0\oplus W_-\,,$$

where W_+, W_0, W_- are continuous, non-zero vector subbundles of $\Omega\times\mathbb{R}^{2n}$ which are invariant with respect to the flow on $\Omega\times\mathbb{R}^{2n}$ determined by $(4.5)_E$ with $E=0$.

Let $\beta_1=\min J_+\le\max J_+=\beta_2$. Using the properties of Σ_0 and the bundles W_+, W_0, W_- established in [22], we see that, if $(\xi,\mathbf{z}_+)\in W_+$ with $\mathbf{z}_+\neq 0$, and if $\mathbf{z}(t)$ is the corresponding solution of $(4.5)_E$ with $E=0$, then

$$\beta_1\le\liminf_{t\to\pm\infty}\frac{1}{t}\ln\|\mathbf{z}(t)\|\le\limsup_{t\to\pm\infty}\frac{1}{t}\ln\|\mathbf{z}(t)\|\le\beta_2<0\,. \tag{4.7}$$

Similarly, let $0<\gamma_1=\min J_-\le\sup J_-=\gamma_2$. Then, if $(\xi,\mathbf{z}_-)\in W_-$ with $\mathbf{z}_-\neq 0$, and if $\mathbf{z}(t)$ is the solution of $(4.5)_E$ with $E=0$ and $\mathbf{z}(0)=\mathbf{z}_-$, we have

$$0<\gamma_1\le\liminf_{t\to\pm\infty}\frac{1}{t}\ln\|\mathbf{z}(t)\|\le\limsup_{t\to\pm\infty}\frac{1}{t}\ln\|\mathbf{z}(t)\|\le\gamma_2\,. \tag{4.8}$$

Next note that, for small values of E, say $|E|<\varepsilon$, there are vector subbundles $W_+(E)$, $W_0(E)$, $W_-(E)$ of $\Omega\times\mathbb{R}^{2n}$, that are invariant with respect to the flow on $\Omega\times\mathbb{R}^{2n}$ determined by $(4.5)_E$ at parameter value E, such that

$$\Omega\times\mathbb{R}^{2n}=W_+(E)\oplus W_0(E)\oplus W_-(E)\,.$$

Moreover, the correspondences $E\mapsto W_-(E)$, $W_0(E)$, $W_+(E)$ are continuous in the natural (Grassmannian) sense. If E is small and negative, then, using Proposition 4.1 and Theorem 4.4 (i), one can show that, for each $\xi\in\Omega$, one has $(\{\xi\}\times\mathbb{R}^{2n})\cap W_\pm(E)\subset F^\pm(\xi)$. Hence, by the continuity property stated in Theorem 4.4 (ii), we have $(\{\xi\}\times\mathbb{R}^{2n})\cap W_\pm\subset F^\pm(\xi)$, where the inclusion now holds at $E=0$.

However, $F^-(\xi)$ cannot contain a vector in $(\{\xi\}\times\mathbb{R}^{2n})\cap W_+$ other than $\mathbf{z}=0$; this follows by considering the behaviour of $\mathbf{z}(t)$ as $t\to-\infty$ in relation (4.7). Similarly, (4.8) implies that $F^+(\xi)$ cannot contain a vector in $(\{\xi\}\times\mathbb{R}^{2n})\cap W_-$ other that $\mathbf{z}=0$. Now, by hypothesis, $N^+(\xi)=N^-(\xi)$ for all $\xi\in\Omega_0$, and this implies that W_+ and W_- must both reduce to the zero vector bundle. This is a contradiction, so Theorem 4.6 is proved. □

Remark 4.7. The proof shows that more is true: if $\dim(F^+(\xi)\cap F^-(\xi))=k$ for $\xi\in\Omega_0$, then the Sacker-Sell spectrum Σ_0 at $E=0$ can contain no more that $2n-2k+1$ intervals, one of which contains $\lambda=0$.

5. L^2-convergence of the principal solutions

We consider a sequence of families of linear Hamiltonian systems

$$\mathbf{z}'=\begin{bmatrix} H_1^k(\xi{\cdot}t) & H_2^k(\xi{\cdot}t)\\ H_3^k(\xi{\cdot}t) & -(H_1^k)^T(\xi{\cdot}t)\end{bmatrix}\mathbf{z}=H^k(\xi{\cdot}t)\,\mathbf{z}\,,\qquad \xi\in\Omega\,,\ k\in\mathbb{N}\,, \tag*{$(5.1)_k$}$$

where H^k are continuous real $2n\times 2n$ matrix-valued functions on Ω with $H^k(\xi)\in\mathfrak{sp}(n,\mathbb{R})$. If all of these systems are disconjugate on $\mathbb{R}$ and $H^k(\xi)$

converges uniformly to $H(\xi)$ with $H_2(\xi) > 0$, it is known (see Theorem 2.11 of Coppel [4]) that the limit system (1.2) is also disconjugate on $\mathbb{R}$. However, the continuity of the principal solutions requires that strong technical conditions be satisfied (see Reid [19]).

In the following theorem we characterize the continuity of the principal solutions in the $L^2(\Omega, m_0)$-topology in terms of the continuity of the Lyapunov exponents of the systems. For a matrix-valued function $A(\xi) \in L^2(\Omega, m_0)$ we set $\|A\|_2 = (\int_\Omega \|A(\xi)\|_s^2 dm_0)^{1/2}$, where $\|A(\xi)\|_s = (\mathrm{tr}[A^T(\xi)A(\xi)])^{1/2}$. Thus, if $A(\xi)$ is a symmetric matrix for every $\xi \in \Omega$, we have $\|A\|_2 = (\int_\Omega \mathrm{tr}[A^2(\xi)]\, dm_0)^{1/2}$.

THEOREM 5.1. *Aassume that systems* $(5.1)_k$ *are disconjugate on* $\mathbb{R}$ *for each* $k \in \mathbb{N}$, $H^k(\xi)$ *converges uniformly to* $H(\xi)$, *and* $H_2(\xi) > 0$ *is positive definite on* Ω. *Denote by* $N_k^\pm(\xi)$, $N^\pm(\xi)$ *the symmetric* $n \times n$ *matrix-valued functions obtained from the principal solutions at* $\pm\infty$, *and by* γ_k, γ *the Lyapunov exponents of* $(5.1)_k$ *and* (1.2), *respectively. The following statements are equivalent:*

(i) $\lim_{k\to\infty} \gamma_k = \gamma$.

(ii) $\lim_{k\to\infty} N_k^\pm(\xi) = N^\pm(\xi)$ *in the* $L^2(\Omega, m_0)$*-topology.*

Proof. (ii) $\Rightarrow$ (i). As shown in Theorem 3.6, we have

$$\begin{aligned} \gamma_k &= \mp \int_\Omega \mathrm{tr}\left[H_1^k(\xi) + H_2^k(\xi)\, N_k^\pm(\xi)\right]\, dm_0\,, \\ \gamma &= \mp \int_\Omega \mathrm{tr}\left[H_1(\xi) + H_2(\xi)\, N^\pm(\xi)\right]\, dm_0\,. \end{aligned} \tag{5.2}$$

From this the implication (ii) $\Rightarrow$ (i) is easily deduced.

(i) $\Rightarrow$ (ii). The convergence of γ_k to γ as $k \to \infty$ and (5.2) yield

$$\lim_{k\to\infty} \int_\Omega \mathrm{tr}\left[H_2^k(\xi)\, N_k^+(\xi)\right]\, dm_0 = \int_\Omega \mathrm{tr}\left[H_2(\xi)\, N^+(\xi)\right]\, dm_0\,. \tag{5.3}$$

Together with the positivity of H_2 and the uniform convergence of $H_2^k(\xi)$ to $H_2(\xi)$ as $k \to \infty$, this shows that there exists a constant $C_0 > 0$ such that

$$\left|\int_\Omega \mathrm{tr}\, N_k^+(\xi)\, dm_0\right| \le C_0 \quad \text{for each } k \in \mathbb{N}\,. \tag{5.4}$$

In addition, each function $N_k^+(\xi)$ is a solution along the flow of the Riccati equation

$$M' = -M H_2^k(\xi{\cdot}t) M - M H_1^k(\xi{\cdot}t) - (H_1^k)^T(\xi{\cdot}t) M + H_3^k(\xi{\cdot}t)\,, \qquad \xi \in \Omega\,. \tag{5.5}$$

This implies

$$\int_\Omega \mathrm{tr}\left[(N_k^+ H_2^k N_k^+)(\xi)\right]\, dm_0 = \int_\Omega \mathrm{tr}\left[(-N_k^+ H_1^k - (H_1^k)^T N_k^+ + H_3^k)(\xi)\right]\, dm_0\,,$$

and using again the positivity of $H_2^k(\xi)$ for sufficiently large k and (5.4), we see that there exists a positive constant $C>0$ such that

$$\int_\Omega \operatorname{tr}\left[(N_k^+)^2(\xi)\right]\,dm_0 \le C \quad \text{for each } k\in\mathbb{N}\,,$$

i.e., $\|N_k^+\|_2^2 \le C$. Therefore, there is a subsequence that is weakly convergent, i.e., convergent in the $\sigma(L^2,L^2)$-topology. For simplicity of notation we continue to write N_k^+ for the subsequence. Thus, we have

$$\lim_{k\to\infty} N_k^+(\xi) = M^+(\xi)$$

in the $\sigma(L^2,L^2)$-topology. Analogously, we obtain

$$\lim_{k\to\infty} N_k^-(\xi) = M^-(\xi)$$

in the $\sigma(L^2,L^2)$-topology.

By the characterization of the principal solutions given at the beginning of Section 3, we have

$$N_k^\pm(\xi) = \lim_{r\to\pm\infty} M_r^k(\xi),$$

where

$$M_r^k(\xi) = F_{2,r}^k(0,\xi)\,(F_{1,r}^k)^{-1}(0,\xi)$$

and $\begin{bmatrix} F_{1,r}^k(t,\xi) \\ F_{2,r}^k(t,\xi)\end{bmatrix}$ is the solution of $(5.1)_k$ with initial conditions $F_{1,r}^k(r,\xi)=0$, $F_{2,r}^k(r,\xi)=I_n$. Moreover, by Proposition 4.2 of [11], we have, for each $r>0$ and $\xi\in\Omega$,

$$M_r^k(\xi) \le N_k^+(\xi) \le N_k^-(\xi) \le M_{-r}^k(\xi)\,.$$

Since $M_r^k(\xi)$ converges uniformly on Ω to $M_r(\xi)$ as $k\to\infty$ and since weak convergence preserves order, we deduce that

$$M_r(\xi) \le M^+(\xi) \le M^-(\xi) \le M_{-r}(\xi)\,.$$

Finally, taking limits as $r\to\infty$ we conclude that

$$N^+(\xi) \le M^+(\xi) \le M^-(\xi) \le N^-(\xi)\,. \tag{5.6}$$

In addition, by (5.2) we have

$$\gamma_k = \frac{1}{2}\int_\Omega \operatorname{tr}\left[H_2^k(\xi)\left(N_k^-(\xi)-N_k^+(\xi)\right)\right]\,dm_0\,,$$

and the continuity of the Lyapunov exponents implies that

$$\lim_{k\to\infty}\int_\Omega \operatorname{tr}\left[H_2^k(\xi)\left(N_k^-(\xi)-N_k^+(\xi)\right)\right]\,dm_0$$
$$= \int_\Omega \operatorname{tr}\left[H_2(\xi)\left(N^-(\xi)-N^+(\xi)\right)\right]\,dm_0.$$

Moreover, the weak convergence of $N_k^\pm(\xi)$ yields

$$\lim_{k\to\infty}\int_\Omega \operatorname{tr}\left[H_2^k(\xi)\left(N_k^-(\xi)-N_k^+(\xi)\right)\right]\,dm_0 = \int_\Omega \operatorname{tr}\left[H_2(\xi)\left(M^-(\xi)-M^+(\xi)\right)\right]\,dm_0.$$

Hence,

$$\int_\Omega \operatorname{tr}\left[H_2(\xi)\left(N^-(\xi)-N^+(\xi)\right)\right]\,dm_0 = \int_\Omega \operatorname{tr}\left[H_2(\xi)\left[M^-(\xi)-M^+(\xi)\right)\right]\,dm_0\,.$$

Finally, the inequalities (5.6) and the positivity of $H_2(\xi)$ yield $N^+(\xi) = M^+(\xi)$ and $N^-(\xi) = M^-(\xi)$ for almost every $\xi \in \Omega$, and consequently

$$\lim_{k\to\infty} N_k^\pm(\xi) = N^\pm(\xi)$$

in the $\sigma(L^2, L^2)$-topology.

Using again the fact that $N^\pm(\xi)$ are solutions to the Riccati equation (5.5) and the weak convergence, we easily deduce that

$$\lim_{k\to\infty}\int_\Omega \operatorname{tr}\left[N_k^\pm H_2^k N_k^\pm(\xi)\right]\,dm_0 = \int_\Omega \operatorname{tr}\left[N^\pm H_2 N^\pm(\xi)\right]\,dm_0\,.$$

Together with the continuity and positivity of $H_2(\xi)$ and $H_2^k(\xi)$ for sufficiently large k, this gives

$$\lim_{k\to\infty}\int_\Omega \operatorname{tr}\left[(N_k^\pm)^2(\xi)\right]\,dm_0 = \int_\Omega \operatorname{tr}\left[(N^\pm)^2(\xi)\right]\,dm_0\,.$$

Therefore,

$$\lim_{k\to\infty}\|N_k^\pm\|_2 = \|N^\pm\|_2,$$

and hence

$$\lim_{k\to\infty} N_k^\pm(\xi) = N^\pm(\xi)$$

in the $L^2(\Omega, m_0)$ topology. This proves the convergence of $N_k^\pm$ to $N^\pm$ in the $L^2(\Omega, m_0)$-topology. Finally, we note that, although we have shown convergence only for a subsequence, since the limit of any convergent subsequence is independent of the subsequence, we have, in fact, convergence of the full sequence. $\square$

References

[1] V. I. Arnold, *On a characteristic class entering in a quantum condition*, Funct. Anal. Appl. **1** (1967), 1–14.

[2] J. H. Barret, *A Prüfer transformation for matrix differential equations*, Proc. Amer. Math. Soc. **8** (1957), 510–518.

[3] F. Battelli and K. Palmer, *Chaos in the Duffing equation*, J. Differential Equations **101** (1993), 276–301.

[4] W. A. Coppel, *Disconjugacy*, Lecture Notes in Mathematics, vol. 220, Springer-Verlag, Berlin, Heidelberg, New York, 1971.

[5] P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964.
[6] D. Hinton and J. Shaw, *On Titchmarsh-Weyl m-functions for linear Hamiltonian systems*, J. Differential Equations **40** (1981), 316–342.
[7] ———, *Hamiltonian systems of limit point or limit circle type with both endpoints singular*, J. Differential Equations **50** (1983), 444–464.
[8] R. Johnson, *m-functions and Floquet exponents for linear differential systems*, Ann. Mat. Pura Appl. **147** (1987), 211–248.
[9] R. Johnson and J. Moser, *The rotation number for almost periodic potentials*, Comm. Math. Phys. **84** (1982), 403–438.
[10] R. Johnson and M. Nerurkar, *Exponential dichotomy and rotation number for linear Hamiltonian systems*, J. Differential Equations **108** (1994), 201–216.
[11] R. Johnson, S. Novo, and R. Obaya, *Ergodic properties and Weyl M-functions for random linear Hamiltonian systems*, Proc. Roy. Soc. Edinburgh Sect. A **130** (2000), 1045–1079.
[12] R. Johnson, K. Palmer, and G. R. Sell, *Ergodic theory of linear dynamical systems*, SIAM J. Math. Anal. **18** (1987), 1–33.
[13] Y. Matsushima, *Differentiable manifolds*, Marcel Dekker, New York 1972.
[14] V. M. Millionščikov, *Proof of the existence of irregular systems of linear differential equations with almost periodic coefficients*, Diff. Urav. **4** (1968), 391–396.
[15] A. S. Mishchenko, V. E. Shatalov, and B. Yu. Sternin, *Lagrangian manifolds and the Maslov operator*, Springer-Verlag, Berlin, Heidelberg, New York, 1990.
[16] S. Novo, C. Núñez, and R. Obaya, *Ergodic properties and rotation number for linear Hamiltonian systems*, J. Differential Equations **148** (1998), 148–185.
[17] V. I. Oseledets, *A multiplicative ergodic theorem, Lyapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc. **19** (1968), 197–231.
[18] W. T. Reid, *A Prüfer transformation for differential systems*, Pacific J. Math. **8** (1958), 575–584.
[19] ———, *A continuity property of principal solutions of linear Hamiltonian differential systems*, Scripta Math. **29** (1973), 337–350.
[20] ———, *Sturmian theory for ordinary differential equations*, Applied Mathematical Sciences, vol. 31, Springer-Verlag, New York, 1980.
[21] R. J. Sacker and G. R. Sell, *Existence of dichotomies and invariant splittings for linear differential systems I*, J. Differential Equations **15** (1974), 429–458.
[22] ———, *A spectral theory for linear differential systems*, J. Differential Equations **27** (1978), 320–358.
[23] J. Selgrade, *Isolated invariant sets for flows on vectors bundles*, Trans. Amer. Math. Soc. **203** (1975), 359–390.
[24] W. A. Veech, *Topological dynamics*, Bull. Amer. Math. Soc. **83** (1977), 775–830.
[25] R. E. Vinograd, *A problem suggested by N. P. Erugin*, Diff. Urav. **11** (1975), 632–638.

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