MAPPINGS WITH CONVEX POTENTIALS AND THE QUASICONFORMAL JACOBIAN PROBLEM

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ABSTRACT. This paper concerns convex functions that arise as potentials of quasiconformal mappings. Several equivalent definitions for such functions are given. We use them to construct quasiconformal mappings whose Jacobian determinants are singular on a prescribed set of Hausdorff dimension less than 1.

1. Introduction

Throughout the paper Ω denotes a convex domain in $\mathbb{R}^n$. We use $D^2u(x)$ to denote the Hessian matrix of a function $u : \Omega \to \mathbb{R}$ at a point $x \in \Omega$. The operator norm of a matrix A is denoted by $\|A\|$; the Euclidean norm of a vector v is denoted by $|v| = \langle v, v \rangle^{1/2}$. As usually, $W^{k,p}_{\mathrm{loc}}$ stands for local Sobolev spaces of real-valued functions.

DEFINITION 1.1. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$. A convex function $u : \Omega \to \mathbb{R}$ is called *quasiuniformly convex* if u is not affine, $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$, and there is a constant $K \in [1, \infty)$ such that

$$(1.1) \qquad \|D^2u(x)\|^n \leq K \det D^2u(x), \quad \text{a.e. } x \in \Omega.$$

When we want to specify the value of K, we call u a K-quasiuniformly convex function, usually using the abbreviation "q.u." Condition (1.1) is equivalent to saying that the ratio of the maximal and minimal eigenvalues of D^2u is essentially bounded on Ω. This should be compared to a related definition of uniformly convex functions [38, p. 59], which imposes a bound on the Hessian eigenvalues themselves rather than on their ratio. Recall that a function $u : \Omega \to \mathbb{R}$ is λ-uniformly convex if $u(x) - \lambda|x|^2/2$ is convex. One can easily see that if $u \in C^2(\Omega)$ is uniformly convex, then u is q.u. convex on every domain Ω' that is compactly contained in Ω. However, Example 2.4 below shows that in general q.u. convex functions are not uniformly convex, even locally.

Our motivation for introducing quasiuniformly convex functions comes from their connection with quasiconformal mappings [24], [37], [39]. In the following definition Df stands for the first-order derivative matrix of a mapping f.

DEFINITION 1.2.   Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$. An injective mapping $f : \Omega \to \mathbb{R}^n$ is called *quasiconformal* if $f \in W^{1,n}_{\text{loc}}(\Omega; \mathbb{R}^n)$ and there is a constant $K \in [1, \infty)$ such that

$$(1.2) \qquad \|Df(x)\|^n \leq K \det Df(x), \quad \text{a.e. } x \in \Omega.$$

A mapping is called K-quasiregular if it verifies all of the above conditions except for injectivity.

If u is a K-q.u. convex function, then its gradient

$$f(x) := \nabla u(x) = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right)$$

is locally in $W^{1,n}$ and (1.2) holds. This means that f is K-quasiregular and therefore continuous. Suppose that $f(x) = f(y)$ for some distinct points $x, y \in \Omega$. Since the graph of u admits only one supporting plane at every point, it follows that $f(z) = f(x)$ for all points z between x and y. This contradicts Reshetnyak's theorem [29], [30], which says that nonconstant quasiregular mappings are open and discrete. Therefore, f is injective, which implies that u is strictly convex (see Corollary 26.3.1 in [31]). To summarize the above, u is K-q.u. convex if and only if f is K-quasiconformal.

In addition to being quasiconformal, f is a monotone mapping [1] in the sense that

$$\langle f(x) - f(y), x - y \rangle \geq 0, \quad x, y \in \Omega.$$

The interplay between monotonicity and quasiconformality is one of the main underlying themes of this paper. While mappings with convex potentials have been a subject of recent research (see [8], [9], [10]), our setup is somewhat different since condition (1.1) is not invariant under affine transformations of $\mathbb{R}^n$. However, the family of all quasiuniformly convex functions is invariant under affine changes of variables.

David and Semmes [12] asked for a characterization of nonnegative functions w such that $C^{-1}w \leq \det Df \leq Cw$ a.e. in $\mathbb{R}^n$ for some quasiconformal mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ and some constant $C > 0$. This question became known as the *quasiconformal Jacobian problem* and drew much interest [4], [5], [21], [26], [32], [33], in part because of its connection to the problem of characterizing bi-Lipschitz images of $\mathbb{R}^n$ [5], [6].

Despite all the efforts, "currently there seems to be no good guess as to what analytic conditions would characterize quasiconformal Jacobians" [5]. One can try to gain a better understanding of this problem by studying the sets on which quasiconformal Jacobians assume the values 0 or ∞. Indeed,

the function w from the previous paragraph must be equal to 0 or ∞ on the same set. In order to make these remarks more precise, we introduce some more notation. Let $\mathcal{L}^n$ stand for the n-dimensional Lebesgue measure on $\mathbb{R}^n$, and let $\dim E$ denote the Hausdorff dimension of a set $E \subset \mathbb{R}^n$. When φ is a function defined on a subset of $\mathbb{R}^n$, we write $\operatorname{ess\,lim}_{y \to x} \varphi(y) = a$ if there is a set $Z \subset \mathbb{R}^n$ such that $\mathcal{L}^n(Z) = 0$ and $\lim_{y \to x,\ y \notin Z} \varphi(y) = a$.

DEFINITION 1.3. A set $E \subset \mathbb{R}^n$ is a *quasiconformal* 0-*set* if there is a quasiconformal mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\operatorname{ess\,lim}_{y \to x} \det Df(y) = 0, \quad x \in E.$$

A *quasiconformal* ∞-*set* is defined in the same way, except that the essential limit is required to be ∞ instead of 0.

Definition 1.3 admits an equivalent formulation in terms of the *precise representative* of $\det Df$. Given a locally integrable function φ, its precise representative $\widetilde{\varphi}$ is defined by

$$\widetilde{\varphi}(x) = \begin{cases} \lim_{r \to 0} \dfrac{1}{\mathcal{L}^n(B(x,r))} \displaystyle\int_{B(x,r)} \varphi(y)\, d\mathcal{L}^n(y) & \text{if the limit exists;} \\ 0 & \text{otherwise.} \end{cases}$$

According to Definition 1.3, E is a quasiconformal 0-set if and only if

$$\lim_{y \to x} \widetilde{\det Df}(y) = 0, \quad x \in E.$$

The same is true with 0 replaced by ∞.

Bonk, Heinonen and Saksman [5] proved that if a set $E \subset \mathbb{R}^2$ has variational 2-capacity zero, then E is both a quasiconformal 0-set and a quasiconformal ∞-set. Note that a set of 2-capacity zero must have Hausdorff dimension zero. In Sections 4 and 5 we combine the tools of convex analysis and potential theory to prove that if $E \subset \mathbb{R}^n$ has Hausdorff dimension strictly smaller than 1, then E is both a quasiconformal 0-set and a quasiconformal ∞-set. It is known that some sets of dimension 1 are not quasiconformal 0-sets (see §5).

Tyson [36] conjectured that for every compact set $E \subset \mathbb{R}^n$ of Hausdorff dimension less than 1 and for every $\varepsilon > 0$ there is a quasiconformal mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $\dim f(E) < \varepsilon$. Theorem 5.6 suggests that such an f might arise as the gradient of a convex function. This idea eventually led to the proof of Tyson's conjecture [25].

2. Definitions and preliminary results

Let $u : \Omega \to \mathbb{R}$ be a convex function. For $z \in \Omega$, the *subdifferential* of u at z is the set

$$\partial u(z) = \{p \in \mathbb{R}^n : u(x) \geq u(z) + \langle p, x - z \rangle \ \forall x \in \Omega\}.$$

If u is differentiable at z, then $\partial u(z)$ consists of only one vector, namely $\nabla u(z)$. For $z \in \Omega$ and $p \in \partial u(z)$ let

$$u_{z,p}(x) = u(x) - u(z) - \langle p, x - z \rangle, \quad x \in \Omega.$$

If u is differentiable at z, then we write u_z instead of $u_{z,\nabla u(z)}$. To avoid possible confusion, we do not use subscripts to denote derivatives in this paper. Following Caffarelli [7], we define the *section* of u with the center $z \in \Omega$, direction $p \in \partial u(z)$, and height $t > 0$ by

$$S_u(z, p, t) = \{x \in \Omega : u_{z,p}(x) < t\}.$$

If u is differentiable at z, then we write $S_u(z, t)$ for $S_u(z, \nabla u(z), t)$. The convex functions whose sections are similar in shape to Euclidean balls $B(z, r) = \{x \in \mathbb{R}^n : |x - z| < r\}$ shall be of primary interest to us.

DEFINITION 2.1.   We say that $u : \mathbb{R}^n \to \mathbb{R}$ *has round sections* if there exists a constant $\tau \in (0, 1)$ with the following property. For every $z \in \mathbb{R}^n$, $p \in \partial u(z)$ and $t > 0$ there is $R > 0$ such that

$$(2.1) \qquad\qquad B(z, \tau R) \subset S_u(z, p, t) \subset B(z, R).$$

In other words, the boundary of every section of u is pinched between two concentric spheres of comparable size.

DEFINITION 2.2.   The sections of a convex function $u : \mathbb{R}^n \to \mathbb{R}$ verify the *engulfing property* with constant C if for every $y \in S_u(x, p, t)$

$$(2.2) \qquad\qquad S_u(x, p, t) \subset S_u(y, q, Ct), \quad \forall q \in \partial u(y).$$

The *Monge-Ampère measure* associated with a convex function $u : \mathbb{R}^n \to \mathbb{R}$ is a Borel measure μ_u defined by $\mu_u(E) = \mathcal{L}^n(\partial u(E))$ for every Borel set $E \subset \mathbb{R}^n$ [18]. Note that μ_u does not change if u is replaced with $u_{z,p}$ for any $z \in \mathbb{R}^n$, $p \in \partial u(z)$. Given a set $E \subset \mathbb{R}^n$ such that $\mathcal{L}^n(E) < \infty$, let x^* be its center of mass and define $\lambda E = \{x^* + \lambda(x - x^*) : x \in E\}$, $\lambda > 0$. In other words, λE is the dilation of E with respect to its center of mass. The Monge-Ampère measure μ_u is called *doubling* if there exist $C > 0$ and $\alpha \in (0, 1)$ such that

$$(2.3) \qquad \mu_u(S_u(z, p, t)) \leq C\mu_u(\alpha S_u(z, p, t)), \quad z \in \mathbb{R}^n, p \in \partial u(p), \ t > 0.$$

If the sections of u are bounded sets, then the doubling condition on μ_u can be proved to be equivalent to the engulfing property of the sections of u (Theorem 2.2 [19] and Theorem 8 [14]). In §3 we will prove that q.u. convex functions have round sections and verify the engulfing property.

DEFINITION 2.3.   A homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is called *quasisymmetric*, or η-quasisymmetric, if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$

such that

$$(2.4) \qquad \frac{|f(x) - f(z)|}{|f(y) - f(z)|} \leq \eta \left(\frac{|x - z|}{|y - z|} \right), \quad z \in \Omega, \ x, y \in \Omega \setminus \{z\}.$$

Note that Definition 2.3 makes sense for all $n \geq 1$, whereas the above definition of quasiconformal mappings is vacuous when $n = 1$. In dimensions 2 or higher, a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is K-quasiconformal if and only if it is η-quasisymmetric. Moreover, K and η depend only on each other and the dimension n (see Theorem 11.14 [20] and [37]). The relation between convex functions and quasisymmetric mappings in one dimension is rather simple: a convex function on a line has a quasisymmetric gradient if and only if its Monge-Ampère measure is doubling [15], [20]. In §3 we explore such connections in higher dimensions. The rest of this section is devoted to several basic facts about q.u. convex functions.

We write I_n for the $n \times n$ identity matrix. When A and B are square matrices, the inequality $A \leq B$ means that $B - A$ is positive semidefinite. See [23] for basic properties of the positive semidefinite ordering.

EXAMPLE 2.4. Let $u(x) = g(|x|)$, where $g : [0, \infty) \to \mathbb{R}$ is a convex increasing function such that g' exists and is absolutely continuous on $[0, M]$ for every $M > 0$. Then

$$D^2 u(x) = \frac{g'(|x|)}{|x|} \left\{ I_n + \left(\frac{|x| g''(|x|)}{g'(|x|)} - 1 \right) \frac{x \otimes x}{|x|^2} \right\}, \quad \text{a.e. } x \in \mathbb{R}^n.$$

Hence for a.e. $x \in \mathbb{R}^n$ we have

$$\min \left\{ \frac{|x| g''(|x|)}{g'(|x|)}, 1 \right\} \frac{g'(|x|)}{|x|} I_n \leq D^2 u \leq \max \left\{ 2 \frac{|x| g''(|x|)}{g'(|x|)} - 1, 1 \right\} \frac{g'(|x|)}{|x|} I_n.$$

Thus $D^2 u$ satisfies (1.1) whenever there exist $C > 0$ such that

$$C \leq \frac{t g''(t)}{g'(t)} \leq C^{-1}, \quad \text{a.e. } t \in [0, \infty).$$

If this condition holds, then integration yields $g'(t) = O(t^C)$, $t \to 0$. It is then easy to see that $u \in W^{2,n}_{\text{loc}}(\mathbb{R}^n)$, and so u is a q.u. convex function in $\mathbb{R}^n$.

In particular, one can take $g(t) = t^{1+\alpha}$ for any $\alpha > 0$. If $0 < \alpha < 1$, then the eigenvalues of $D^2 u(x)$ grow indefinitely as $x \to 0$. If $\alpha > 1$, then they vanish as $x \to 0$, which implies that u is not uniformly convex in any neighborhood of the origin.

The following lemma summarizes some convergence properties of q.u. convex functions.

LEMMA 2.5. *Let Ω be a convex domain in $\mathbb{R}^n$, and let u_k, $k = 1, 2, \ldots$ be K-q.u. convex functions. If the sequence $\{u_k\}$ converges pointwise on a dense subset of Ω, then*

(i) *there is $u \in C^1(\Omega)$ such that $u_k \to u$ locally uniformly in Ω;*
(ii) *$\nabla u_k \to \nabla u$ locally uniformly in Ω;*
(iii) *u is either K-q.u. convex or affine;*
(iv) *$\det D^2 u_k \to \det D^2 u$ weakly in $L^1_{\mathrm{loc}}(\Omega)$;*
(v) *if u is affine, then $u_k \to u$ in $W^{2,n}_{\mathrm{loc}}(\Omega)$.*

Proof. By Theorem 10.8 [31] $u_k \to u$ locally uniformly in Ω. By Theorem 10.6 [31] the functions u_k are equi-Lipschitz on every compact subset of Ω (i.e., they have uniformly bounded Lipschitz constants). Hence the gradient mappings ∇u_k are locally uniformly bounded in Ω. By Theorem 20.5 [37] there is a subsequence $\{\nabla u_{k_j}\}$ that converges locally uniformly in Ω. Hence $u \in C^1(\Omega)$. Now by Theorem 25.7 [31] we have $\nabla u_k \to \nabla u$ locally uniformly in Ω. Since the mappings ∇u_k are K-quasiconformal, ∇u is either constant or a K-quasiconformal mapping [37], [39]. Statement (iv) follows from [29, p. 141] and Theorem II.9.1 [29]. If ∇u is constant, then the functions $u_k - u$ are K-q.u. convex and $\nabla(u_k - u) \to 0$ locally uniformly in Ω. Theorem II.9.1 [29] now implies that $\nabla(u_k - u) \to 0$ in $W^{1,n}_{\mathrm{loc}}(\Omega)$. $\square$

It is obvious that the class of all convex functions on a given set is a convex cone. In other words, if u and v are convex, then so is $\alpha u + \beta v$ for any positive coefficients α and β. This property is shared by the class of K-q.u. convex functions.

LEMMA 2.6.   *The set of all K-q.u. convex functions in a convex domain $\Omega \subset \mathbb{R}^n$ is a convex cone.*

Proof. Let u and v be K-q.u. convex functions in Ω, and let $\alpha, \beta > 0$. Then for a.e. $x \in \Omega$

$$\|\alpha D^2 u(x) + \beta D^2 v(x)\| \le \alpha \|D^2 u(x)\| + \beta \|D^2 v(x)\|$$
$$\le K^{1/n}\left(\alpha(\det D^2 u(x))^{1/n} + \beta(\det D^2 v(x))^{1/n}\right)$$
$$\le K^{1/n} \det\left(\alpha D^2 u(x) + \beta D^2 v(x)\right)^{1/n},$$

where the last step is based on Minkowski's determinantal inequality [23, 7.8.8]. Since the function $\alpha u + \beta v$ belongs to $W^{2,n}_{\mathrm{loc}}(\Omega)$, it is K-q.u. convex. $\square$

Lemma 2.6 allows us to build q.u. convex functions by adding together several translated and rescaled copies of simpler functions, such as the ones in Example 2.4. For instance, if u is a K-q.u. convex function and μ is a Radon measure with compact support in $\mathbb{R}^n$, then the convolution $u * \mu$ is K-q.u. convex. Other operations that preserve the class of K-q.u. convex functions are described in Lemmas 2.7 and 2.8 below. Their proofs rely on the strict convexity of q.u. convex functions which was proved in §1.

LEMMA 2.7. *Let $u : \Omega \to \mathbb{R}$ be a q.u. convex function. Suppose that $\mathcal{H} \subset \mathbb{R}^n$ is a hyperplane of dimension $d \geq 2$ such that $\Omega \cap \mathcal{H} \neq \varnothing$. Then the restriction of u to $\Omega \cap \mathcal{H}$ is K-q.u. convex.*

Proof. Let $\mathcal{H}^\perp$ be the orthogonal complement of $\mathcal{H}$ in $\mathbb{R}^n$. Let $G = \{\xi \in \mathcal{H}^\perp : \Omega \cap (\mathcal{H} + \xi) \neq \varnothing\}$. By virtue of Fubini's theorem, for $\mathcal{L}^{n-d}$-a.e. points $\xi \in G$ the restriction of u to $\mathcal{H} + \xi$ is K-q.u. convex. Choose a sequence of such points ξ_k so that $\xi_k \to 0$. By Lemma 2.5 the restriction of u to $\mathcal{H}$ is either K-q.u. convex or affine. The second case cannot occur since u is strictly convex. $\square$

LEMMA 2.8. *Suppose that u and v are K-q.u. convex in $\mathbb{R}^n$. Then*

(i) *the Legendre transform of u*

$$u^*(x) := \sup_{y \in \mathbb{R}^n} \{\langle x, y \rangle - u(y)\},$$

is K-q.u. convex in $\mathbb{R}^n$;

(ii) *the infimal convolution of u and v*

$$(u \square v)(x) := \inf_{y \in \mathbb{R}^n} \{u(x - y) + v(y)\},$$

is K-q.u. convex in $\mathbb{R}^n$.

Proof. (i) Since $\nabla u : \mathbb{R}^n \to \mathbb{R}^n$ is quasiconformal, it follows that $|\nabla u(x)| \to \infty$ as $|x| \to \infty$. This means that u is *co-finite* [31], i.e., its epigraph contains no nonvertical halflines. Since u is also strictly convex, it follows by Theorem 26.5 [31] that u^* is finite and differentiable in $\mathbb{R}^n$. Furthermore, the gradient mapping $\nabla u^* : \mathbb{R}^n \to \mathbb{R}^n$ is the inverse of u, which implies that ∇u^* is K-quasiconformal. Thus u is K-q.u. convex.

(ii) Since $u \square v = (u^* + v^*)^*$ [31, 16.4], the statement follows from (i) and Lemma 2.6. $\square$

We conclude this section with a remark on the regularity of q.u. convex functions. Since K-quasiconformal mappings are locally Hölder continuous with exponent $\alpha = 1/K$ (e.g., [24], [39]), it follows that K-q.u. convex functions are locally $C^{1,\alpha}$. The sharpness of the exponent α is demonstrated by the function $u(x) = |x|^{1+\alpha}$, which is K-q.u. convex according to Example 2.4. The problem of Sobolev regularity is more difficult. Gehring [16] proved that K-quasiconformal mappings are locally $W^{1,p}$ for some $p > n$, where p depends only on n and K. He conjectured that p can be taken arbitrarily close to $nK/(K-1)$. So far, Gehring's conjecture has been proved only when $n = 2$ [2]. If this conjecture is true in all dimensions, it will yield a sharp $W^{2,p}$ estimate for q.u. convex functions, since the above function u does not belong to $W^{2,p}_{\text{loc}}$ when $p = nK/(K-1)$.

3. Definitions of quasiuniform convexity

The following theorem provides several equivalent definitions of q.u. convex functions.

THEOREM 3.1.   *Let $n \geq 2$, and let $u : \mathbb{R}^n \to \mathbb{R}$ be a convex function. The following are equivalent:*

   (i) *u is a quasiuniformly convex function;*
   (ii) *u is differentiable and $\nabla u : \mathbb{R}^n \to \mathbb{R}^n$ is quasiconformal;*
   (iii) *u is differentiable but not affine; in addition, there exists a homeo-morphism $\eta : [0, \infty) \to [0, \infty)$ such that*

$$(3.1) \qquad \frac{u_z(x)}{u_z(y)} \leq \frac{|x-z|}{|y-z|}\eta\left(\frac{|x-z|}{|y-z|}\right), \quad z \in \mathbb{R}^n, \ x, y \in \mathbb{R}^n \setminus \{z\};$$

   (iv) *u is differentiable but not affine; in addition, there exists $H < \infty$ such that*

$$(3.2) \qquad \max_{|x-z|=r} u_z(x) \leq H \min_{|x-z|=r} u_z(x), \quad z \in \mathbb{R}^n, \ r > 0;$$

   (v) *u has round sections.*

The equivalence is quantitative in the sense that the constants and functions involved in each statement depend only on each other and n, but not on u.

The most interesting equivalence here is (ii)⇔(v). Note that in (v) the function u is not assumed to be differentiable, whereas (ii) implies that u is locally $C^{1,\alpha}$ and $W^{2,p}$ for some $\alpha > 0$, $p > n$ (see the end of §2). Because the proof of (v)⇒(ii) is somewhat involved, we isolate a part of it in the following lemma.

LEMMA 3.2.   *If a convex function $u : \mathbb{R}^n \to \mathbb{R}$ has round sections, then it is differentiable and strictly convex. Furthermore, there is a constant C such that for any $z, w \in \mathbb{R}^n$*

$$(3.3) \qquad\qquad u_z(z + 2w) \leq Cu_z(z + w).$$

The constant C depends only on τ in Definition 2.1.

Proof. A convex function is differentiable if and only if its restriction to every line is such [31, 25.2]. It therefore suffices to prove the lemma for $n = 1$.

Suppose, by way of contradiction, that u is not strictly convex. Let $(a, b) \subset \mathbb{R}$ be a maximal (nonempty) open interval on which u is affine. Since u is assumed not to be affine on $\mathbb{R}$, at least one of the endpoints of (a, b) is finite; for the sake of definiteness suppose that a is finite. Let u'_+ and u'_- denote the one-sided derivatives of u. For $t > 0$ the section $S(a, u'_+(a), t)$ is an open interval (a_t, b_t) which contains (a, b). Therefore, $\lim_{t \to 0}(b_t - a) \geq b - a$, which

implies $\lim_{t\to 0}(a - a_t) \geq \tau(b - a) > 0$. Let $a' = \lim_{t\to 0} a_t < a$. For $x \in (a', a)$ we have

$$u(x) < u(a) + u'_+(a)(x - a) + t, \quad \text{for all } t > 0.$$

This means that $u(x) = u(a) + u'_+(a)(x - a)$ for all $x \in (a', b)$, which contradicts the maximality of the interval (a, b).

Our next goal is to prove that $u'_-(z) = u'_+(z)$ for all $z \in \mathbb{R}$. Again, let (a_t, b_t) denote the section $S_u(z, u'_+(z), t)$. Note that

$$u(a_t) = u(z) + u'_+(z)(a_t - z) + t \quad \text{and} \quad u(b_t) = u(z) + u'_+(z)(b_t - z) + t.$$

Since u is strictly convex, we have $a_t, b_t \to z$ as $t \to 0$. Hence

$$(3.4) \qquad u'_+(z) = \lim_{t\to 0} \frac{u(b_t) - u(z)}{b_t - z} = u'_+(z) + \lim_{t\to 0} \frac{t}{b_t - z};$$

$$(3.5) \qquad u'_-(z) = \lim_{t\to 0} \frac{u(a_t) - u(z)}{a_t - z} = u'_+(z) + \lim_{t\to 0} \frac{t}{a_t - z}.$$

From (3.4) we have

$$\lim_{t\to 0} \frac{t}{|b_t - z|} = 0,$$

which implies

$$\lim_{t\to 0} \frac{t}{|a_t - z|} \leq \tau^{-1} \lim_{t\to 0} \frac{t}{|b_t - z|} = 0.$$

This and (3.5) yield $u'_-(z) = u'_+(z)$.

It remains to prove (3.3). In doing so we may assume that $u = u_z$, $z = 0$, and $w = 1$. Let $\zeta = (1 + \tau)^{-1/2}$. The convexity of u implies

$$(3.6) \qquad u'(\zeta) \leq \frac{u(1) - u(\zeta)}{1 - \zeta} \leq \frac{u(1)}{1 - \zeta}.$$

Let $t = u_\zeta(0) = \zeta u'(\zeta)$. Since the closure of $S_u(\zeta, t)$ contains 0, it also contains the point $(1 + \tau)\zeta$ due to roundedness of sections. Thus $u_\zeta((1 + \tau)\zeta) \leq t$. Combining this with (3.6), we obtain

$$(3.7) \qquad u((1 + \tau)\zeta) \leq u(\zeta) + \tau\zeta u'(\zeta) + t \leq \left\{ 1 + \frac{\zeta(1 + \tau)}{1 - \zeta} \right\} u(1).$$

Since $(1 + \tau)\zeta = \sqrt{1 + \tau} > 1$, inequality (3.3) follows from (3.7) by iteration.
□

Proof of Theorem 3.1. The equivalence of (i) and (ii) was already observed in §1. We shall prove that (ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(ii).

(ii)$\Rightarrow$(iii). Since ∇u is nonconstant, u is not affine. Let $f = \nabla u_z$, $r_1 = |x - z|$, $r_2 = |y - z|$. By the results mentioned in §2 the mapping f is η_0-quasisymmetric for some $\eta_0 : [0, \infty) \to [0, \infty)$. Integrating f along a line segment connecting x to z, we obtain

$$(3.8) \qquad u_z(x) \leq r_1 \max_{B(z,r_1)} |f|$$

Let γ be the curve of steepest descent from y to z with respect to u_z. See [22, VIII.3.4] for the existence, uniqueness, and rectifiability of γ. Let $\tilde{\gamma}$ be the part of γ lying in $B(z, r_2) \setminus B(z, r_2/2)$. Then

$$(3.9) \qquad u_z(y) = \int_\gamma |f(s)| ds \geq \int_{\tilde{\gamma}} |f(s)| ds \geq \frac{r_2}{2} \min_{\partial B(z, r_2/2)} |f|.$$

From (3.8) and (3.9) we find

$$\frac{u_z(x)}{u_z(y)} \leq \frac{2r_1}{r_2} \eta_0 \left(\frac{2r_1}{r_2} \right).$$

Therefore, (3.1) holds with $\eta(t) = 2\eta_0(2t)$.

(iii)$\Rightarrow$(iv). Set $H = \eta(1)$ with η as in (3.1).

(iv)$\Rightarrow$(v). Let $S_u(z, t)$ be a section of u. Suppose that $S_u(z, t)$ is unbounded. Being an unbounded convex set, it must contain a halfline l emanating from z. The restriction of u_z to l is a bounded convex function vanishing at z. Therefore, u_z vanishes on l. By (3.2) u_z vanishes identically, which contradicts the assumption that u is not affine. Thus $S_u(z, t)$ is bounded.

Let $R = \max\{|x - z| : x \in \partial S_u(z, t)\}$. Pick a point $x \in \partial B(z, R) \cap \partial S_u(z, t)$ and let $y = z + H^{-1}(x - z)$. Since u_z is convex,

$$u_z(y) \leq H^{-1} u_z(x) + (1 - H^{-1}) u_z(z) = H^{-1} t.$$

By (3.2) we have $u_z(w) \leq t$ whenever $|w - z| = H^{-1} R$. Therefore,

$$B(z, H^{-1} R) \subset S_u(z, t) \subset B(z, R).$$

(v)$\Rightarrow$(ii). By Lemma 3.2 u is differentiable, and therefore continuously differentiable [31, 25.5]. The strict convexity of u implies that the mapping $f = \nabla u$ is injective. To prove that f is quasiconformal, it suffices to show that there is a constant H such that

$$(3.10) \qquad |f(x) - f(z)| \leq H |f(y) - f(z)|$$

whenever $z \in \Omega$ and $|x - z| = |y - z| = r > 0$ (see [37] or Ch. 10–11 of [20]). Note that

$$(3.11) \qquad f(x) - f(z) = \nabla u_z(x) \quad \text{and} \quad f(y) - f(z) = \nabla u_z(y).$$

Considering the restriction of u to the line passing through z and y, we see that the quotient $u_z(y)/r$ does not exceed the slope of u_z at y. Hence

$$(3.12) \qquad u_z(y) \leq r |\nabla u_z(y)|.$$

Next, let $L(w) = \langle \nabla u_z(x), w - x \rangle + u_z(x)$ be the tangent plane of u_z at x. Since $u_z(x) > 0$, we have

$$(3.13) \qquad r |\nabla u_z(x)| \leq \max\{L(w) : |w - x| = r\}$$
$$\leq \max\{L(w) : |w - z| = 2r\}$$
$$\leq \max\{u_z(w) : |w - z| = 2r\} =: M.$$

The section $S_u(z, M)$ does not contain the closed ball $\overline{B}(z, 2r)$ and is therefore contained in $B(z, 2\tau^{-1}r)$, where τ is as in Definition 2.1. Let $y' = z + 2\tau^{-1}y$. Then $u_z(y') \geq M$. Using (3.11), (3.12), (3.13) and Lemma 3.2, we obtain

$$(3.14) \qquad \frac{|f(x) - f(z)|}{|f(y) - f(z)|} = \frac{|\nabla u_z(x)|}{|\nabla u_z(y)|} \leq \frac{u_z(y')}{u_z(y)} \leq H,$$

where H depends only on τ. $\square$

REMARK 3.3.   When $n = 1$, part (ii) of Theorem 3.1 can be interpreted as

(ii$'$) $\nabla u : \mathbb{R} \to \mathbb{R}$ exists and is quasisymmetric.

The proof of Theorem 3.1 applies verbatim to the equivalence between (ii$'$), (iii), (iv), and (v).

REMARK 3.4.   One could also state a version of Theorem 3.1 for convex functions defined in an arbitrary convex domain $\Omega \subset \mathbb{R}^n$. Statements (iii), (iv) and (v) would then contain a condition that the considered points and sections lie in a ball B such that $2B \subset \Omega$. We do not pursue this matter here.

REMARK 3.5.   Part (iv) of Theorem 3.1 implies that q.u. convex functions are *quasisymmetrically convex* in the sense of [3]. The notion of quasisymmetric convexity was introduced in [3] to characterize the convex domains in $\mathbb{R}^n$ that are Gromov hyperbolic in the Hilbert metric.

REMARK 3.6.   By Theorem 11.3 [20] the function η in part (iii) can be taken equal to $\eta(t) = C \max\{t^\alpha, t^{1/\alpha}\}$ for some $C \geq 1$ and $0 < \alpha \leq 1$.

Theorem 3.1 enables us to easily establish connections between quasiuniform convexity and some better known properties of convex functions.

DEFINITION 3.7.   The Monge-Ampère measure μ_u verifies *the condition* μ_∞ if for any $\delta_1 \in (0, 1)$ there exists $\delta_2 \in (0, 1)$ such that for every section $S = S_u(z, t)$ and every Borel set $E \subset S$,

$$\frac{\mathcal{L}^n(E)}{\mathcal{L}^n(S)} < \delta_2 \Rightarrow \frac{\mu_u(E)}{\mu_u(S)} < \delta_1.$$

The condition μ_∞, which is stronger than the doubling condition, plays an important role in the proof of Harnack's inequality for non-negative solutions to the linearized Monge-Ampère equation [11].

COROLLARY 3.8.   *Let* $u : \mathbb{R}^n \to \mathbb{R}$ *be a q.u. convex function. Then its Monge-Ampère measure* μ_u *verifies the condition* μ_∞.

Proof. By Gehring's theorem [16] μ_u is an A_∞ weight. Since u has round sections by Theorem 3.1, the μ_∞ property follows. $\square$

The following example, due to Wang [40], demonstrates that the converse of Corollary 3.8 is false. Consider the following strictly convex function in $\mathbb{R}^2$:

$$u(x, y) = \begin{cases} x^4 + \frac{3}{2}x^{-2}y^2, & |y| \leq |x|^3; \\ \frac{1}{2}x^2|y|^{2/3} + 2|y|^{4/3}, & |y| > |x|^3. \end{cases}$$

The Monge-Ampère measure of u is absolutely continuous with respect to the Lebesgue measure, and its density is bounded away from 0 and ∞ [40]. Hence u verifies the condition μ_∞. Since the second derivative of u in y

$$\frac{\partial^2 u}{\partial y^2} = \begin{cases} 3x^{-2}, & |y| < |x|^3; \\ -\frac{1}{9}x^2|y|^{-4/3} + \frac{8}{9}|y|^{-2/3}, & |y| > |x|^3, \end{cases}$$

is unbounded near the origin, so is $\|D^2 u\|$. Thus u is not quasiuniformly convex.

It would be interesting to characterize quasiuniformly convex functions in terms of their Monge-Ampère measure. Such a characterization would be a step toward the solution of the quasiconformal Jacobian problem.

THEOREM 3.9. *If a convex function $u : \mathbb{R}^n \to \mathbb{R}$ has round sections, then its sections verify the engulfing property with a constant depending only on τ in Definition 2.1.*

Proof. In this proof C stands for various constants which depend only on τ. According to Definition 2.2, we must find C such that for all $z \in \mathbb{R}^n$ and all $t > 0$

$$S_u(z, t) \subset S_u(y, Ct), \quad \forall y \in S_u(z, t).$$

Let R be such that $B(z, \tau R) \subset S_u(z, t) \subset B(z, R)$. For every $y \in \partial S_u(z, t)$ inequality (3.13) implies

$$|\nabla u_x(y)| \leq \frac{\max\{u_z(w) : |w - z| = 2|y - z|\}}{|y - z|} \leq \frac{\max\{u_z(w) : |w - z| = 2R\}}{\tau R}.$$

Applying (3.3) to u_z we obtain

$$\max\{u_z(w) : |w - z| = 2R\} \leq C \max\{u_z(w) : |w - z| = \tau R\} \leq Ct.$$

Hence for every $y \in \partial S_u(z, t)$ we have

$$|\nabla u_z(y)| \leq Ct/R;$$

this estimate is also valid in $S_u(z, t)$ by the convexity of u_z. Now if $x, y \in S_u(z, t)$, then

$$u_y(x) = (u_z)_y(x) = u_z(x) - u_z(y) - \langle \nabla u_z(y), x - y \rangle \leq t + |\nabla u_z(y)||x - y| \leq Ct.$$

Thus $x \in S_u(y, Ct)$ for all $x \in S_u(z, t)$, as required. $\square$

4. Quasiconformal ∞-sets

For $p \in \mathbb{R}$ let $\mathcal{M}^p$ denote the space of all Radon measures on $\mathbb{R}^n$ such that $\int (1 + |x|)^p d\mu(x) < \infty$. In other words, $\mathcal{M}^0$ is the space of all finite Radon measures, and $\mathcal{M}^p$ is the image of $\mathcal{M}^0$ under multiplication by the weight $(1 + |x|)^{-p}$. The space $\mathcal{M}^0$ is naturally equipped with the weak* topology since $\mathcal{M}^0 \subset C_0(\mathbb{R}^n)^*$. For $p \neq 0$ the topology on $\mathcal{M}^p$ is induced from $\mathcal{M}^0$ by the above multiplication map. The spaces $\mathcal{M}^p$ are ordered by inclusion: $\mathcal{M}^p \subset \mathcal{M}^q$ if $p > q$. In this section we are mainly concerned with the case $p < 0$, while in §5 we shall consider $\mathcal{M}^p$ with $p > 0$.

For every measure $\mu \in \mathcal{M}^p$ its Riesz potential

$$\mathcal{I}_\beta \mu(x) = \int |x - y|^{\beta - n} d\mu(y)$$

is finite a.e. in $\mathbb{R}^n$ as long as $0 < \beta \leq n + p$ [27, p. 61].

LEMMA 4.1. *Suppose that $\mu \in \mathcal{M}^{\alpha - 1}$, $0 < \alpha < 1$, and $\mu(\mathbb{R}^n) > 0$. Then there exists a q.u. convex function $u : \mathbb{R}^n \to \mathbb{R}$ such that*

$$(4.1) \qquad \alpha^n (\mathcal{I}_{n+\alpha-1}\mu)^n \leq \det D^2 u \leq (\mathcal{I}_{n+\alpha-1}\mu)^n$$

a.e. in $\mathbb{R}^n$.

Proof. We shall obtain u by convolving μ with the kernel

$$\mathcal{K}_\alpha(x, y) = \frac{|x - y|^{\alpha+1} - |y|^{\alpha+1}}{\alpha + 1} + \langle x, y \rangle |y|^{\alpha-1}, \quad x, y \in \mathbb{R}^n,$$

which is q.u. convex in x by Example 2.4. Using the asymptotic expansion

$$|x - y|^{\alpha+1} = |y|^{\alpha+1} \left| \frac{x}{|y|} - \frac{y}{|y|} \right|^{\alpha+1} = |y|^{\alpha+1} \left\{ 1 - (\alpha+1)\frac{\langle x, y \rangle}{|y|^2} + O\left(\frac{|x|^2}{|y|^2} \right) \right\}$$

$$= |y|^{\alpha+1} - (\alpha+1)\langle x, y \rangle |y|^{\alpha-1} + O(|x|^2 |y|^{\alpha-1}), \quad |y| \to \infty,$$

we obtain

$$(4.2) \qquad \mathcal{K}_\alpha(x, y) = O(|y|^{\alpha-1}), \quad |y| \to \infty,$$

locally uniformly in x. Similarly,

$$(4.3) \quad \nabla_x \mathcal{K}_\alpha(x, y) = (x - y)|x - y|^{\alpha-1} + y|y|^{\alpha-1} = O(|y|^{\alpha-1}), \quad |y| \to \infty,$$

locally uniformly in x. Let

$$(4.4) \qquad u(x) = \int \mathcal{K}_\alpha(x, y) d\mu(y), \quad x \in \mathbb{R}^n.$$

Since the integral converges locally uniformly in x, it follows from Lemma 2.5 that u is a q.u. convex function. Differentiating (4.4) yields

$$(4.5) \qquad \nabla u(x) = \int \nabla_x \mathcal{K}_\alpha(x, y) d\mu(y), \quad x \in \mathbb{R}^n,$$

and

$$(4.6) \qquad D^2 u(x) = \int |x - y|^{\alpha - 1} \left(I_n + (\alpha - 1) \frac{(x - y) \otimes (x - y)}{|x - y|^2} \right) d\mu(y),$$

for a.e. $x \in \mathbb{R}^n$. For any $z \in \mathbb{R}^n$ we have

$$\alpha I_n \leq I_n + (\alpha - 1) \frac{z \otimes z}{|z|^2} \leq I_n$$

in the sense of the positive semidefinite ordering. This and (4.6) readily imply (4.1). □

THEOREM 4.2. *For every set $E \subset \mathbb{R}^n$ of Hausdorff dimension less than 1 there is a q.u. convex function $u : \mathbb{R}^n \to \mathbb{R}$ such that*

$$(4.7) \qquad \operatorname*{ess\,lim}_{y \to x} \det D^2 u(y) = \infty, \quad x \in E.$$

Furthermore, there is a set of Hausdorff dimension 1 for which no such u exists.

Proof. Since Hausdorff measures are Borel regular [28, 4.5], we can replace E with a Borel superset of the same dimension. Thus we may assume that E is Borel and therefore capacitable [27, 2.8]. Choose γ so that $\dim E < \gamma < 1$. By Theorem 3.14 [27] the set E has $(n - \gamma)$-capacity zero. This means that there exists a sequence of open sets

$$G_1 \supset G_2 \supset \cdots \supset E$$

such that their $(n - \gamma)$-capacity tends to 0. Let $E' = \bigcap_k G_k$; this is a G_δ-set of zero $(n - \gamma)$-capacity. Let $E_j = E' \cap B(0, j)$, $j = 1, 2, \dots$. By Theorem 3.1 [27] for each j there exists a finite measure μ_j with compact support on $\mathbb{R}^n$ such that its Riesz potential $\mathcal{I}_{n - \gamma} \mu_j$ is infinite at every point of E_j. Since Riesz potentials are lower semicontinuous, it follows that

$$(4.8) \qquad \lim_{y \to x} \mathcal{I}_{n - \gamma} \mu_j(y) = \infty, \quad x \in E_j.$$

Let $\mu = \sum_j c_j \mu_j$, where the coefficients $c_j > 0$ are sufficiently small so that $\mu \in \mathcal{M}^{-\gamma}$. (In fact, we can achieve $\mu \in \mathcal{M}^p$ for any $p \in \mathbb{R}$ by making c_j very small.) Applying Lemma 4.1 to μ with $\alpha = 1 - \gamma$, we obtain the first part of the theorem.

It remains to show that the condition $\dim E < 1$ cannot be replaced with $\dim E \leq 1$. Let E be the line segment $\{t e_1 : 0 \leq t \leq 1\}$, where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. Suppose that $u : \mathbb{R}^n \to \mathbb{R}$ is a K-q.u. convex function such that (4.7) holds. Then for any given $M > 0$ there exists $\varepsilon > 0$ such that $\det D^2 u \geq M$ a.e. in the ε-neighborhood of E. Since $u \in W^{2,n}_{\text{loc}}(\mathbb{R}^n)$, for a.e. $x \in \mathbb{R}^n$ the function $\varphi_x(t) = u(x + t e_1)$, $t \in \mathbb{R}$, has locally absolutely

continuous derivative φ'_x. Hence

$$\varphi'_x(1) - \varphi'_x(0) = \int_0^1 \varphi''_x(t)dt, \quad \text{a.e. } x \in \mathbb{R}^n.$$

If $D^2 u$ exists at the point $x + te_1$, then $\varphi''_x(t)$ is bounded from below by the minimal eigenvalue of $D^2 u$, which is in turn bounded from below by $K^{-1}(\det D^2 u)^{1/n}$. Therefore,

(4.9) $\qquad |\nabla u(x + e_1) - \nabla u(x)| \geq K^{-1} M^{1/n}, \quad \text{a.e. } x \in B(0, \varepsilon).$

In fact, the latter inequality holds for all $x \in B(0, \varepsilon)$ because ∇u is continuous. On the other hand, the continuity of ∇u implies that it is locally bounded. Since the constant M in (4.9) can be arbitrarily large, we arrive at a contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

COROLLARY 4.3. *Every set $E \subset \mathbb{R}^n$ of Hausdorff dimension less than 1 is a quasiconformal ∞-set.*

It seems likely that the condition $\dim E < 1$ in Corollary 4.3 can be considerably weakened. In fact, it is possible that every set of n-dimensional measure zero is a quasiconformal ∞-set. See [5], [21].

5. Quasiconformal 0-sets

By an argument similar to the second part of the proof of Theorem 4.2, a quasiconformal 0-set cannot contain a line segment [5], [12]. In this section we prove that every set of Hausdorff dimension less than 1 is a quasiconformal 0-set. The idea of the proof is to apply the Legendre transform to the convex functions constructed in the previous section. The main difficulty lies in finding a q.u. convex function whose Legendre transform has a vanishing Hessian matrix on a prescribed set.

Let $\mathcal{M}^p$, $p \in \mathbb{R}$, be as in §4. Recall that $\mu_k \to \mu$ in $\mathcal{M}^p$ if and only if $w^p \mu_k \xrightarrow{w^*} w^p \mu$, where $w(x) = (1 + |x|)$. Given $0 < \alpha < 1$ and $\mu \in \mathcal{M}^\alpha$, we define

$$\rho_{\alpha,\mu}(x) = \frac{|x|^2}{2} + \int \frac{|x - y|^{\alpha+1} - |y|^{\alpha+1}}{\alpha + 1} \, d\mu(y), \quad x \in \mathbb{R}^n;$$

$$F_{\alpha,\mu}(x) = \nabla \rho_{\alpha,\mu}(x) = x + \int (x - y)|x - y|^{\alpha-1} \, d\mu(y), \quad x \in \mathbb{R}^n.$$

Since the integrands are majorized by $|y|^\alpha$ locally uniformly in x, both integrals converge. Consequently, $\rho_{\alpha,\mu}$ is q.u. convex. The reader might wonder why we did not use the kernel $\mathcal{K}_\alpha$ from §4, which would allow for measures from a larger space $\mathcal{M}^{\alpha-1}$. The reason is that for the proof of Lemma 5.5 to work, we need a kernel $K(x, y)$ such that its gradient $F = \nabla_x K$ is antisymmetric, i.e., $F(x, y) = -F(y, x)$. Unfortunately, $\mathcal{K}_\alpha$ does not possess this property.

LEMMA 5.1. *Let $\mu \in \mathcal{M}^\alpha$, $0 < \alpha < 1$. Then*

(5.1) $$|F_{\alpha,\mu}(x) - F_{\alpha,\mu}(y)| \geq |x - y|, \quad x, y \in \mathbb{R}^n,$$

and

(5.2) $$|x| - C_1 \leq |F_{\alpha,\mu}(x)| \leq C_2|x| + C_3, \quad x \in \mathbb{R}^n,$$

where the constants C_1, C_2 and C_3 are positive and depend only on α and $\int w^\alpha d\mu$.

Proof. The mapping $x \mapsto F_{\alpha,\mu}(x) - x$ is monotone, being the gradient of a convex function. Since

$$\langle F_{\alpha,\mu}(x) - F_{\alpha,\mu}(y) - (x - y), x - y \rangle \geq 0,$$

it follows that $\langle F_{\alpha,\mu}(x) - F_{\alpha,\mu}(y), x - y \rangle \geq |x-y|^2$, which implies (5.1). Using the elementary inequality $|x - y|^\alpha \leq |x|^\alpha + |y|^\alpha$, we obtain

$$|F_{\alpha,\mu}(x)| \leq |x| + \int (|x|^\alpha + |y|^\alpha)\, d\mu(y).$$

On the other hand, (5.1) implies

$$|F_{\alpha,\mu}(x)| \geq |x| - |F_{\alpha,\mu}(0)| \geq |x| - \int w^\alpha\, d\mu.$$

This proves (5.2). $\qquad\square$

LEMMA 5.2. *Suppose that $\mu_k \to \mu$ in $\mathcal{M}^\alpha$, $0 < \alpha < 1$, and $\int w^\alpha d\mu_k \to \int w^\alpha d\mu$. Then $F_{\alpha,\mu_k} \to F_{\alpha,\mu}$ and $F_{\alpha,\mu_k}^{-1} \to F_{\alpha,\mu}^{-1}$ locally uniformly in $\mathbb{R}^n$.*

Proof. Let $\tilde{\mu}_k = w^\alpha \mu_k$ and $\tilde{\mu} = w^\alpha \mu$. For any bounded continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$ we have $\int \varphi\, d\tilde{\mu}_k \to \int \varphi\, d\tilde{\mu}$ [13, p. 225]. For any fixed $x \in \mathbb{R}^n$

$$\rho_{\alpha,\mu_k}(x) = \frac{|x|^2}{2} + \int \frac{|x - y|^{\alpha+1} - |y|^{\alpha+1}}{(\alpha + 1)w(y)^\alpha}\, d\tilde{\mu}_k(y),$$

where the integrand is bounded and continuous on $\mathbb{R}^n$. Hence $\rho_{\alpha,\mu_k} \to \rho_{\alpha,\mu}$ pointwise in $\mathbb{R}^n$. Since the pointwise convergence of differentiable convex functions implies locally uniform convergence of their gradients [31, 25.7], $F_{\alpha,\mu_k} \to F_{\alpha,\mu}$ locally uniformly in $\mathbb{R}^n$. Since $F_{\alpha,\mu}^{-1}$ is a contraction by (i), it follows that $F_{\alpha,\mu}^{-1} \circ F_{\alpha,\mu_k} \to \mathrm{id}$ locally uniformly. Let $M = \sup_k |F_{\alpha,\mu_k}(0)|$. Since F_{α,μ_k}^{-1} is a contraction,

$$|F_{\alpha,\mu_k}^{-1}(x)| \leq |x - F_{\alpha,\mu_k}(0)| \leq |x| + M, \quad x \in \mathbb{R}^n.$$

It remains to observe that for every $R > 0$

$$\sup_{|x| \leq R} |F_{\alpha,\mu_k}^{-1}(x) - F_{\alpha,\mu}^{-1}(x)| \leq \sup_{|y| \leq R+M} |y - F_{\alpha,\mu}^{-1}(F_{\alpha,\mu_k}(y))| \to 0$$

as $k \to \infty$. $\qquad\square$

Consider the pushforward of μ under $F_{\alpha,\mu}$:

$$\mathcal{F}_\alpha(\mu) := (F_{\alpha,\mu})_{\#}\mu.$$

By (5.2) $\mathcal{F}_\alpha$ maps $\mathcal{M}^p$ into itself for all $p \geq \alpha$.

LEMMA 5.3.   *Let* $0 < \alpha < 1$ *and let* $p \geq \alpha$. *If* $\mu_k \to \mu$ *in* $\mathcal{M}^p$ *and* $\int w^p \, d\mu_k \to \int w^p \, d\mu$, *then* $\mathcal{F}_\alpha(\mu_k) \to \mathcal{F}_\alpha(\mu)$ *in* $\mathcal{M}^p$.

Proof. Let $\tilde{\mu}_k = w^p \mu_k$ and $\tilde{\mu} = w^p \mu$. Pick a test function $\varphi \in C_0(\mathbb{R}^n)$. We must prove that

$$\int \varphi w^p \, d\mathcal{F}_\alpha(\mu_k) \to \int \varphi w^p \, d\mathcal{F}_\alpha(\mu),$$

which is equivalent to

$$(5.3) \qquad \int (\varphi \circ F_{\alpha,\mu_k}) \frac{(w \circ F_{\alpha,\mu_k})^p}{w^p} \, d\tilde{\mu}_k \to \int (\varphi \circ F_{\alpha,\mu}) \frac{(w \circ F_{\alpha,\mu})^p}{w^p} \, d\tilde{\mu}.$$

Since $\tilde{\mu}_k \xrightarrow{w^*} \tilde{\mu}$ and $\tilde{\mu}_k(\mathbb{R}^n) \to \tilde{\mu}(\mathbb{R}^n)$, it suffices to prove that the integrands in (5.3) converge uniformly in $\mathbb{R}^n$. For this we use Lemma 5.2, whose assumptions are satisfied because $p \geq \alpha$. Since the functions φ and w are uniformly continuous on $\mathbb{R}^n$, Lemma 5.2 implies

$$\varphi \circ F_{\alpha,\mu_k} \to \varphi \circ F_{\alpha,\mu}$$

and

$$\frac{(w \circ F_{\alpha,\mu_k})^p}{w^p} \to \frac{(w \circ F_{\alpha,\mu})^p}{w^p}$$

locally uniformly. Using (5.2), we find that

$$\lim_{|x| \to \infty} \sup_k (\varphi \circ F_{\alpha,\mu_k})(x) = 0$$

and

$$\sup_{x \in \mathbb{R}^n} \sup_k \frac{(w \circ F_{\alpha,\mu_k})(x)}{w(x)} < \infty,$$

hence

$$\lim_{|x| \to \infty} \sup_k (\varphi \circ F_{\alpha,\mu_k})(x) \frac{(w \circ F_{\alpha,\mu_k})^p(x)}{w^p(x)} \to 0.$$

Locally uniform convergence together with uniform vanishing at infinity imply uniform convergence of the integrands in (5.3). $\square$

REMARK 5.4.   The assumptions of Lemma 5.3 are satisfied whenever $\mu_k \to \mu$ in $\mathcal{M}^q$ for some $q > p$.

LEMMA 5.5.   *Let* $0 < \alpha < 1$. *The mapping* $\mathcal{F}_\alpha : \mathcal{M}^p \to \mathcal{M}^p$ *is surjective for all* $p \geq 2$.

Proof. Let us introduce the spaces of finite atomic measures

$$\mathcal{M}_{c,m} = \left\{ c \sum_{k=1}^{m} \delta_{a_k} \ : \ a_k \in \mathbb{R}^n \right\},$$

where $c > 0$ and $m \geq 1$. To avoid double indices, write $\delta_A = \sum_{k=1}^{m} \delta_{a_k}$ where $A = (a_1, \ldots, a_m) \in \mathbb{R}^{mn}$. Let $\mu = c\delta_A$ for some $A \in \mathbb{R}^{mn}$. Then $\mathcal{F}_\alpha(\mu) = c\delta_B$, where

$$(5.4) \qquad b_k = a_k + c \sum_{l \neq k} (a_k - a_l)|a_k - a_l|^{\alpha-1}, \quad k = 1, \ldots, m.$$

Define $\Phi : \mathbb{R}^{mn} \to \mathbb{R}$ by

$$\Phi(A) = \frac{1}{2} \sum_{k=1}^{m} |a_k|^2 + \frac{c}{\alpha+1} \sum_{1 \leq k < l \leq m} |a_k - a_l|^{\alpha+1}.$$

It is easy to see that Φ is a co-finite, strictly convex function and therefore $\nabla\Phi : \mathbb{R}^{mn} \to \mathbb{R}^{mn}$ is a bijection. Furthermore, $\nabla\Phi(A) = B$, where A and B are related by (5.4). This shows that $\mathcal{F}_\alpha : \mathcal{M}_{c,m} \to \mathcal{M}_{c,m}$ is a bijection.

By the same argument as in the first part of Lemma 5.1 we have

$$|\nabla\Phi(A) - \nabla\Phi(A')| \geq |A - A'|, \quad A, A' \in \mathbb{R}^{mn}.$$

Since $\nabla\Phi(0) = 0$, it follows that $|\nabla\Phi(A)| \geq |A|$ for all $A \in \mathbb{R}^{mn}$. To put it another way,

$$\int |x|^2 \, d\mathcal{F}_\alpha(\mu)(x) \geq \int |x|^2 \, d\mu(x), \quad \mu \in \mathcal{M}_{c,m}.$$

This and $\mu(\mathbb{R}^n) = cm = \mathcal{F}_\alpha(\mu)(\mathbb{R}^n)$ imply

$$(5.5) \qquad \int (1 + |x|^2) \, d\mathcal{F}_\alpha(\mu)(x) \geq \int (1 + |x|^2) \, d\mu(x), \quad \mu \in \mathcal{M}_{c,m}.$$

Given $\nu \in \mathcal{M}^p$, let $\tilde{\nu} = w^p \nu$ and find a sequence $\{\tilde{\nu}_k\} \subset \bigcup_{c,m} \mathcal{M}_{c,m}$ such that $\tilde{\nu}_k \xrightarrow{w^*} \tilde{\nu}$ and $\tilde{\nu}_k(\mathbb{R}^n) \to \tilde{\nu}(\mathbb{R}^n)$. It follows that the family of measures $\{\tilde{\nu}_k\}$ is *tight* in the sense that

$$(5.6) \qquad \lim_{R \to \infty} \sup_k \tilde{\nu}_k(\mathbb{R}^n \setminus B(0, R)) = 0.$$

For each k let $\nu_k = w^{-p} \tilde{\nu}_k$ and let $\mu_k \in \bigcup_{c,m} \mathcal{M}_{c,m}$ be such that $\mathcal{F}_\alpha(\mu_k) = \nu_k$. By virtue of (5.5)

$$(5.7) \qquad \sup_k \int w^2(x) \, d\mu_k(x) \leq 2 \sup_k \int (1 + |x|^2) \, d\nu_k(x)$$

$$\leq 2 \sup_k \int w^p d\nu_k < \infty.$$

Let $F_k = F_{\alpha,\mu_k}$. By (5.2) and (5.7) there is a constant C such that $F_k(x) \geq |x| - C$ for all k. Since $\nu_k = (F_k)_{\#}\mu_k$, it follows that

$$\int_{|x|>R} (1+|x|)^p d\mu_k(x) = \int_{|F_k^{-1}(x)|>R} (1+|F_k^{-1}(x)|)^p d\nu_k(x)$$
$$\leq \int_{|x|>R-C} (1+|x|-C)^p d\nu_k(x)$$
$$\leq \tilde{\nu}_k(\mathbb{R}^n \setminus B(0, R-C)), \quad R > C.$$

Combining this with (5.6), we conclude that the measures $\tilde{\mu}_k = w^p \mu_k$ form a tight family. It follows (see, e.g., [34, p. 133]) that there exists a weak*-convergent subsequence $\{\tilde{\mu}_{k_j}\}$ with a limit $\tilde{\mu}$ such that $\tilde{\mu}_{k_j}(\mathbb{R}^n) \to \tilde{\mu}(\mathbb{R}^n)$. The measure $\mu = w^{-p}\tilde{\mu}$ evidently belongs to $\mathcal{M}^p$. By Lemma 5.3 we have $\mathcal{F}_\alpha(\mu) = \nu$. □

THEOREM 5.6. *Let $E \subset \mathbb{R}^n$ be a set of Hausdorff dimension less than 1. Then there exists a q.u. convex function $u : \mathbb{R}^n \to \mathbb{R}$ such that*

(5.8) $$\operatorname*{ess\,lim}_{y \to x} \det D^2 u(y) = 0, \quad x \in E.$$

Proof. As in the proof of Theorem 4.2, we can find $\alpha \in (0,1)$ and $\nu \in \mathcal{M}^2$ such that $\mathcal{I}_{n+\alpha-1}\nu = \infty$ on E. Let $\mu \in \mathcal{M}^2$ be such that $\mathcal{F}_\alpha(\mu) = \nu$. By (5.1) the mapping $T = F_{\alpha,\mu}^{-1}$ is a contraction. Since $\mu = T_{\#}\nu$, for every $x \in E$ we have

$$\int |T(x) - y|^{\alpha-1} d\mu(y) = \int |T(x) - T(y)|^{\alpha-1} d\nu(y) \geq \int |x - y|^{\alpha-1} d\nu(y) = \infty.$$

In other words, the Riesz potential $\mathcal{I}_{n+\alpha-1}\mu$ is infinite on $T(E)$. As in the proof of Theorem 4.2, we obtain

$$\operatorname*{ess\,lim}_{y \to x} \det D^2 \rho_{\alpha,\mu}(y) = \infty, \quad x \in T(E).$$

Since $\rho_{\alpha,\mu}$ is q.u. convex, so is $\rho_{\alpha,\mu}^*$, by virtue of Lemma 2.8. Finally, using the fact that $\nabla \rho_{\alpha,\mu}^* = T$ is the inverse of $\nabla \rho_{\alpha,\mu}$, we obtain

$$\operatorname*{ess\,lim}_{y \to x} \det D^2 \rho_{\alpha,\mu}^*(y) = 0, \quad x \in E.$$

This completes the proof. □

COROLLARY 5.7. *Every set $E \subset \mathbb{R}^n$ of Hausdorff dimension less than 1 is a quasiconformal 0-set.*

Corollary 5.7 is a more satisfactory result than Corollary 4.3, because the former is sharp as far as the dimension of E is concerned. However, there exist quasiconformal 0-sets of Hausdorff dimension greater than 1 [17], [35], which means that quasiconformal 0-sets cannot be characterized by their size alone.

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