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DUALIZING COMPLEX OF THE INCIDENCE ALGEBRA OF A FINITE REGULAR CELL COMPLEX

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ABSTRACT. Let Σ be a finite regular cell complex with $\emptyset \in \Sigma$, and regard it as a *poset* (i.e., partially ordered set) by inclusion. Let R be the incidence algebra of the poset Σ over a field k. Corresponding to the Verdier duality for constructible sheaves on Σ, we have a dualizing complex $\omega^\bullet \in D^b(\mathrm{mod}_{R\otimes_k R})$ giving a duality functor from $D^b(\mathrm{mod}_R)$ to itself. This duality is somewhat analogous to the Serre duality for a projective scheme ($\emptyset \in \Sigma$ plays a role similar to that of "irrelevant ideals"). If $H^i(\omega^\bullet) \neq 0$ for exactly one i, then the underlying topological space of Σ is Cohen-Macaulay (in the sense of the Stanley-Reisner ring theory). The converse also holds if Σ is a simplicial complex. R is always a Koszul ring with $R^! \cong R^{\mathsf{op}}$. The relation between the Koszul duality for R and the Verdier duality is discussed.

1. Introduction

Let Σ be a finite regular cell complex, and $X := \bigcup_{\sigma\in\Sigma} \sigma$ its underlying topological space. The order given by $\sigma > \tau \overset{\text{def}}{\Longleftrightarrow} \bar{\sigma} \supset \tau$ makes Σ a finite partially ordered set (*poset*, for short). Here $\bar{\sigma}$ is the closure of σ in X. Let R be the incidence algebra of the poset Σ over a field k. For a ring A, mod_A denotes the category of finitely generated left A-modules. In this paper, we study the bounded derived category $D^b(\mathrm{mod}_R)$ using the theory of constructible sheaves (e.g., Poincaré-Verdier duality). For the sheaf theory, consult [6], [7], [14]. We basically use the same notation as [6].

Let $\mathrm{Sh}_c(X)$ be the category of k-constructible sheaves on X with respect to the cell decomposition Σ. We have an exact functor $(-)^\dagger : \mathrm{mod}_R \to \mathrm{Sh}_c(X)$. For $M \in \mathrm{mod}_R$, we have a natural decomposition $M = \bigoplus_{\sigma\in\Sigma} M_\sigma$ as a k-vector space. If $p \in \sigma \subset X$, the stalk $(M^\dagger)_p$ of $M^\dagger$ at the point p is isomorphic to M_σ.

Let $\Sigma' := \Sigma \setminus \emptyset$ be an induced subposet of Σ, and T the incidence algebra of Σ' over k. Then we have a category equivalence $\mathrm{mod}_T \cong \mathrm{Sh}_c(X)$, which is well

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known to specialists (see, for example, [8], [11], [14]). However, in this paper, $\emptyset \in \Sigma$ plays a role. Although $\text{mod}_R \not\cong \text{Sh}_c(X)$, mod_R has several interesting properties which mod_T does not possess. In some sense, $\emptyset$ is analogous to the "irrelevant ideal" of a commutative Noetherian homogeneous k-algebra (i.e., the homogeneous coordinate ring of a projective scheme over k).

We have a left exact functor $\Gamma_\emptyset : \text{mod}_R \to \text{vect}_k$ defined by $\Gamma_\emptyset(M) = \{ x \in M_\emptyset \mid Rx \subset M_\emptyset \}$. We denote its ith right derived functor by $H^i_\emptyset(-)$. For $M \in \text{mod}_R$, Theorem 2.2 states that

$$H^i(X, M^\dagger) \cong H^{i+1}_\emptyset(M) \quad \text{for all } i \geq 1,$$

$$0 \to H^0_\emptyset(M) \to M_\emptyset \to H^0(X, M^\dagger) \to H^1_\emptyset(M) \to 0 \quad \text{(exact)}.$$

Here $H^\bullet(X, M^\dagger)$ stands for the sheaf cohomology (cf. [7], [6]).

The above fact is clearly analogous to the relation between graded modules over a commutative Noetherian homogeneous k-algebra A and the quasi-coherent sheaves on the projective scheme $\text{Proj}(A)$. There are other resemblances between these topics. In the final section of this paper, we give a list of the similarities.

Let A and B be k-algebras. Recently, several authors studied a dualizing complex $C^\bullet \in D^b(\text{mod}_{A\otimes_k B})$ giving duality functors between $D^b(\text{mod}_A)$ and $D^b(\text{mod}_B)$. (Note that if $M \in \text{mod}_A$ and $N \in \text{mod}_{A\otimes_k B}$, then $\text{Hom}_A(M, N)$ has a left B-module structure.) In typical cases, it is assumed that $B = A^{\mathsf{op}}$. But, in this paper, from Verdier's dualizing complex $\mathcal{D}^\bullet_X \in D^b(\text{Sh}_c(X))$ on X, we construct a dualizing complex $\omega^\bullet \in D^b(\text{mod}_{R\otimes R})$ which gives the duality functor $\mathbf{R}\,\text{Hom}_R(-, \omega^\bullet)$ from $D^b(\text{mod}_R)$ to itself. Theorem 3.2 states that

$$\mathbf{R}\,\text{Hom}_R(M^\bullet, \omega^\bullet)^\dagger \cong \mathbf{R}\mathcal{H}om((M^\bullet)^\dagger, \mathcal{D}^\bullet_X)$$

in $D^b(\text{Sh}_c(X))$ for all $M^\bullet \in D^b(\text{mod}_R)$. The dualizing complex $\omega^\bullet$ satisfies the Auslander condition in the sense of [19].

Corollary 3.5 states that

$$\text{Ext}^i_R(M^\bullet, \omega^\bullet)_\emptyset \cong H^{-i+1}_\emptyset(M^\bullet)^\vee.$$

This corresponds to the (global) Verdier duality on X. But, since $H^i_\emptyset(-)$ can be seen as an analog of a local cohomology over a commutative Noetherian homogeneous k-algebra, the above isomorphism can be seen as an imitation of the Serre duality. In Theorem 5.3 (1), $\emptyset \in \Sigma$ is also essential. It states that, for a simplicial complex Σ, $H^i(\omega^\bullet) = 0$ for all $i \neq -\dim X$ if and only if X is Cohen-Macaulay in the sense of the Stanley-Reisner ring theory. If we use the convention that $\emptyset \notin \Sigma$, then the Cohen-Macaulay property cannot be characterized in this way.

Under the assumption that a subset Ψ of Σ gives the open subset $U_\Psi := \bigcup_{\sigma\in\Psi} \sigma$ of X, Theorem 5.3 describes the cohomology $H^i(U_\Psi, M^\dagger|_{U_\Psi})$ using the duality functor $\mathbf{R}\,\text{Hom}_R(-, \omega^\bullet)$. Note that the cohomology with compact support $H_c^i(U_\Psi, M^\dagger|_{U_\Psi})$ is much easier to treat in our context, as shown in Lemma 5.1.

We can regard R as a graded ring in a natural way. Then R is always Koszul, and the quadratic dual ring $R^!$ is isomorphic to the opposite ring R^{op} (Proposition 7.1). Koszul duality (cf. [1]) gives an equivalence $D^b(\mathrm{mod}_R) \cong D^b(\mathrm{mod}_{R^{\mathsf{op}}})$ of triangulated categories. The functors giving this equivalence coincide with the compositions of the duality functors $\mathbf{R}\operatorname{Hom}_R(-,\omega^\bullet)$ and $\operatorname{Hom}_k(-,k)$. This result is an "augmented" version of Vybornov [14].

It is well known that the Möbius function of a finite poset is a very important tool in combinatorics. In Proposition 6.1, generalizing [13, Proposition 3.8.9], we describes the Möbius function $\mu(\sigma,\hat{1})$ of the poset $\hat{\Sigma} := \Sigma \amalg \{\hat{1}\}$ in terms of cohomology with compact support. As shown in [2], some finite posets arising from purely combinatorial/algebraic topics (e.g., Bruhat order) are isomorphic to the posets of finite regular cell complexes. So the author expects that the results in the present paper will play a role in a combinatorial study of these posets.

2. Preparation

A *finite regular cell complex* (cf. [3, §6.2] and [4]) is a non-empty topological space X together with a finite set Σ of subsets of X such that the following conditions are satisfied:

(i) $\emptyset \in \Sigma$ and $X = \bigcup_{\sigma\in\Sigma} \sigma$;
(ii) the subsets $\sigma \in \Sigma$ are pairwise disjoint;
(iii) for each $\sigma \in \Sigma$, $\sigma \neq \emptyset$, there exists a homeomorphism from an i-dimensional disc $B^i = \{x \in \mathbb{R}^i \mid ||x|| \leq 1\}$ onto the closure $\bar{\sigma}$ of σ which maps the open disc $U^i = \{x \in \mathbb{R}^i \mid ||x|| < 1\}$ onto σ.
(iv) For any $\sigma \in \Sigma$, the closure $\bar{\sigma}$ can be written as the union of some cells in Σ.

Note that X is compact in this case. An element $\sigma \in \Sigma$ is called a *cell*. We regard Σ as a poset with the order given by $\sigma > \tau \overset{\text{def}}{\Longleftrightarrow} \bar{\sigma} \supset \tau$. The combinatorics of posets of this type is discussed in [2]. If $\sigma \in \Sigma$ is homeomorphic to U^i, we write $\dim \sigma = i$ and call σ an *i-cell*. We define $\dim \emptyset = -1$ and set $d := \dim X = \max\{\, \dim\sigma \mid \sigma \in \Sigma\,\}$.

A finite simplicial complex is a primary example of a finite regular cell complex. When Σ is a finite simplicial complex, we sometimes identify Σ with the corresponding abstract simplicial complex. That is, we identify a cell $\sigma \in \Sigma$ with the set $\{\, \tau \mid \tau \text{ is a 0-cell with } \tau \leq \sigma\,\}$. In this case, Σ is a subset of the power set 2^V, where V is the set of the vertices (i.e., 0-cells) of Σ. Under this identification, for $\sigma \in \Sigma$, we let $\mathrm{st}_\Sigma\, \sigma := \{\, \tau \in \Sigma \mid \tau \cup \sigma \in \Sigma\,\}$ and $\mathrm{lk}_\Sigma\, \sigma := \{\, \tau \in \mathrm{st}_\Sigma\, \sigma \mid \tau \cap \sigma = \emptyset\,\}$ be subcomplexes of Σ.

Let $\sigma, \sigma' \in \Sigma$. If $\dim\sigma = i+1$, $\dim\sigma' = i-1$ and $\sigma' < \sigma$, then there are exactly two cells $\sigma_1, \sigma_2 \in \Sigma$ between σ' and σ. (Here $\dim\sigma_1 = \dim\sigma_2 = i$.) A remarkable property of a regular cell complex is the existence of an *incidence function* ε (cf. [4, II. Definition 1.8]). The definition of an incidence function is the following.

(i) To each pair (σ, σ') of cells, ε assigns a number $\varepsilon(\sigma, \sigma') \in \{0, \pm 1\}$.
(ii) $\varepsilon(\sigma, \sigma') \neq 0$ if and only if $\dim \sigma' = \dim \sigma - 1$ and $\sigma' < \sigma$.
(iii) If $\dim \sigma = 0$, then $\varepsilon(\sigma, \emptyset) = 1$.
(iv) If $\dim \sigma = i+1$, $\dim \sigma' = i-1$ and $\sigma' < \sigma_1$, $\sigma_2 < \sigma$, $\sigma_1 \neq \sigma_2$, then we have $\varepsilon(\sigma, \sigma_1)\, \varepsilon(\sigma_1, \sigma') + \varepsilon(\sigma, \sigma_2)\, \varepsilon(\sigma_2, \sigma') = 0$.

We can compute the (co)homology groups of X using the cell decomposition Σ and an incidence function ε.

Let P be a finite poset. The incidence algebra R of P over a field k is the k-vector space with a basis $\{e_{x,\,y} \mid x, y \in P \text{ with } x \geq y\}$. The k-bilinear multiplication defined by $e_{x,\,y}\, e_{z,\,w} = \delta_{y,\,z}\, e_{x,\,w}$ makes R a finite dimensional associative k-algebra. Set $e_x := e_{x,\,x}$. Then $1 = \sum_{x \in P} e_x$ and $e_x\, e_y = \delta_{x,y}\, e_x$. We have $R \cong \bigoplus_{x \in P} Re_x$ as a left R-module, and each Re_x is indecomposable.

Denote the category of finitely generated left R-modules by mod_R. If $N \in \operatorname{mod}_R$, we have $N = \bigoplus_{x \in P} N_x$ as a k-vector space, where $N_x := e_x N$. Note that $e_{x,\,y}\, N_y \subset N_x$ and $e_{x,\,y}\, N_z = 0$ for $y \neq z$. If $f : N \to N'$ is a morphism in mod_R, then $f(N_x) \subset N'_x$.

For each $x \in P$, we can construct an indecomposable injective module $E_R(x) \in \operatorname{mod}_R$. (When there is no possibility of confusion, we simply denote it by $E(x)$.) Let $E(x)$ be the k-vector space with a basis $\{e(x)_y \mid y \leq x\}$. Then we can regard $E(x)$ as a left R-module by

$$e_{z,\,w}\, e(x)_y = \begin{cases} e(x)_z & \text{if } y = w \text{ and } z \leq x, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Note that $E(x)_y = k\, e(x)_y$ if $y \leq x$, and $E(x)_y = 0$ otherwise. An indecomposable injective in mod_R is of the form $E(x)$ for some $x \in P$. Since $\dim_k R < \infty$, mod_R has enough projectives and injectives. It is well known that R has finite global dimension.

Let Σ be a finite regular cell complex, and X its underlying topological space. We make Σ a poset as above. In the rest of this paper, R is the incidence algebra of Σ over k. For $M \in \operatorname{mod}_R$, we have $M = \bigoplus_{\sigma \in \Sigma} M_\sigma$ as a k-vector space, where $M_\sigma := e_\sigma M$.

Let $\operatorname{Sh}(X)$ be the category of sheaves of finite dimensional k-vector spaces on X. We say $\mathcal{F} \in \operatorname{Sh}(X)$ is a *constructible sheaf* with respect to the cell decomposition Σ, if $\mathcal{F}|\sigma$ is a constant sheaf for all $\emptyset \neq \sigma \in \Sigma$. Here, $\mathcal{F}|\sigma$ denotes the inverse image $j^*\mathcal{F}$ of $\mathcal{F}$ under the embedding map $j : \sigma \to X$. Let $\operatorname{Sh}_c(X)$ be the full subcategory of $\operatorname{Sh}(X)$ consisting of constructible sheaves with respect to Σ. It is well known that $D^b(\operatorname{Sh}_c(X)) \cong D^b_{\operatorname{Sh}_c(X)}(\operatorname{Sh}(X))$. (See [7, Theorem 8.1.11]. There, it is assumed that Σ is a simplicial complex. However, this assumption is irrelevant. In fact, the key lemma [7, Corollay 8.1.5] also holds for regular cell complexes. See also [11, Lemma 5.2.1].) So we will freely identify these categories.

There is a functor $(-)^\dagger : \mathrm{mod}_R \to \mathrm{Sh}_c(X)$, which is well known to specialists (see, for example, [14, Theorem A]), but for the reader's convenience we give a precise construction here. See [14], [17] for details.

For $M \in \mathrm{mod}_R$, set

$$\mathrm{Sp\acute{e}}(M) := \bigcup_{\emptyset \neq \sigma \in \Sigma} \sigma \times M_\sigma.$$

Let $\pi : \mathrm{Sp\acute{e}}(M) \to X$ be the projection map which sends $(p, m) \in \sigma \times M_\sigma \subset \mathrm{Sp\acute{e}}(M)$ to $p \in \sigma \subset X$. For an open subset $U \subset X$ and a map $s : U \to \mathrm{Sp\acute{e}}(M)$, we will consider the following conditions:

- $(*)$ $\pi \circ s = \mathrm{Id}_U$ and $s_q = e_{\tau,\sigma} \cdot s_p$ for all $p \in \sigma$, $q \in \tau$ with $\tau \geq \sigma$. Here s_p (resp. s_q) is the element of M_σ (resp. M_τ) with $s(p) = (p, s_p)$ (resp. $s(q) = (q, s_q)$).
- $(**)$ There is an open covering $U = \bigcup_{\lambda \in \Lambda} U_\lambda$ such that the restriction of s to U_λ satisfies $(*)$ for all $\lambda \in \Lambda$.

Now we define a sheaf $M^\dagger \in \mathrm{Sh}_c(X)$ from M as follows. For an open set $U \subset X$, set

$$M^\dagger(U) := \{\, s \mid s : U \to \mathrm{Sp\acute{e}}(M) \text{ is a map satisfying } (**) \,\}$$

and the restriction map $M^\dagger(U) \to M^\dagger(V)$ is the natural one. It is easy to see that $M^\dagger$ is a constructible sheaf. For $\sigma \in \Sigma$, let $U_\sigma := \bigcup_{\tau \geq \sigma} \tau$ be an open set of X. Then we have $M^\dagger(U_\sigma) \cong M_\sigma$. Moreover, if $\sigma \leq \tau$, then we have $U_\sigma \supset U_\tau$ and the restriction map $M^\dagger(U_\sigma) \to M^\dagger(U_\tau)$ corresponds to the multiplication map $M_\sigma \ni x \mapsto e_{\tau,\sigma} x \in M_\tau$. For a point $p \in \sigma$, the stalk $(M^\dagger)_p$ of $M^\dagger$ at p is isomorphic to M_σ. This construction gives the functor $(-)^\dagger : \mathrm{mod}_R \to \mathrm{Sh}_c(X)$. Let $0 \to M' \to M \to M'' \to 0$ be a complex in mod_R. The complex $0 \to (M')^\dagger \to M^\dagger \to (M'')^\dagger \to 0$ is exact if and only if $0 \to M'_\sigma \to M_\sigma \to M''_\sigma \to 0$ is exact for all $\emptyset \neq \sigma \in \Sigma$. Hence $(-)^\dagger$ is an exact functor. We also remark that $M_\emptyset$ is irrelevant to $M^\dagger$.

For example, we have $E(\sigma)^\dagger \cong j_* \underline{k}_{\bar\sigma}$, where j is the embedding map from the closure $\bar\sigma$ of σ to X and $\underline{k}_{\bar\sigma}$ is the constant sheaf on $\bar\sigma$. We also have that $E(\sigma)^\dagger \cong j_* \underline{k}_{\bar\sigma} \cong i_* \underline{k}_\sigma$, where $i : \sigma \to X$ is the embedding map and $\underline{k}_\sigma$ is the constant sheaf on σ. Similarly, we have $(Re_\sigma)^\dagger \cong h_! \underline{k}_{U_\sigma}$, where h is the embedding map from the open subset $U_\sigma = \bigcup_{\tau \geq \sigma} \tau$ to X.

Remark 2.1. Let $\Sigma' := \Sigma \setminus \emptyset$ be an induced subposet of Σ, and T its incidence algebra over k. Then we have a functor $\mathrm{mod}_T \to \mathrm{Sh}_c(X)$ defined in a similar way as $(-)^\dagger$, and it gives an equivalence $\mathrm{mod}_T \cong \mathrm{Sh}_c(X)$ (cf. [14, Theorem A]). On the other hand, by virtue of $\emptyset \in \Sigma$, our functor $(-)^\dagger : \mathrm{mod}_R \to \mathrm{Sh}_c(X)$ is neither full nor faithful, but we will see that mod_R has several interesting properties which mod_T does not possess.

For $M \in \mathrm{mod}_R$, set $\Gamma_\emptyset(M) := \{ x \in M_\emptyset \mid Rx \subset M_\emptyset \}$. It is easy to see that $\Gamma_\emptyset(M) \cong \mathrm{Hom}_R(k, M)$. Here we regard k as a left R-module by $e_{\sigma,\tau}\, k = 0$ for all $e_{\sigma,\tau} \neq e_\emptyset$. Clearly, $\Gamma_\emptyset$ gives a left exact functor from mod_R to itself (or vect_k). We denote the ith right derived functor of $\Gamma_\emptyset(-)$ by $H^i_\emptyset(-)$. In other words, $H^i_\emptyset(-) = \mathrm{Ext}^i_R(k, -)$.

THEOREM 2.2 (cf. [17, Theorem 3.3]). *For $M \in \mathrm{mod}_R$, we have an isomorphism*

$$H^i(X, M^\dagger) \cong H^{i+1}_\emptyset(M) \quad \textit{for all } i \geq 1,$$

and an exact sequence

$$0 \to H^0_\emptyset(M) \to M_\emptyset \to H^0(X, M^\dagger) \to H^1_\emptyset(M) \to 0.$$

Here $H^\bullet(X, M^\dagger)$ stands for the cohomology with coefficients in the sheaf $M^\dagger$.

Proof. Let $I^\bullet$ be an injective resolution of M, and consider the exact sequence

$$0 \to \Gamma_\emptyset(I^\bullet) \to I^\bullet \to I^\bullet/\Gamma_\emptyset(I^\bullet) \to 0 \tag{2.2}$$

of cochain complexes. Put $J^\bullet := I^\bullet/\Gamma_\emptyset(I^\bullet)$. Each component of $J^\bullet$ is a direct sum of copies of $E(\sigma)$ for various $\emptyset \neq \sigma \in \Sigma$. Since $E(\sigma)^\dagger$ is the constant sheaf on $\bar\sigma$ which is homeomorphic to a closed disc, we have $H^i(X, E(\sigma)^\dagger) = H^i(\bar\sigma; k) = 0$ for all $i \geq 1$. Hence $(J^\bullet)^\dagger\, (\cong (I^\bullet)^\dagger\,)$ gives a $\Gamma(X, -)$-acyclic resolution of $M^\dagger$. It is easy to see that $[J^\bullet]_\emptyset \cong \Gamma(X, (J^\bullet)^\dagger)$. So the assertions follow from (2.2), since $H^0(I^\bullet) \cong M$ and $H^i(I^\bullet) = 0$ for all $i \geq 1$. □

REMARK 2.3. (1) If $M_\emptyset = 0$, then we have $H^i(X, M^\dagger) \cong H^{i+1}_\emptyset(M)$ for all i.

(2) Let A be a commutative Noetherian homogeneous k-algebra (i.e., $A = \bigoplus_{i \geq 0} A_i$ is a graded commutative ring satisfying: (1) $A_0 = k$, (2) $\dim_k A_1 < \infty$, (3) A is generated by A_1 as a k-algebra). For a graded A-module M, we have the algebraic quasi-coherent sheaf $\tilde M$ on the projective scheme $Y := \mathrm{Proj}\, A$. It is well known that $H^i(Y, \tilde M) \cong [H^{i+1}_{\mathfrak m}(M)]_0$ for all $i \geq 1$, and

$$0 \to [H^0_{\mathfrak m}(M)]_0 \to M_0 \to H^0(Y, \tilde M) \to [H^1_{\mathfrak m}(M)]_0 \to 0 \qquad \text{(exact)}.$$

Here $H^i_{\mathfrak m}(M)$ stands for the local cohomology module with support in the irrelevant ideal $\mathfrak m = \bigoplus_{i \geq 1} A_i$, and $[H^i_{\mathfrak m}(M)]_0$ is its degree 0 component ($H^i_{\mathfrak m}(M)$ has a natural $\mathbb Z$-grading). See also Remark 4.6 (2) below and the list given in §8.

(3) Assume that Σ is a simplicial complex with n vertices. The *Stanley-Reisner ring* $k[\Sigma]$ of Σ is the quotient ring of the polynomial ring $k[x_1, \ldots, x_n]$ by the squarefree monomial ideal I_Σ corresponding to Σ (see [3], [12] for details). In [16], we defined *squarefree* $k[\Sigma]$-modules which are certain $\mathbb N^n$-graded $k[\Sigma]$-modules. For example, $k[\Sigma]$ itself is squarefree. The category $\mathrm{Sq}(\Sigma)$ of squarefree $k[\Sigma]$-modules is equivalent to mod_R of the present paper

(see [18]). Let $\Phi : \mathrm{mod}_R \to \mathrm{Sq}(\Sigma)$ be the functor giving this equivalence. In [17], we defined a functor $(-)^+ : \mathrm{Sq}(\Sigma) \to \mathrm{Sh}_c(X)$. For example, $k[\Sigma]^+ \cong \underline{k}_X$. The functor $(-)^+$ is essentially same as the functor $(-)^\dagger : \mathrm{mod}_R \to \mathrm{Sh}_c(X)$ of the present paper. More precisely, $(-)^\dagger \cong (-)^+ \circ \Phi$. For $M \in \mathrm{mod}_R$, we have $H^i_\emptyset(M) \cong [H^i_{\mathfrak{m}}(\Phi(M))]_0$. So the above theorem is a variation of [17, Theorem 3.3].

3. Dualizing complexes

Let $D^b(\mathrm{mod}_R)$ be the bounded derived category of mod_R. For $M^\bullet \in D^b(\mathrm{mod}_R)$ and $i \in \mathbb{Z}$, $M^\bullet[i]$ denotes the ith translation of $M^\bullet$, that is, $M^\bullet[i]$ is the complex with $M^\bullet[i]^j = M^{i+j}$. So, if $M \in \mathrm{mod}_R$, $M[i]$ is the cochain complex $\cdots \to 0 \to M \to 0 \to \cdots$, where M sits in the $(-i)$th position.

In this section, from Verdier's dualizing complex $\mathcal{D}^\bullet_X \in D^b(\mathrm{Sh}_c(X))$, we construct a cochain complex $\omega^\bullet$ of injective left $(R \otimes_k R)$-modules which gives a duality functor from $D^b(\mathrm{mod}_R)$ to itself. Let M be a left $(R \otimes_k R)$-module. When we regard M as a left R-module via the ring homomorphism $R \ni x \mapsto x \otimes 1 \in R \otimes_k R$ (resp. $R \ni x \mapsto 1 \otimes x \in R \otimes_k R$), we denote it by ${}_RM$ (resp. $M_{R^{\mathsf{op}}}$).

For $i \leq 1$, the ith component ω^i of $\omega^\bullet$ has a k-basis

$$\{\, e(\sigma)^\tau_\rho \mid \sigma, \tau, \rho \in \Sigma,\, \dim \sigma = -i,\, \sigma \geq \tau, \rho \,\},$$

and its module structure is defined by

$$(e_{\sigma',\,\tau'} \otimes 1) \cdot e(\sigma)^\tau_\rho = \begin{cases} e(\sigma)^\tau_{\sigma'} & \text{if } \tau' = \rho \text{ and } \sigma' \leq \sigma, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(1 \otimes e_{\sigma',\,\tau'}) \cdot e(\sigma)^\tau_\rho = \begin{cases} e(\sigma)^{\sigma'}_\rho & \text{if } \tau' = \tau \text{ and } \sigma' \leq \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have ${}_R(\omega^i) \cong (\omega^i)_{R^{\mathsf{op}}} \cong \bigoplus_{\dim \sigma = -i} E(\sigma)^{\mu(\sigma)}$ as left R-modules, where $\mu(\sigma) := \#\{\tau \in \Sigma \mid \tau \leq \sigma\}$. Note that $R \otimes_k R$ is isomorphic to the incidence algebra of the poset $\Sigma \times \Sigma$. For each $\sigma \in \Sigma$ with $\dim \sigma = -i$, we let $I(\sigma)$ be the subspace $\langle\, e(\sigma)^\tau_\rho \mid \tau,\, \rho \leq \sigma \,\rangle$ of ω^i. Then, as a left $R \otimes_k R$-module, $I(\sigma)$ is isomorphic to the injective module $E_{R \otimes_k R}(\,(\sigma, \sigma)\,)$, and $\omega^i \cong \bigoplus_{\dim \sigma = -i} I(\sigma)$. Thus $\omega^\bullet$ is of the form

$$0 \to \omega^{-d} \to \omega^{-d+1} \to \cdots \to \omega^1 \to 0,$$
$$\omega^i = \bigoplus_{\substack{\sigma \in \Sigma \\ \dim \sigma = -i}} E_{R \otimes_k R}(\,(\sigma, \sigma)\,).$$

The differential of $\omega^\bullet$ given by

$$\omega^i \ni e(\sigma)^\tau_\rho \longmapsto \sum_{\sigma' \geq \tau,\, \rho} \varepsilon(\sigma, \sigma') \cdot e(\sigma')^\tau_\rho \in \omega^{i+1}$$

makes $\omega^\bullet$ a complex of left $(R \otimes_k R)$-modules.

Let $M \in \operatorname{mod}_R$. Using the left R-module structure $I(\sigma)_{R^{\mathrm{op}}}$, we can regard $\operatorname{Hom}_R(M, {}_RI(\sigma))$ also as a left R-module. Moreover, we have the following.

Lemma 3.1. *For $M \in \operatorname{mod}_R$, we have $\operatorname{Hom}_R(M, {}_RI(\sigma)) \cong E(\sigma) \otimes_k (M_\sigma)^\vee$ as left R-modules. Here $(M_\sigma)^\vee$ is the dual vector space $\operatorname{Hom}_k(M_\sigma, k)$ of M_σ.*

Proof. First, we show that if $M_\sigma = 0$ then $\operatorname{Hom}_R(M, {}_RI(\sigma)) = 0$. Assume the contrary. If $0 \neq f \in \operatorname{Hom}_R(M, {}_RI(\sigma))$, there is some $x \in M_\tau$, $\tau < \sigma$, such that $f(x) \neq 0$. But we have $f(e_{\sigma,\tau}\, x) = e_{\sigma,\tau}\, f(x) \neq 0$. This contradicts the fact that $e_{\sigma,\tau}\, x \in M_\sigma = 0$.

For a general $M \in \operatorname{mod}_R$, let $M_{\geq \sigma} = \bigoplus_{\tau \in \Sigma,\, \tau \geq \sigma} M_\tau$ be a submodule of M. By the short exact sequence $0 \to M_{\geq\sigma} \to M \to M/M_{\geq\sigma} \to 0$ we have

$$0 \to \operatorname{Hom}_R(M/M_{\geq\sigma}, {}_RI(\sigma)) \to \operatorname{Hom}_R(M, {}_RI(\sigma)) \to \operatorname{Hom}_R(M_{\geq\sigma}, {}_RI(\sigma)) \to 0.$$

Since $(M/M_{\geq\sigma})_\sigma = 0$, we have $\operatorname{Hom}_R(M, {}_RI(\sigma)) = \operatorname{Hom}_R(M_{\geq\sigma}, {}_RI(\sigma))$. So we may assume that $M = M_{\geq\sigma}$. Let $\{f_1, \dots, f_n\}$ be a k-basis of $(M_\sigma)^\vee$. Since $({}_RI(\sigma))_\tau = 0$ for $\tau > \sigma$, $\operatorname{Hom}_R(M_{\geq\sigma}, {}_RI(\sigma))$ has a k-basis $\{\, e(\sigma)^\tau_\sigma \otimes f_i \mid \tau \leq \sigma,\ 1 \leq i \leq n \,\}$. By the module structure of $I(\sigma)_{R^{\mathrm{op}}}$, we have the expected isomorphism. □

Since each ${}_R\omega^i$ is injective, $\mathbf{D}(-) := \operatorname{Hom}^\bullet_R(-, {}_R\omega^\bullet) \cong \mathbf{R}\operatorname{Hom}_R(-, {}_R\omega^\bullet)$ gives a contravariant functor from $D^b(\operatorname{mod}_R)$ to itself. In the sequel, we simply denote $\operatorname{Hom}_R(-, {}_R\omega^i)$ by $\operatorname{Hom}_R(-, \omega^i)$, etc.

We can describe $\mathbf{D}(M^\bullet)$ explicitly. Since $\omega^i \cong \bigoplus_{\dim\sigma = -i} I(\sigma)$, we have

$$\operatorname{Hom}_R(M, \omega^i) \cong \bigoplus_{\dim\sigma=-i} \operatorname{Hom}_R(M, I(\sigma)) \cong \bigoplus_{\dim\sigma=-i} E(\sigma) \otimes_k (M_\sigma)^\vee$$

for $M \in \operatorname{mod}_R$ by Lemma 3.1. So we can easily check that $\mathbf{D}(M)$ is of the form

$$\mathbf{D}(M) : 0 \longrightarrow \mathbf{D}^{-d}(M) \longrightarrow \mathbf{D}^{-d+1}(M) \longrightarrow \cdots \longrightarrow \mathbf{D}^1(M) \longrightarrow 0,$$
$$\mathbf{D}^i(M) = \bigoplus_{\dim\sigma=-i} E(\sigma) \otimes_k (M_\sigma)^\vee.$$

Here the differential sends $e(\sigma)_\rho \otimes f \in E(\sigma) \otimes_k (M_\sigma)^\vee$ to

$$\sum_{\tau\in\Sigma,\, \tau\geq\rho} \varepsilon(\sigma,\tau)\cdot e(\tau)_\rho \otimes f(e_{\sigma,\tau}-) \ \in \bigoplus_{\dim\tau=\dim\sigma-1} E(\tau) \otimes_k (M_\tau)^\vee.$$

For a bounded cochain complex $M^\bullet$ of objects in mod_R, we have

$$\mathbf{D}^t(M^\bullet) = \bigoplus_{i-j=t} \mathbf{D}^i(M^j) = \bigoplus_{-\dim\sigma - j = t} E(\sigma) \otimes_k (M^j_\sigma)^\vee,$$

and the differential is given by

$$\mathbf{D}^t(M^\bullet) \supset E(\sigma) \otimes_k (M^j_\sigma)^\vee \ni x \otimes y \mapsto d(\, x \otimes y \,) + (-1)^t (x \otimes \partial^\vee(y)) \in \mathbf{D}^{t+1}(M^\bullet),$$

where $\partial^\vee : (M^j_\sigma)^\vee \to (M^{j-1}_\sigma)^\vee$ is the k-dual of the differential ∂ of $M^\bullet$, and d is the differential of $\mathbf{D}(M^j)$.

Since the underlying space X of Σ is locally compact and finite dimensional, it admits Verdier's dualizing complex $\mathcal{D}^\bullet_X \in D^b(\mathrm{Sh}(X))$ with coefficients in k (see [6, V. §2]).

THEOREM 3.2. *For $M^\bullet \in D^b(\mathrm{mod}_R)$, we have*

$$\mathbf{D}(M^\bullet)^\dagger \cong \mathbf{R}\mathcal{H}om((M^\bullet)^\dagger, \mathcal{D}^\bullet_X) \quad \text{in } D^b(\mathrm{Sh}_c(X)).$$

Proof. An explicit description of $\mathbf{R}\mathcal{H}om((M^\bullet)^\dagger, \mathcal{D}^\bullet_X)$ is given in the unpublished thesis [11] of A. Shepard. When Σ is a simplicial complex, this description is treated in [14, §2.4], and also follows from the author's previous paper [17] (and [18]). The general case can be reduced to the simplicial complex case using the barycentric subdivision.

Shepard's description of $\mathbf{R}\mathcal{H}om((M^\bullet)^\dagger, \mathcal{D}^\bullet_X)$ is the same thing as the above description of $\mathbf{D}(M^\bullet)$ under the functor $(-)^\dagger$. □

LEMMA 3.3. *For each $\sigma \in \Sigma$, the natural map $E(\sigma) \to \mathbf{D} \circ \mathbf{D}(E(\sigma))$ is an isomorphism in $D^b(\mathrm{mod}_R)$.*

Proof. We may assume that $\sigma \neq \emptyset$. Let $\Sigma|_\sigma := \{\, \tau \in \Sigma \mid \tau \leq \sigma \,\}$ be a subcomplex of Σ. It is easy to see that $\mathbf{D}(E(\sigma))_\emptyset$ is isomorphic to the chain complex $C_\bullet(\Sigma|_\sigma, k)$ of $\Sigma|_\sigma$. Thus $H^i(\mathbf{D}(E(\sigma)))_\emptyset = \tilde{H}_{-i}(\bar{\sigma}; k)$ for all i, where $\tilde{H}_\bullet(\bar{\sigma}; k)$ stands for the reduced homology group of the closure $\bar{\sigma}$ of σ. Hence $H^i(\mathbf{D}(E(\sigma)))_\emptyset = 0$ for all i.

By Theorem 3.2 and the Verdier duality, we have

$$\mathbf{D}(E(\sigma))^\dagger \cong \mathbf{R}\mathcal{H}om(j_*\underline{k}_\sigma, \mathcal{D}^\bullet_X) \cong j_!\underline{k}_\sigma[\dim \sigma].$$

Here $j : \sigma \to X$ is the embedding map.

Let M be a simple R-module with $M = M_\sigma \cong k$. Combining the above observations, we have $\mathbf{D}(E(\sigma)) \cong M[\dim \sigma]$. So $\mathbf{D} \circ \mathbf{D}(E(\sigma)) \cong \mathbf{D}(M[\dim \sigma]) \cong E(\sigma)$, and the natural map $E(\sigma) \to \mathbf{D} \circ \mathbf{D}(E(\sigma))$ is an isomorphism. □

THEOREM 3.4.

(1) $\omega^\bullet \in D^b(\mathrm{mod}_{R \otimes_k R})$ *is a dualizing complex in the sense of* [19, Definition 1.1]. *Hence $\mathbf{D}(-)$ is a duality functor from $D^b(\mathrm{mod}_R)$ to itself.*
(2) *The dualizing complex $\omega^\bullet$ satisfies the Auslander condition in the sense of* [19, Definition 2.1]. *That is, if we set*

$$j_\omega(M) := \inf \{\, i \mid \mathrm{Ext}^i_R(M, \omega^\bullet) \neq 0 \,\} \in \mathbb{Z} \cup \{\infty\},$$

then, for all $i \in \mathbb{Z}$ and all $M \in \mathrm{mod}_R$, any submodule N of $\mathrm{Ext}^i_R(M, \omega^\bullet)$ satisfies $j_\omega(N) \geq i$.

Proof. (1) The conditions (i) and (ii) of [19, Definition 1.1] obviously hold in our case, so it remains to prove that condition (iii) also holds. To see this, it suffices to show that the natural morphism $R \to \mathbf{D} \circ \mathbf{D}(R)$ is an isomorphism. But it follows from "Lemma on Way-out Functors" ([5, Proposition 7.1]) and Lemma 3.3.

(2) We may assume that $M \neq 0$. By the description of $\mathbf{D}(M)$, we have

$$j_\omega(M) = -\max\{\, \dim \sigma \mid \sigma \in \Sigma, M_\sigma \neq 0 \,\}$$

and $\operatorname{Ext}^i_R(M, \omega^\bullet)_\sigma = 0$ for $\sigma \in \Sigma$ with $\dim \sigma > -i$. Hence, any submodule $N \subset \operatorname{Ext}^i_R(M, \omega^\bullet)$ satisfies $j_\omega(N) \geq i$. □

Corollary 3.5. *We have* $\operatorname{Ext}^i_R(M^\bullet, \omega^\bullet)_\emptyset \cong H^{-i+1}_\emptyset(M^\bullet)^\vee$ *for all* $i \in \mathbb{Z}$ *and all* $M^\bullet \in D^b(\operatorname{mod}_R)$.

Proof. Since $\mathbf{D} \circ \mathbf{D}(M^\bullet)$ is an injective resolution of $M^\bullet$, we have $\mathbf{R}\Gamma_\emptyset(M^\bullet) = \Gamma_\emptyset(\mathbf{D} \circ \mathbf{D}(M^\bullet))$. By the structure of $\mathbf{D}(-)$, we have $\Gamma_\emptyset(\mathbf{D} \circ \mathbf{D}(M^\bullet)) = (\mathbf{D}(M^\bullet)_\emptyset)^\vee[-1]$. So we are done. □

4. Categorical Remarks

For $M, N \in \operatorname{mod}_R$ and $\sigma \in \Sigma$, set $\underline{\operatorname{Hom}}_R(M,N)_\sigma := \operatorname{Hom}_R(M_{\geq \sigma}, N)$. We make $\underline{\operatorname{Hom}}_R(M,N) := \bigoplus_{\sigma \in \Sigma} \underline{\operatorname{Hom}}_R(M,N)_\sigma$ a left R-module as follows: For $f \in \underline{\operatorname{Hom}}_R(M,N)_\sigma$ and a cell τ with $\tau \geq \sigma$, we let $e_{\tau,\sigma} f$ be the restriction of f into the submodule $M_{\geq \tau}$ of $M_{\geq \sigma}$.

Lemma 4.1. *For* $M \in \operatorname{mod}_R$*, we have* $\underline{\operatorname{Hom}}_R(M, E(\sigma)) \cong E(\sigma) \otimes_k (M_\sigma)^\vee$.

Proof. Similar to Lemma 3.1. □

If a complex $M^\bullet$ is exact, then so is $\underline{\operatorname{Hom}}_R(M^\bullet, E(\sigma))$ by Lemma 4.1. By the usual argument on double complexes, if $M^\bullet$ is bounded and exact, and $I^\bullet$ is bounded and each I^i is injective, then $\underline{\operatorname{Hom}}^\bullet_R(M^\bullet, I^\bullet)$ is exact.

Note that Σ is a *meet-semilattice* (see [13, §3.3]) as a poset if and only if, for any two cells $\sigma, \tau \in \Sigma$ with $\bar{\sigma} \cap \bar{\tau} \neq \emptyset$, there is a cell $\rho \in \Sigma$ with $\bar{\sigma} \cap \bar{\tau} = \bar{\rho}$. If Σ is a simplicial complex, or more generally, a polyhedral complex, then it is a meet-semilattice. If Σ is a meet-semilattice, for two cells $\sigma, \tau \in \Sigma$, either there is no upper bound for σ and τ (i.e., no cell $\rho \in \Sigma$ satisfies $\rho \geq \sigma$ and $\rho \geq \tau$), or there is the least element $\sigma \vee \tau$ in $\{\, \rho \in \Sigma \mid \rho \geq \sigma, \tau \,\}$ (cf. [13, Proposition 3.3.1]).

Assume that Σ is a meet-semilattice. Consider $\underline{\operatorname{Hom}}_R(Re_\sigma, N)_\tau$ for $N \in \operatorname{mod}_R$ and $\tau \in \Sigma$. If $\sigma \vee \tau$ exists, then we have $\underline{\operatorname{Hom}}_R(Re_\sigma, N)_\tau = N_{\sigma \vee \tau}$. Otherwise, there is no upper bound for σ and τ, and $\underline{\operatorname{Hom}}_R(Re_\sigma, N)_\tau = 0$. Hence the complex $\underline{\operatorname{Hom}}_R(Re_\sigma, N^\bullet)$ is exact for an exact complex $N^\bullet$. Hence if $N^\bullet$ is bounded and exact, and $P^\bullet$ is bounded and each P^i is projective, then $\underline{\operatorname{Hom}}^\bullet_R(P^\bullet, N^\bullet)$ is exact.

By the above remarks, we have the following lemma (see [7, I.1.10] for the derived functor of a bifunctor).

LEMMA 4.2. *For $M^\bullet, N^\bullet \in D^b(\mathrm{mod}_R)$, we have:*

(1) *If $I^\bullet$ is an injective resolution of $N^\bullet$, then*

$$\mathbf{R}\underline{\mathrm{Hom}}_R(M^\bullet, N^\bullet) \cong \underline{\mathrm{Hom}}^\bullet_R(M^\bullet, I^\bullet).$$

(2) *If Σ is a meet-semilattice as a poset (e.g., Σ is a simplicial complex), then*

$$\mathbf{R}\underline{\mathrm{Hom}}_R(M^\bullet, N^\bullet) \cong \underline{\mathrm{Hom}}^\bullet_R(P^\bullet, N^\bullet)$$

for a projective resolution $P^\bullet$ of $M^\bullet$.

EXAMPLE 4.3. The additional assumption in Lemma 4.2 (2) is indeed necessary, that is, $\mathbf{R}\underline{\mathrm{Hom}}_R(M^\bullet, N^\bullet) \not\cong \underline{\mathrm{Hom}}^\bullet_R(P^\bullet, N^\bullet)$ in general.

For example, let X be a closed 2 dimensional disc, and Σ a regular cell decomposition of X consisting of one 2-cell (say, σ), two 1-cells (say, τ_1, τ_2), and two 0-cells (say, ρ_1, ρ_2). Since $\bar\tau_1 \cap \bar\tau_2 = \rho_1 \cup \rho_2$, Σ is not a meet-semilattice.

Let N be a left R-module with $N = N_\sigma = k$. Then an injective resolution of N is of the form

$$I^\bullet : 0 \to E(\sigma) \to E(\tau_1) \oplus E(\tau_2) \to E(\rho_1) \oplus E(\rho_2) \to E(\emptyset) \to 0.$$

We have

$$\underline{\mathrm{Hom}}_R(Re_{\rho_1}, E(\sigma))_{\rho_2} = \underline{\mathrm{Hom}}_R(Re_{\rho_1}, E(\tau_1))_{\rho_2} = \underline{\mathrm{Hom}}_R(Re_{\rho_1}, E(\tau_2))_{\rho_2} = k$$

and

$$\underline{\mathrm{Hom}}_R(Re_{\rho_1}, E(\rho_1))_{\rho_2} = \underline{\mathrm{Hom}}_R(Re_{\rho_1}, E(\rho_2))_{\rho_2} = 0.$$

Thus $\underline{\mathrm{Ext}}^1_R(Re_{\rho_1}, N)_{\rho_2} = H^1(\underline{\mathrm{Hom}}(Re_{\rho_1}, I^\bullet))_{\rho_2} \neq 0$, while Re_{ρ_1} is a projective module.

PROPOSITION 4.4. *If $M^\bullet \in D^b(\mathrm{mod}_R)$, then*

$$\mathbf{D}(M^\bullet) \cong \mathbf{R}\underline{\mathrm{Hom}}_R(M^\bullet, \mathbf{D}(Re_\emptyset)).$$

Proof. Since $\mathbf{D}(Re_\emptyset)$ is of the form

$$0 \to D^{-d} \to D^{-d+1} \to \cdots \to D^1 \to 0$$

with $D^i = \bigoplus_{\dim\sigma = -i} E(\sigma)$, the assertion follows from Lemmas 4.1 and 4.2. □

Since $(Re_\emptyset)^\dagger \cong \underline{k}_X$, we have $\mathcal{D}^\bullet_X \cong \mathbf{D}(\underline{k}_X) \cong \mathbf{D}(Re_\emptyset)^\dagger$ by Proposition 4.4.

If $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}_c(X)$, then it is easy to see that $\mathcal{H}om(\mathcal{F}, \mathcal{G}) \in \mathrm{Sh}_c(X)$. For $M, N \in \mathrm{mod}_R$ and $\emptyset \neq \sigma \in \Sigma$, we have

$$\begin{aligned}\mathcal{H}om(M^\dagger, N^\dagger)(U_\sigma) &= \mathrm{Hom}_{\mathrm{Sh}(U_\sigma)}(M^\dagger|_{U_\sigma}, N^\dagger|_{U_\sigma}) \cong \mathrm{Hom}_R(M_{\geq\sigma}, N_{\geq\sigma}) \\ &= \mathrm{Hom}_R(M_{\geq\sigma}, N) = \underline{\mathrm{Hom}}_R(M, N)_\sigma.\end{aligned}$$

Hence

$$\underline{\mathrm{Hom}}_R(M,N)^\dagger \cong \mathcal{H}om(M^\dagger, N^\dagger).$$

For $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D^b(\mathrm{Sh}_c(X))$, it is known that $\mathbf{R}\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \in D^b(\mathrm{Sh}_c(X))$ (see [7, Proposition 8.4.10]). Thus we can use an injective resolution of $\mathcal{G}^\bullet$ in $D^b(\mathrm{Sh}_c(X))$ to compute $\mathbf{R}\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$. If $I^\bullet$ is an injective resolution of $N^\bullet \in D^b(\mathrm{mod}_R)$, then $(I^\bullet)^\dagger$ is an injective resolution of $(N^\bullet)^\dagger$ in $D^b(\mathrm{Sh}_c(X))$. Hence we have the following.

Proposition 4.5 ([11, Theorem 5.2.5]). *If* $M^\bullet, N^\bullet \in D^b(\mathrm{mod}_R)$*, then*

$$\mathbf{R}\underline{\mathrm{Hom}}_R(M^\bullet, N^\bullet)^\dagger \cong \mathbf{R}\mathcal{H}om((M^\bullet)^\dagger, (N^\bullet)^\dagger).$$

By Lemma 4.2 (2), if Σ is a meet-semilattice, then $\mathbf{R}\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ for $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D^b(\mathrm{Sh}_c(X))$ can be computed using a projective resolution of $\mathcal{F}^\bullet$ in $D^b(\mathrm{Sh}_c(X))$.

Remark 4.6. (1) Let J be the left ideal of R generated by $\{ e_{\sigma,\emptyset} \mid \sigma \neq \emptyset \}$. Note that $J^\dagger \cong \underline{k}_X$. Then we have that $\underline{\mathrm{Hom}}_R(J,M)^\dagger \cong M^\dagger$ and $\underline{\mathrm{Hom}}_R(J,M)_\emptyset \cong \Gamma(X, M^\dagger)$. Moreover, we have $\underline{\mathrm{Ext}}^i_R(J,M) = \underline{\mathrm{Ext}}^i_R(J,M)_\emptyset \cong H^i(X, M^\dagger)$ for all $i \geq 1$ by an argument similar to that in the proof of Theorem 2.2.

(2) Let $\mathrm{mod}_\emptyset$ be the full subcategory of mod_R consisting of modules M with $M_\sigma = 0$ for all $\sigma \neq \emptyset$. Then $\mathrm{mod}_\emptyset$ is a *dense subcategory* of mod_R. That is, for a short exact sequence $0 \to M' \to M \to M'' \to 0$ in mod_R, M is in $\mathrm{mod}_\emptyset$ if and only if M' and M'' are in $\mathrm{mod}_\emptyset$. So we have the quotient category $\mathrm{mod}_R / \mathrm{mod}_\emptyset$ by [10, Theorem 4.3.3]. Let $\pi : \mathrm{mod}_R \to \mathrm{mod}_R / \mathrm{mod}_\emptyset$ be the canonical functor. It is easy to see that $\pi(M) \cong \pi(M')$ if and only if $M_{>\emptyset} \cong M'_{>\emptyset}$. Moreover, we have $\mathrm{Sh}_c(X) \cong \mathrm{mod}_R / \mathrm{mod}_\emptyset$.

Let the notation be as in (1) of this remark. Then $\underline{\mathrm{Hom}}_R(J, -)$ gives a functor $\eta : \mathrm{mod}_R / \mathrm{mod}_\emptyset \to \mathrm{mod}_R$ with $\pi \circ \eta = \mathrm{Id}$. Moreover, η is a *section functor* (cf. [10, §4.4]) and $\mathrm{mod}_\emptyset$ is a *localizing subcategory* of mod_R.

Let $A = \bigoplus_{i \geq 0} A_i$ be a commutative Noetherian homogeneous k-algebra as in Remark 2.3 (2) and Gr_A the category of graded A-modules. We say $M \in \mathrm{Gr}_A$ is a *torsion* module if for all $x \in M$ there is some $i \in \mathbb{N}$ with $A_{\geq i} \cdot x = 0$. Let Tor_A be the full subcategory of Gr_A consisting of torsion modules. Clearly, Tor_A is dense in Gr_A. It is well known that the category $\mathrm{Qco}(Y)$ of quasi-coherent sheaves on the projective scheme $Y := \mathrm{Proj}\, A$ is equivalent to the quotient category $\mathrm{Gr}_A / \mathrm{Tor}_A$, and we have the section functor $\mathrm{Qco}(Y) \to \mathrm{Gr}_A$ given by $\mathcal{F} \mapsto \bigoplus_{i \in \mathbb{Z}} H^0(Y, \mathcal{F}(i))$. So Tor_A is a localizing subcategory of Gr_A. In this sense, our $\mathrm{Sh}_c(X) \cong \mathrm{mod}_R / \mathrm{mod}_\emptyset$ is a small imitation of $\mathrm{Qco}(Y) \cong \mathrm{Gr}_A / \mathrm{Tor}_A$.

5. Cohomologies of sheaves on open subsets

Let $\Psi \subset \Sigma$ be an order filter of the poset Σ. That is, $\sigma \in \Psi$, $\tau \in \Sigma$, and $\tau \geq \sigma$ imply $\tau \in \Psi$. Then $U_\Psi := \bigcup_{\sigma \in \Psi} \sigma$ is an open subset of X. If $M \in \mathrm{mod}_R$, $M_\Psi := \bigoplus_{\sigma \in \Psi} M_\sigma$ is a submodule of M. It is easy to see that $(M_\Psi)^\dagger \cong j_! j^* M^\dagger$, where $j : U_\Psi \to X$ is the embedding map. If $\Psi = \{\tau \mid \tau \geq \sigma\}$ for some $\sigma \in \Sigma$, then U_Ψ and M_Ψ are denoted by U_σ and $M_{\geq \sigma}$, respectively.

Lemma 5.1. *Let $\Psi \subset \Sigma$ be an order filter with $\Psi \not\ni \emptyset$. Then we have the following isomorphisms for all $i \in \mathbb{Z}$ and $M \in \mathrm{mod}_R$.*

(1) $H^{i+1}_\emptyset(M_\Psi) \cong H^i_c(U_\Psi, M^\dagger|_{U_\Psi})$ *for all i.*

(2) $\mathrm{Ext}^i_R(M, \omega^\bullet)_\sigma \cong H^{-i+1}_\emptyset(M_{\geq\sigma})^\vee \cong H^{-i}_c(U_\sigma, M^\dagger|_{U_\sigma})^\vee$ *for all $\emptyset \neq \sigma \in \Sigma$.*

Proof. (1) We have

$$H^{i+1}_\emptyset(M_\Psi) \cong H^i(X, (M_\Psi)^\dagger) \cong H^i(X, j_! j^* M^\dagger) \cong H^i_c(U_\Psi, M^\dagger|_{U_\Psi}).$$

Here, by Remark 2.3 (1), the first isomorphism holds even if $i = 0$.

(2) By the description of $\mathbf{D}(M)$, we have $\mathbf{D}(M)_\sigma \cong \mathbf{D}(M_{\geq\sigma})_\emptyset$. Hence we have

$$\mathrm{Ext}^i_R(M, \omega^\bullet)_\sigma \cong \mathrm{Ext}^i_R(M_{\geq\sigma}, \omega^\bullet)_\emptyset \cong H^{-i+1}_\emptyset(M_{\geq\sigma})^\vee \cong H^{-i}_c(U_\sigma, M^\dagger|_{U_\sigma})^\vee.$$

Here the second isomorphism follows from Corollary 3.5. □

Proposition 5.2. *For any $\sigma \in \Sigma$, $\mathbf{D}(Re_\sigma)^\dagger \cong \mathbf{R}j_* \mathcal{D}^\bullet_{U_\sigma}$, where $j : U_\sigma \to X$ is the embedding map. In particular, $\mathbf{D}(Re_\emptyset)^\dagger \cong \mathcal{D}^\bullet_X$.*

Proof. Set $U := U_\sigma$. Since $(Re_\sigma)^\dagger \cong j_! \underline{k}_U$, we have

$$\begin{aligned}\mathbf{D}(Re_\sigma)^\dagger &\cong \mathbf{R}\mathcal{H}om(\, j_! \underline{k}_U,\, \mathcal{D}^\bullet_X\,) && \text{(by Theorem 3.2)}\\ &\cong \mathbf{R}j_* \mathbf{R}\mathcal{H}om(\, \underline{k}_U,\, j^* \mathcal{D}^\bullet_X\,) && \text{(by [6, VII. Theorem 5.2])}\\ &\cong \mathbf{R}j_* \mathbf{R}\mathcal{H}om(\, \underline{k}_U,\, \mathcal{D}^\bullet_U\,) \cong \mathbf{R}j_* \mathcal{D}^\bullet_U.\end{aligned}$$

□

Motivated by Lemma 5.1, we give a formula for the ordinary (not compact support) cohomology $H^i(U_\Psi, M^\dagger|_{U_\Psi})$.

Theorem 5.3. *Let $\Psi \subset \Sigma$ be an order filter with $\Psi \not\ni \emptyset$. We have*

$$H^i(U_\Psi, M^\dagger|_{U_\Psi}) \cong [\, \mathrm{Ext}^i_R(\, \mathbf{D}(M)_\Psi,\, \omega^\bullet\,)\,]_\emptyset$$

for all $i \in \mathbb{N}$ and $M \in \mathrm{mod}_R$.

Proof. For simplicity set $U := U_\Psi$. Let $\mathcal{F}^\bullet \in D^b(\mathrm{Sh}(U))$. Taking a complex in the isomorphic class of $\mathcal{F}^\bullet$, we may assume that each component $\mathcal{F}^i$ is a direct sum of sheaves of the form $h_! \underline{k}_V$, where V is an open subset of U with the embedding map $h : V \to U$ (see [6, II. Proposition 7.4]). Since each component $\mathcal{D}_U^i$ of $\mathcal{D}_U^\bullet$ is an injective sheaf, $h^*\mathcal{D}_U^i$ is also injective by [6, II. Corollary 6.10], and we have

$$\mathcal{H}om(h_!\underline{k}_V, \mathcal{D}_U^i) \cong \mathbf{R}h_*\mathbf{R}\mathcal{H}om(\underline{k}_V, h^*\mathcal{D}_U^i) \cong \mathbf{R}h_*(h^*\mathcal{D}_U^i) \cong h_*h^*\mathcal{D}_U^i$$

by [6, VII, Theorem 5.2]. Since the sheaf $h_*h^*\mathcal{D}_U^i$ is flabby, $\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{D}_U^\bullet)$ is a complex of flabby sheaves. Hence we have

$$\begin{aligned}\operatorname{Ext}^i_{\operatorname{Sh}(U)}(\,\mathcal{F}^\bullet, \mathcal{D}_U^\bullet\,) &\cong H^i(\,\Gamma(U, \mathcal{H}om^\bullet(\,\mathcal{F}^\bullet,\, \mathcal{D}_U^\bullet\,)\,)\\ &\cong \mathbf{R}^i\Gamma(\,U, \mathbf{R}\mathcal{H}om(\,\mathcal{F}^\bullet,\, \mathcal{D}_U^\bullet\,)\,).\end{aligned}$$

Since $\mathbf{R}\mathcal{H}om(\,\mathbf{R}\mathcal{H}om(M^\dagger|_U, \mathcal{D}_U^\bullet),\, \mathcal{D}_U^\bullet\,) \cong M^\dagger|_U$ in $D^b(\operatorname{Sh}(U))$, we have

$$\begin{aligned}H^i(U, M^\dagger|_U) &\cong \mathbf{R}^i\Gamma(\,U,\, \mathbf{R}\mathcal{H}om(\,\mathbf{R}\mathcal{H}om(M^\dagger|_U, \mathcal{D}_U^\bullet),\, \mathcal{D}_U^\bullet)\,)\\ &\cong \operatorname{Ext}^i_{\operatorname{Sh}(U)}(\,\mathbf{R}\mathcal{H}om(M^\dagger|_U, \mathcal{D}_U^\bullet),\, \mathcal{D}_U^\bullet\,)\\ &\cong \mathbf{R}^{-i}\Gamma_c(U, \mathbf{R}\mathcal{H}om(M^\dagger|_U, \mathcal{D}_U^\bullet))^\vee \quad \text{(by [6, V, Theorem 2.1])}\\ &\cong \mathbf{R}^{-i}\Gamma_c(U, \mathbf{R}\mathcal{H}om(M^\dagger, \mathcal{D}_X^\bullet)|_U)^\vee\\ &\cong \mathbf{R}^{-i}\Gamma_c(U, \mathbf{D}(M)^\dagger|_U)^\vee\\ &\cong \mathbf{R}^{-i+1}\Gamma_\emptyset(U, \mathbf{D}(M)_\Psi)^\vee \qquad \text{(by Lemma 5.1)}\\ &\cong (\,\operatorname{Ext}^i_R(\,\mathbf{D}(M)_\Psi,\, \omega^\bullet\,)_\emptyset\,) \qquad \text{(by Corollary 3.5).} \ \square\end{aligned}$$

Example 5.4. Assume that X is a d-dimensional manifold (in this paper, the word "manifold" always means a manifold with or without boundary, as in [6]) and $\Psi \subset \Sigma$ is an order filter with $\Psi \not\ni \emptyset$. We denote the *orientation sheaf* of X over k (cf. [6, V.§3]) by or_X. Thus we have $or_X[d] \cong \mathcal{D}_X^\bullet$ in $D^b(\operatorname{Sh}(X))$. Let $U := U_\Psi$ be an open subset with the embedding map $j : U \to X$. We have

$$(\mathbf{D}(Re_\emptyset)_\Psi)^\dagger \cong j_!j^*\mathbf{D}(Re_\emptyset)^\dagger \cong j_!j^*\mathcal{D}_X^\bullet \cong j_!\mathcal{D}_U^\bullet \cong (j_!or_U)[d].$$

Thus

$$[\,\operatorname{Ext}^i_R(\,\mathbf{D}(Re_\emptyset)_\Psi,\, \omega^\bullet\,)\,]_\emptyset \cong H_\emptyset^{-i+1}(\mathbf{D}(Re_\emptyset)_\Psi)^\vee \cong H_c^{d-i}(U, or_U)^\vee.$$

But we have $H^i(U;k) \cong H_c^{d-i}(U, or_U)^\vee$ by the Poincaré duality. So equality in Theorem 5.3 can actually hold.

For a finite poset P, the *order complex* $\Delta(P)$ is the set of chains of P. Recall that a subset C of P is a *chain* if any two elements of C are comparable. Obviously, $\Delta(P)$ is an (abstract) simplicial complex. The geometric realization of the order complex $\Delta(\Sigma')$ of $\Sigma' := \Sigma \setminus \emptyset$ is homeomorphic to the underlying space X of Σ.

We say a finite regular cell complex Σ is *Cohen-Macaulay* (resp. *Buchsbaum*) over k if $\Delta(\Sigma')$ is Cohen-Macaulay (resp. Buchsbaum) over k in the sense of [12, II.§§3-4] (resp. [12, II.§8]). (If Σ is a simplicial complex, we can use Σ directly instead of $\Delta(\Sigma')$.) These are topological properties of the underlying space X. In fact, Σ is Buchsbaum if and only if $\mathcal{H}^i(\mathcal{D}_X^\bullet) = 0$ for all $-i \neq d := \dim X$ (see [17, Corollary 4.7]). For example, if X is a manifold, Σ is Buchsbaum. Similarly, Σ is Cohen-Macaulay if and only if it is Buchsbaum and $\tilde{H}^i(X;k) = 0$ for all $i < d$.

We have

$$H^i(\mathbf{D}(Re_\emptyset))_\emptyset = \mathrm{Ext}^i_R(Re_\emptyset, \omega^\bullet)_\emptyset \cong H_\emptyset^{-i+1}(Re_\emptyset)^\vee \cong \tilde{H}^{-i}(X;k)^\vee$$

for all $i \in \mathbb{Z}$ by Corollary 3.5 and Theorem 2.2. Recall that $\mathbf{D}(Re_\emptyset)^\dagger \cong \mathcal{D}_X^\bullet$. So $H^i(\mathbf{D}(Re_\emptyset)) = 0$ for all $i \neq -d$ if and only if X is Cohen-Macaulay over k. In general, $H^i(\omega^\bullet)^\dagger$ can be non-zero for some $i \neq -d$ even if X is Cohen-Macaulay. For example, let X be a closed 2-dimensional disc, and Σ the regular cell decomposition of X given in Example 4.3. Then the "ρ_1-ρ_2 component" $(\omega^\bullet)^{\rho_1}_{\rho_2}$ of $\omega^\bullet$ is of the form

$$0 \to E(\sigma)^{\rho_1}_{\rho_2} \to E(\tau_1)^{\rho_1}_{\rho_2} \oplus E(\tau_2)^{\rho_1}_{\rho_2} \to 0.$$

Thus $H^{-1}(\omega^\bullet)^{\rho_1}_{\rho_2} \neq 0$, while X is Cohen-Macaulay. However, we have the following result.

Proposition 5.5. *Assume that Σ is a meet-semilattice as a poset (e.g., Σ is a simplicial complex). Then we have:*

(1) $H^i(\omega^\bullet) = 0$ *for all $i \neq -d$ if and only if Σ is Cohen-Macaulay over k.*
(2) $H^i(\omega^\bullet)^\dagger = 0$ *for all $i \neq -d$ if and only if Σ is Buchsbaum over k.*

Proof. (1) Since $\omega^\bullet \cong \mathbf{D}(R) \cong \bigoplus_{\sigma\in\Sigma} \mathbf{D}(Re_\sigma)$, the "only if" part is clear by the argument preceding the proposition. To prove the "if" part, we assume that Σ is Cohen-Macaulay. Set $\Omega := H^{-d}(\mathbf{D}(Re_\emptyset))$. Then $\Omega[d] \cong \mathbf{D}(Re_\emptyset)$ in $D^b(\mathrm{mod}_R)$. By Proposition 4.4, we have $\mathbf{D}(Re_\sigma) \cong \mathbf{R}\underline{\mathrm{Hom}}_R(Re_\sigma, \Omega[d])$. Since Re_σ is a projective module, we have $\underline{\mathrm{Ext}}^i_R(Re_\sigma, \Omega) = 0$ for all $i > 0$ by Lemma 4.2. Thus $H^i(\mathbf{D}(Re_\sigma)) = 0$ for all $i \neq -d$.

(2) Similar to (1). □

Remark 5.6. By [17, Proposition 4.10], Proposition 5.5 (1) states that if Σ is a Cohen-Macaulay simplicial complex, the relative simplicial complex $(\Sigma, \mathrm{del}_\Sigma(\sigma))$ is Cohen-Macaulay in the sense of [12, III.§7] for all $\sigma \in \Sigma$. Here $\mathrm{del}_\Sigma(\sigma) := \{\, \tau \in \Sigma \mid \tau \not\geq \sigma \,\}$ is a subcomplex of Σ.

Example 5.7. (1) We say that a finite regular cell complex Σ of dimension d is *Gorenstein** over k (see [12, p. 67]), if the order complex $\Delta := \Delta(\Sigma')$ of $\Sigma' := \Sigma \setminus \emptyset$ is Cohen-Macaulay over k (that is, $\tilde{H}_i(\mathrm{lk}_\Delta\, \sigma; k) = 0$ for all $\sigma \in \Sigma$ and all $i \neq d - \dim\sigma - 1$; see [12, II. Corollary 4.2]) and $\tilde{H}_{d-\dim\sigma-1}(\mathrm{lk}_\Delta\, \sigma; k) = k$ for all $\sigma \in \Delta$. (If Σ is a simplicial complex, we can use Σ directly instead of Δ.) This is a topological property of the underlying space X. For example, if X is homeomorphic to a d-dimensional sphere, then Σ is Gorenstein* (over any k). Note that Σ is Gorenstein* over k if and only if it is Cohen-Macaulay over k and Eulerian (cf. [13]) as a poset.

It is easy to see that $\mathbf{D}(Re_\emptyset) \cong (Re_\emptyset)[d]$ in $D^b(\mathrm{mod}_R)$ if and only if X is Gorenstein*. If Σ is a Gorenstein* *simplicial* complex, then $\omega^\bullet \cong \Omega[d]$ for some $\Omega \in \mathrm{mod}_{R\otimes_k R}$ by Proposition 5.5. Moreover, we can describe Ω explicitly. In fact, Ω has a k-basis $\{\, e^\sigma_\tau \mid \sigma, \tau \in \Sigma, \sigma \cup \tau \in \Sigma \,\}$ and its module structure is defined by

$$(e_{\sigma',\tau'} \otimes 1) \cdot e^\tau_\rho = \begin{cases} e^\tau_{\sigma'} & \text{if } \tau' = \rho \text{ and } \sigma' \cup \tau \in \Sigma, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(1 \otimes e_{\sigma',\tau'}) \cdot e^\tau_\rho = \begin{cases} e^{\sigma'}_\rho & \text{if } \tau' = \tau \text{ and } \sigma' \cup \rho \in \Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

To check this, note that the "τ-ρ component" $(\omega^\bullet)^\tau_\rho$ of $\omega^\bullet = \langle\, e(\sigma)^\tau_\rho \mid \sigma \geq \tau, \rho \,\rangle$ is isomorphic to $\tilde{C}_{-n-\bullet}(\mathrm{lk}_\Sigma(\tau \cup \rho))$ as a complex of k-vector spaces, where $\tilde{C}_\bullet(\mathrm{lk}_\Sigma(\tau \cup \rho))$ is the augmented chain complex of $\mathrm{lk}_\Sigma(\tau \cup \rho)$ and $n = \dim(\tau \cup \rho) + 1$. So the description follows from the Gorenstein* property of Σ. It is easy to see that $\mathbf{D}(Re_\sigma) \cong \langle\, e^\sigma_\tau \mid \tau \in \mathrm{st}_\Sigma\, \sigma \,\rangle$. So we have $\mathbf{R}j_*\mathcal{D}^\bullet_{U_\sigma} \cong j_*\underline{k}_{U_\sigma}[d]$, where $j : U_\sigma \to X$ is the embedding map ($j_*\underline{k}_{U_\sigma}$ is essentially the constant sheaf on the closure $\bar{U}_\sigma$ of U_σ).

(2) Let Σ be a finite simplicial complex of dimension d, and V the set of its vertices. Assume that Σ is Gorenstein in the sense of [12, II.§5]. Then there is a subset $W \subset V$ and a Gorenstein* simplicial complex $\Delta \subset 2^{V\setminus W}$ such that $\Sigma = 2^W * \Delta$, where "$*$" stands for the simplicial join. (The Gorenstein property depends on the particular simplicial decomposition of X.) Since a Gorenstein simplicial complex is Cohen-Macaulay, there is $\Omega \in \mathrm{mod}_{R\otimes_k R}$ such that $\omega^\bullet \cong \Omega[d]$. By an argument similar to (1), Ω has a k-basis $\{\, e^\sigma_\tau \mid \sigma \cup \tau \in \Sigma,\ \sigma \cup \tau \supset W \,\}$ and its left $R \otimes_k R$-module structure is obtained in a similar way as in (1).

Assume that Σ is the d-simplex 2^V. Then Σ is Gorenstein and Ω has a k-basis $\{\, e^\sigma_\tau \mid \sigma \cup \tau = V \,\}$. Moreover, we have a ring isomorphism given by $\varphi : R \ni e_{\sigma,\tau} \mapsto e_{\tau^{\mathsf{c}},\sigma^{\mathsf{c}}} \in R^{\mathsf{op}}$, where R^{op} is the opposite ring of R, and $\sigma^{\mathsf{c}} := V \setminus \sigma$. Thus R has a left $(R \otimes_k R)$-module structure given by $(x \otimes y) \cdot r = x \cdot r \cdot \varphi(y)$. Then a map given by $R \ni e_{\sigma,\tau} \mapsto e^{\tau^{\mathsf{c}}}_\sigma \in \Omega$ is an isomorphism of $(R \otimes R)$-modules. So R is an Auslander regular ring in this case. See [18, Remark 3.3].

(3) Assume that Σ is a simplicial complex and X is a d-dimensional manifold which is orientable (i.e., $or_X \cong \underline{k}_X$) and connected. Then $H^i(\omega^\bullet)^\dagger = 0$ for all $i \neq -d$. It is easy to see that $\Omega := H^{-d}(\omega^\bullet) \in \mathrm{mod}_{R\otimes_k R}$ has a k-basis $\{\, e^\sigma_\tau \mid \sigma \cup \tau \in \Sigma \,\}$ and the module structure is give by the same way as (1).

6. The Möbius function of the poset $\hat{\Sigma}$

The *Möbius function* of a finite poset P is a function

$$\mu : \{ (x, y) \mid x \leq y \text{ in } P \} \to \mathbb{Z}$$

recursively defined by $\mu(x, x) = 1$ for all $x \in P$ and $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$ for all $x, y \in P$ with $x < y$. See [13, Chapter 3] for a general theory of this function.

For a finite regular cell complex Σ, let $\hat{\Sigma}$ be the poset obtained from Σ by adjoining the greatest element $\hat{1}$ (even if Σ already possess a greatest element, we add a new one). Then the Möbius function μ of $\hat{\Sigma}$ has a topological meaning. For example, we have $\mu(\emptyset, \hat{1}) = \tilde{\chi}(X)$, where $\tilde{\chi}(X)$ is the reduced Euler characteristic $\sum_{i \geq 0} (-1)^i \dim_k \tilde{H}^i(X; k)$ of X. When the underlying space X is a manifold, the Möbius function of $\hat{\Sigma}$ is completely determined in [13, Proposition 3.8.9]. Here we study the general case.

For $\sigma \in \Sigma$ with $\dim \sigma > 0$, $\{ \sigma' \in \Sigma \mid \sigma' < \sigma \}$ is a regular cell decomposition of $\bar{\sigma} - \sigma$ which is homeomorphic to a sphere of dimension $\dim \sigma - 1$. Hence we have $\mu(\tau, \sigma) = (-1)^{l(\tau, \sigma)}$ for $\tau \in \Sigma$ with $\tau \leq \sigma$ by [13, Proposition 3.8.9], where $l(\tau, \sigma) := \dim \sigma - \dim \tau$. So it remains to describe $\mu(\sigma, \hat{1})$ for $\sigma \neq \emptyset$.

PROPOSITION 6.1. *For a cell $\emptyset \neq \sigma \in \Sigma$ with $j := \dim \sigma$, we have*

$$\mu(\sigma, \hat{1}) = \sum_{i \geq j} (-1)^{i-j+1} \dim_k H_c^i(U_\sigma; k).$$

Here $H_c^i(U_\sigma; k)$ is the cohomology with compact support of the open set $U_\sigma = \bigcup_{\rho \geq \sigma} \rho$ of X.

Proof. The assertion follows from the following computation:

$$\begin{aligned}
\mu(\sigma, \hat{1}) &= -\sum_{\rho \in \Sigma,\, \rho \geq \sigma} \mu(\sigma, \rho) \\
&= \sum_{i \geq j} (-1)^{i-j+1} \cdot \#\{ \rho \in \Sigma \mid \rho \geq \sigma,\, \dim \rho = i \} \\
&= \sum_{i \geq j} (-1)^{i-j+1} \dim_k \mathcal{H}^{-i}(\mathcal{D}_X^\bullet)(U_\sigma) \\
&= \sum_{i \geq j} (-1)^{i-j+1} \dim_k H_c^i(U_\sigma; k).
\end{aligned}$$

Here the second equality follows from the fact that $\mu(\sigma, \rho) = (-1)^{l(\sigma, \rho)}$; the third equality follows from $\mathcal{D}_X^\bullet \cong \mathbf{D}(Re_\emptyset)^\dagger$ and the description of $\mathbf{D}(Re_\emptyset)$ (recall also that $M^\dagger(U_\sigma) \cong M_\sigma$); and the last equality follows from the Verdier duality. $\square$

Assume that X is a manifold of dimension d. If $\sigma \neq \emptyset$ is contained in the boundary of X, then U_σ is homeomorphic to $(\mathbb{R}^{d-1} \times \mathbb{R}_{\geq 0})$ and $H_c^i(U_\sigma; k) = 0$ for all i. Thus $\mu(\sigma, \hat{1}) = 0$ in this case. If σ is not contained in the boundary of X, then U_σ is homeomorphic to $\mathbb{R}^d$ and $H_c^i(U_\sigma; k) = 0$ for all $i \neq d$ and $H_c^d(U_\sigma; k) = k$. Hence we have $\mu(\sigma, \hat{1}) = (-1)^{d-\dim\sigma+1}$. So Proposition 6.1 recovers [13, Proposition 3.8.9].

7. Relation to Koszul duality

Let $A = \bigoplus_{i\geq 0} A_i$ be an $\mathbb{N}$-graded associative k-algebra such that $\dim_k A_i < \infty$ for all i and $A_0 \cong k^n$ for some $n \in \mathbb{N}$ as an algebra. Then $\mathfrak{r} := \bigoplus_{i>0} A_i$ is the graded Jacobson radical. We say A is *Koszul* if a left A-module $A/\mathfrak{r}$ admits a graded projective resolution

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \to A/\mathfrak{r} \to 0$$

such that P^{-i} is generated by its degree i component as an A-module (i.e., $P^{-i} = AP_i^{-i}$). If A is Koszul, it is a quadratic ring, and its *quadratic dual ring* $A^!$ (see [1, Definition 2.8.1]) is Koszul again, and isomorphic to the opposite ring of the Yoneda algebra $\operatorname{Ext}_A^\bullet(A/\mathfrak{r}, A/\mathfrak{r})$.

Note that the incidence algebra R of Σ is a graded ring with $\deg(e_{\sigma,\sigma'}) = \dim\sigma - \dim\sigma'$. So we can discuss the Koszul property of R.

Proposition 7.1 (cf. [18, Lemma 4.5]). *The incidence algebra R of a finite regular cell complex Σ is always Koszul. Moreover, the quadratic dual ring $R^!$ is isomorphic to R^{op}.*

When Σ is a simplicial complex, the above result was proved by Polishchuk [8] in much wider context (but $\emptyset \notin \Sigma$ in his convention). More precisely, Polishchuk introduced a new partial order on the set $\Sigma \setminus \emptyset$ associated with a perversity function p, and constructed two rings from this new poset. Then he proved that these two rings are Koszul and quadratic dual rings of each other. Our rings R and R^{op} correspond to the case when p is the top (or bottom) perversity. In the middle perversity case, Σ has to be a *simplicial* complex to make their rings Koszul.

Proof. By [9], [15], R is Koszul if and only if the order complex $\Delta(I)$ is Cohen-Macaulay over k for any open interval I of Σ. Set $\Sigma' := \Sigma \setminus \emptyset$. Note that $\Delta(I) = \operatorname{lk}_{\Delta(\Sigma')} F$ for some $F \in \Delta(\Sigma')$ containing a maximal cell $\sigma \in \Sigma$. Set $\Delta := \operatorname{st}_{\Delta(\Sigma')} \sigma$. Then $\Delta(I) = \operatorname{lk}_\Delta F$. Since the underlying space of Δ is the closed disc $\bar{\sigma}$, Δ is Cohen-Macaulay. Hence $\operatorname{lk}_\Delta F$ is also. So R is Koszul.

Let

$$T := T_{R_0} R_1 = R_0 \oplus R_1 \oplus (R_1 \otimes_{R_0} R_1) \oplus \cdots = \bigoplus_{i\geq 0} R_1^{\otimes i}$$

be the tensor ring of

$$R_1 = \langle\, e_{\sigma,\tau} \mid \sigma, \tau \in \Sigma,\ \sigma > \tau,\ \dim\sigma = \dim\tau + 1\,\rangle$$

over R_0. Then $R \cong T/I$, where

$$I = (\, e_{\sigma,\rho_1} \otimes e_{\rho_1,\tau} - e_{\sigma,\rho_2} \otimes e_{\rho_2,\tau} \mid \sigma > \rho_i > \tau,\ \dim\sigma = \dim\tau + 2\,)$$

is a two-sided ideal. Let $R_1^* := \mathrm{Hom}_{R_0}(R_1, R_0)$ be the dual of the *left* R_0-module R_1. Then R_1^* has a right R_0-module structure such that $(fa)(v) = (f(v))a$, and a left R_0-module structure such that $(af)(v) = f(va)$, where $a \in R_0$, $f \in R_1^*$, $v \in R_1$. As a left (or right) R_0-module, R_1^* is generated by $\{\, e^*_{\tau,\sigma} \mid \sigma > \tau,\ \dim\sigma = \dim\tau + 1\,\}$, where $e^*_{\tau,\sigma}(e_{\sigma',\tau'}) = \delta_{\sigma,\sigma'} \cdot \delta_{\tau,\tau'} \cdot e_\sigma$.

Let $T^* = T_{R_0} R_1^*$ be the tensor ring of R_1^*. Note that $e^*_{\tau,\sigma} \otimes e^*_{\tau',\sigma'} \in R_1^* \otimes_{R_0} R_1^*$ is non-zero if and only if $\sigma = \tau'$. We have that $(R_1^* \otimes_{R_0} R_1^*)$ is isomorphic to $(R_1 \otimes_{R_0} R_1)^* = \mathrm{Hom}_{R_0}(R_1 \otimes_{R_0} R_1, R_0)$ via $(f \otimes g)(v \otimes w) = g(vf(w))$, where $f, g \in R_1^*$ and $v, w \in R_1$. In particular, $(e^*_{\tau,\rho} \otimes e^*_{\rho,\sigma})(e_{\sigma,\rho} \otimes e_{\rho,\tau}) = e_\sigma$. Recall that if $\sigma, \tau \in \Sigma$, $\sigma > \tau$ and $\dim\sigma = \dim\tau + 2$, then there are exactly two cells $\rho_1, \rho_2 \in \Sigma$ between σ and τ. So an easy computation shows that the quadratic dual ideal

$$I^\perp = (\, f \in R_1^* \otimes R_1^* \mid f(v) = 0 \text{ for all } v \in I_2 \subset R_1 \otimes R_1 = T_2\,) \subset T^*$$

of I is equal to

$$(\, e^*_{\tau,\rho_1} \otimes e^*_{\rho_1,\sigma} + e^*_{\tau,\rho_2} \otimes e^*_{\rho_2,\sigma} \mid \rho_1 \neq \rho_2,\ \sigma > \rho_i > \tau,\ \dim\sigma = \dim\tau + 2\,).$$

The k-algebra homomorphism $R^{\mathsf{op}} \to R^! = T^*/I^\perp$ defined by the identity map on $R_0 = T_0 = (T^*)_0 = (R^!)_0$ and $R_1 \ni e_{\sigma,\tau} \mapsto \varepsilon(\sigma,\tau) \cdot e^*_{\tau,\sigma} \in R^!_1$ is a graded isomorphism. Here ε is an incidence function of Σ. □

Since $R^! \cong R^{\mathsf{op}}$, $\mathrm{Hom}_k(-,k)$ gives duality functors $\mathbf{D}_k : \mathrm{mod}_R \to \mathrm{mod}_{R^!}$ and $\mathbf{D}_k^{\mathsf{op}} : \mathrm{mod}_{R^!} \to \mathrm{mod}_R$. These functors are exact, and they can be extended to duality functors between $D^b(\mathrm{mod}_R)$ and $D^b(\mathrm{mod}_{R^!})$.

Note that $R^!$ is a graded ring with $\deg e^*_{\tau,\sigma} = \dim\sigma - \dim\tau$. Let gr_R (resp. $\mathrm{gr}_{R^!}$) be the category of finitely generated graded left R-modules (resp. $R^!$-modules). Note that we can regard the functor $\mathbf{D}$ (resp. $\mathbf{D}_k$ and $\mathbf{D}_k^{\mathsf{op}}$) as a functor from $D^b(\mathrm{gr}_R)$ to itself (resp. $D^b(\mathrm{gr}_R) \to D^b(\mathrm{gr}_{R^!})$ and $D^b(\mathrm{gr}_{R^!}) \to D^b(\mathrm{gr}_R)$).

For each $i \in \mathbb{Z}$, let $\mathrm{gr}_R(i)$ be the full subcategory of gr_R consisting of modules M with $\deg M_\sigma = \dim\sigma - i$. For any $M \in \mathrm{gr}_R$, there are modules $M^{(i)} \in \mathrm{gr}_R(i)$ such that $M \cong \bigoplus_{i\in\mathbb{Z}} M^{(i)}$. The forgetful functor gives an equivalence $\mathrm{gr}_R(i) \cong \mathrm{mod}_R$ for all $i \in \mathbb{Z}$, and $D^b(\mathrm{gr}_R(i))$ is a full subcategory of $D^b(\mathrm{gr}_R)$. Similarly, let $\mathrm{gr}_{R^!}(i)$ be the full subcategory of $\mathrm{gr}_{R^!}$ consisting of modules M with $\deg M_\sigma = -\dim\sigma - i$. The above mentioned facts on $\mathrm{gr}_R(i)$ also hold for $\mathrm{gr}_{R^!}(i)$.

Let $DF : D^b(\mathrm{gr}_R) \to D^b(\mathrm{gr}_{R^!})$ and $DG : D^b(\mathrm{gr}_{R^!}) \to D^b(\mathrm{gr}_R)$ be the functors defined in [1, Theorem 2.12.1]. Since R and $R^!$ are Artinian, DF

and DG give an equivalence $D^b(\mathrm{gr}_R) \cong D^b(\mathrm{gr}_{R^!})$ by the Koszul duality ([1, Theorem 2.12.6]).

For the case when Σ is a simplicial complex the following result was proved by Vybornov [14] (under the convention that $\emptyset \notin \Sigma$). Independently, the author also proved a similar result ([18, Theorem 4.7]).

Theorem 7.2 (cf. Vybornov, [14, Corollary 4.3.5]). *Under the above notation, if $M^\bullet \in D^b(\mathrm{gr}_R(0))$, then we have $DF(M^\bullet) \in D^b(\mathrm{gr}_{R^!}(0))$. Similarly, if $N^\bullet \in D^b(\mathrm{gr}_{R^!}(0))$, then $DG(N^\bullet) \in D^b(\mathrm{gr}_R(0))$. Under the equivalence $\mathrm{gr}_R(0) \cong \mathrm{mod}_R$ and $\mathrm{gr}_{R^!}(0) \cong \mathrm{mod}_{R^!}$, we have $DF \cong \mathbf{D}_k \circ \mathbf{D}$ and $DG \cong \mathbf{D} \circ \mathbf{D}_k^{\mathsf{op}}$.*

Proof. Recall that $(R^!)_0 = R_0$. Let $N \in \mathrm{mod}_{R^!}$. For the functor DG, we need the left R-module structure on $\mathrm{Hom}_{R_0}(R, N_\sigma)$ given by $(xf)(y) := f(yx)$. The R-homomorphism given by $\mathrm{Hom}_{R_0}(R, N_\sigma) \ni f \longmapsto \sum_{\tau \leq \sigma} e(\sigma)_\tau \otimes_k f(e_{\sigma,\tau}) \in E(\sigma) \otimes_k N_\sigma$ gives an isomorphism $\mathrm{Hom}_{R_0}(R, N_\sigma) \cong E(\sigma) \otimes_k N_\sigma$. Under this isomorphism, for cells $\tau < \sigma$, the morphism $\mathrm{Hom}_{R_0}(R, N_\sigma) \to \mathrm{Hom}_{R_0}(R, N_\tau)$ given by $f \mapsto [x \mapsto e^*_{\tau,\sigma} f(e_{\sigma,\tau} x)]$ corresponds to the morphism $E(\sigma) \otimes_k N_\sigma \to E(\tau) \otimes_k N_\tau$ given by $e(\sigma)_\rho \otimes y \mapsto e(\tau)_\rho \otimes e^*_{\tau,\sigma} y$. (Here $e(\tau)_\rho = 0$ if $\tau \not\geq \rho$.)

Let $N \in \mathrm{gr}_{R^!}$. By the explicit description of $\mathbf{D}$ given in §3, we have

$$(\mathbf{D} \circ \mathbf{D}_k^{\mathsf{op}})^i(N) = \bigoplus_{\substack{\sigma \in \Sigma \\ \dim \sigma = -i}} E(\sigma) \otimes_k N_\sigma = \bigoplus_{\substack{\sigma \in \Sigma \\ \dim \sigma = -i}} \mathrm{Hom}_{R_0}(R, N_\sigma)$$

and the differential map defined by

$$E(\sigma) \otimes_k N_\sigma \ni e(\sigma)_\rho \otimes y \mapsto \sum_{\substack{\tau \in \Sigma \\ \dim \tau = -i-1}} \varepsilon(\sigma, \tau)\,(\,e(\tau)_\rho \otimes e^*_{\tau,\sigma}\, y) \in (\mathbf{D} \circ \mathbf{D}_k^{\mathsf{op}})^{i+1}(N).$$

So, if we forget the grading of modules, we have $DG(N) \cong (\mathbf{D} \circ \mathbf{D}_k^{\mathsf{op}})(N)$. Similarly, we can obtain an isomorphism $DG(N^\bullet) \cong (\mathbf{D} \circ \mathbf{D}_k^{\mathsf{op}})(N^\bullet)$ for a complex $N^\bullet \in D^b(\mathrm{gr}_{R^!})$.

Assume that $N \in \mathrm{gr}_{R^!}(0)$. Then the degree of $e(\sigma)_\tau \otimes y \in E(\sigma) \otimes_k N_\sigma \subset DG(N)$ is $(\dim \tau - \dim \sigma) + \dim \sigma = \dim \tau$ (see the proof of [1, Theorem 2.12.1] for the grading of $DG(N)$). Thus we have $DG(N) \in \mathrm{gr}_R(0)$.

We can prove the statement on DF in a similar (and easier) way. □

The results corresponding to Proposition 7.1 and Theorem 7.2 also hold for the incidence algebra of the poset $\Sigma \setminus \emptyset$. In other words, Vybornov [14, Corollary 4.3.5] and the "top perversity case" of Polishchuk [8] can be generalized directly into regular cell complexes.

8. Summary

For the reader's convenience, we give a list of the similarities between the subjects investigated in this paper and (quasi-)coherent sheaves on a projective scheme.

For a cell complex and related concepts, we use the same notation as before (Σ, X, R, mod_R, $\mathrm{Sh}_c(X)$ and so on). Readers who skipped the preceding sections are recommended to see §1 for a review of this notation. Let $A = \bigoplus_{i\geq 0} A_i$ be a commutative Noetherian homogeneous algebra over a field k. We denote the graded maximal ideal $\bigoplus_{i\geq 1} A_i$ by $\mathfrak{m}$. Let Gr_A be the category of graded A-modules, and gr_A its full subcategory consisting of finitely generated modules. For $M \in \mathrm{gr}_A$ and $N \in \mathrm{Gr}_A$, $\mathrm{Hom}_A(M, N)$ has a natural graded A-module structure. By $\mathrm{Qco}(Y)$ (resp. $\mathrm{Coh}(Y)$) we denote the category of quasi-coherent (resp. coherent) sheaves on the projective scheme $Y = \mathrm{Proj}(A)$

In the following list, the item (nR) for $n = 1, 2, \ldots$, states the property of mod_R corresponding to the property of Gr_A (or gr_A) stated in the item (nA). Of course, the situations of (nR) are much simpler than those of (nA).

(1R) We have an exact functor $(-)^\dagger : \mathrm{mod}_R \to \mathrm{Sh}_c(X)$ with $M^\dagger(U_\sigma) \cong M_\sigma$ for each $\emptyset \neq \sigma \in \Sigma$. Here U_σ denotes the open set $\bigcup_{\tau\geq\sigma} \tau$ of X.

(2R) We have a left exact functor $\Gamma_\emptyset : \mathrm{mod}_R \to \mathrm{mod}_R$ (or vect_k) whose derived functor $H^i_\emptyset(-)$ satisfies $H^i(X, M^\dagger) \cong H^{i+1}_\emptyset(M)$ for all $i \geq 1$ and $0 \to H^0_\emptyset(M) \to M_\emptyset \to H^0(X, M^\dagger) \to H^1_\emptyset(M) \to 0$ (exact).

(3R) If $\mathrm{mod}_\emptyset$ is the full subcategory of mod_R consisting of modules M with $\Gamma_\emptyset(M) = M$ (equivalently, $M \in \mathrm{mod}_\emptyset \iff M^\dagger = 0$), then this is a localizing subcategory with $\mathrm{mod}_R / \mathrm{mod}_\emptyset \cong \mathrm{Sh}_c(X)$.

(4R) We have a dualizing complex $\omega^\bullet \in D^b(\mathrm{mod}_{R\otimes_k R})$ giving the duality functor $\mathbf{R}\mathrm{Hom}_R(-, \omega^\bullet)$ from $D^b(\mathrm{mod}_R)$ to itself. We have a direct summand $\overline{\omega}^\bullet$ of $\omega^\bullet$ such that $(\overline{\omega}^\bullet)^\dagger \in D^b(\mathrm{Sh}_c(X))$ is the dualizing complex $\mathcal{D}^\bullet_X$ of X (e.g., if X is a manifold of dimension d, then $H^{-d}(\overline{\omega}^\bullet)^\dagger$ is the orientation sheaf of X). For $M^\bullet \in D^b(\mathrm{mod}_R)$, we have $\mathbf{R}\mathrm{Hom}_R(M^\bullet, \omega^\bullet)^\dagger \cong \mathbf{R}\mathcal{H}om((M^\bullet)^\dagger, \mathcal{D}^\bullet_X)$ in $D^b(\mathrm{Sh}_c(X))$. Moreover, $\mathbf{R}\mathcal{H}om(-, \omega^\bullet)^\dagger$ corresponds to the Verdier duality for $D^b(\mathrm{Sh}_c(X))$.

(5R) For $M^\bullet \in D^b(\mathrm{mod}_R)$, we have $\mathrm{Ext}^i_R(M^\bullet, \omega^\bullet)_\emptyset \cong H^{-i+1}_\emptyset(M^\bullet)^\vee$.

(6R) The dualizing complex $\omega^\bullet$ satisfies the Auslander condition of [19]. For $0 \neq M \in \mathrm{mod}_R$, we have

$$\max\{\dim\sigma \mid M_\sigma \neq 0\} = -\min\{i \mid \mathrm{Ext}^i_R(M, \omega^\bullet) \neq 0\}.$$

(1A) We have a well known exact functor $(-)^\sim : \mathrm{Gr}_A \to \mathrm{Qco}(Y)$. If $M \in \mathrm{gr}_A$, then $\tilde{M}$ is coherent.

(2A) We have a left exact functor $\Gamma_\mathfrak{m} : \mathrm{Gr}_A \to \mathrm{Gr}_A$ whose derived functor (i.e., the *local cohomology functor*) $H^i_\mathfrak{m}(-)$ satisfies $H^i(Y, \tilde{M}) \cong$

$[H^{i+1}_{\mathfrak{m}}(M)]_0$ for all $i \geq 1$ and $0 \to [H^0_{\mathfrak{m}}(M)]_0 \to M_0 \to H^0(Y, \tilde{M}) \to [H^1_{\mathfrak{m}}(M)]_0 \to 0$ (exact).

(3A) If Tor_A is the full subcategory of Gr_A consisting of modules M with $\Gamma_{\mathfrak{m}}(M) = M$ (equivalently, $M \in \mathrm{Tor}_A \iff \tilde{M} = 0$), then this is a localizing subcategory with $\mathrm{Gr}_A / \mathrm{Tor}_A \cong \mathrm{Qco}(Y)$.

(4A) We have a dualizing complex $\omega^\bullet_A \in D^b(\mathrm{gr}_A)$ which gives the duality functor $\mathbf{R}\,\mathrm{Hom}_A(-, \omega^\bullet_A)$ from $D^b(\mathrm{gr}_A)$ to itself. If we use the convention that $H^i_{\mathfrak{m}}(\omega^\bullet_A) \neq 0 \iff i = 1$, then $(\omega^\bullet_A)^\sim \in D^b(\mathrm{Coh}(Y))$ is the dualizing complex $\mathcal{D}^\bullet_Y$ of Y. For $M^\bullet \in D^b(\mathrm{gr}_A)$, we have $\mathbf{R}\,\mathrm{Hom}_A(M^\bullet, \omega^\bullet_A)^\sim \cong \mathbf{R}\mathcal{H}om((M^\bullet)^\sim, \mathcal{D}^\bullet_Y)$ in $D^b(\mathrm{Coh}(Y))$. Moreover, $\mathbf{R}\,\mathrm{Hom}_A(-, \omega^\bullet_A)^\sim$ corresponds to the Serre duality for $D^b(\mathrm{Coh}(Y))$.

(5A) For $M^\bullet \in D^b(\mathrm{gr}_A)$, we have $\mathrm{Ext}^i_A(M^\bullet, \omega^\bullet_A) \cong H^{-i+1}_{\mathfrak{m}}(M^\bullet)^\vee$, where $(-)^\vee$ stands for the graded k-dual. (Note that $\mathbf{R}\Gamma_{\mathfrak{m}}(\omega^\bullet_A) \cong A^\vee[-1]$ in our convention.)

(6A) The dualizing complex $\omega^\bullet_A$ satisfies the Auslander condition (this condition is always satisfied in the commutative case). For $0 \neq M \in \mathrm{gr}_A$, we have $\text{Krull-dim}(M) - 1 = -\min\{\, i \mid \mathrm{Ext}^i_A(M, \omega^\bullet_A) \neq 0 \,\}$. Recall that if $M \notin \mathrm{Tor}_A$, then $\dim \tilde{M} = \text{Krull-dim}(M) - 1$.

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