Illinois Journal of Mathematics
Volume 47, Number 4, Winter 2003, Pages 1303–1326
S 0019-2082

TYPES OF RADON-NIKODYM PROPERTIES FOR THE PROJECTIVE TENSOR PRODUCT OF BANACH SPACES

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ABSTRACT. Let X and Y be Banach spaces such that X has a boundedly complete basis. Then $X\hat{\otimes}Y$, the projective tensor product of X and Y, has the Radon-Nikodym property (resp. the analytic Radon-Nikodym property, the near Radon-Nikodym property, contains no copy of c_0) if and only if Y has the same property.

1. Preliminaries

Throughout this paper G will denote a compact metrizable abelian group, $\mathcal{B}(G)$ is the σ-algebra of Borel subsets of G, and λ is normalized Haar measure on G. The dual group of G will be denoted by Γ.

Let X be a real or complex Banach space. We denote by $L_1(G,X)$ (respectively, $L_\infty(G,X)$) the Banach space of (all equivalence classes of) λ-Bochner integrable functions on G with values in X (respectively, (all equivalence classes of) λ-measurable X-valued functions that are essentially bounded).

If μ is a countably additive X-valued measure on $\mathcal{B}(G)$, we say that it is of bounded variation if $\sup\sum_{A\in\pi}\|\mu(A)\|<\infty$, where the supremum is taken over all finite measurable partitions of G. The measure μ is said to be of bounded average range if there is a positive constant c so that $\|\mu(A)\|\le c\lambda(A)$, for every $A\in\mathcal{B}(G)$.

We will denote by $\mathcal{M}_1(G,X)$ the space of all X-valued measures on $\mathcal{B}(G)$ that are of bounded variation, and $\mathcal{M}_\infty(G,X)$ will denote the space of all X-valued measures on $\mathcal{B}(G)$ that are of bounded average range.

For $\gamma\in\Gamma$ and $f\in L_1(G,X)$, we define the Fourier coefficient of f at γ by

$$\hat{f}(\gamma)=\int_G f(t)\overline{\gamma}(t)d\lambda(t).$$

Received December 17, 2002; received in final form April 15, 2003.

2000 *Mathematics Subject Classification.* Primary 46M05, 46B22, 46E40.

The research of Eve Oja was partially supported by Estonian Science Foundation Grant 4400.

Similarly, if $\mu \in \mathcal{M}_1(G, X)$, we define the Fourier coefficient of μ at γ by

$$\hat{\mu}(\gamma) = \int_G \overline{\gamma}(t) d\mu(t).$$

Let Λ be a subset of Γ. A measure $\mu \in \mathcal{M}_1(G, X)$ will be called a Λ-measure if $\hat{\mu}(\gamma) = 0$ for all $\gamma \notin \Lambda$.

Definition 1 ([17], [15]). Let G be a compact metrizable abelian group, let Λ be a subset of Γ, and let X be a Banach space. We say that X has type I-Λ-Radon-Nikodym property (I-Λ-RNP) if every Λ-measure μ in $\mathcal{M}_\infty(G, X)$ is differentiable; that is, there is a function $f \in L_1(G, X)$ such that $\mu(E) = \int_E f d\lambda$ for all $E \in \mathcal{B}(G)$.

Definition 2 ([15]). Let G be a compact metrizable abelian group, let Λ be a subset of Γ, and let X be a Banach space. We say that X has type II-Λ-Radon-Nikodym property (II-Λ-RNP) if every λ-continuous, Λ-measure in $\mathcal{M}_1(G, X)$ is differentiable.

Remark 1. Let G be the Cantor group, that is, $G = \{-1, 1\}^{\mathbb{N}}$. Then $\Gamma = \{-1, 1\}^{(\mathbb{N})}$ and Fourier coefficients of measures on $\mathcal{B}(G)$ with values in a real or complex Banach space are well-defined. If $\Lambda = \Gamma$, then I-Λ-RNP, II-Λ-RNP and the usual Radon-Nikodym property are all equivalent for both real and complex Banach spaces. Since Γ is infinite and discrete, it contains an infinite Sidon subset [41, page 126]. If Λ is such an infinite Sidon set, then by [16] a real or complex Banach space has I-Λ-RNP if and only if it has II-Λ-RNP if and only if it does not contain a copy of c_0.

Remark 2. If $G = \mathbb{T}$, the circle group, then $\Gamma = \mathbb{Z}$. Let X be a complex Banach space. If $\Lambda = \mathbb{Z}$, then X has I-Λ-RNP if and only if X has II-Λ-RNP if and only if X has the Radon-Nikodym property. If $\Lambda = \mathbb{N} \cup \{0\}$, then X has I-Λ-RNP if and only if X has II-Λ-RNP if and only if X has the analytic Radon-Nikodym property (see [15]). If Λ is an infinite Sidon set (for example $\{2^n : n \in \mathbb{N}\}$), then X has I-Λ-RNP if and only if X has II-Λ-RNP if and only if X does not contain a subspace isomorphic to c_0 (see [16]) .

Another Radon-Nikodym property that we will deal with is the near Radon-Nikodym property, which was introduced in [26].

Definition 3. Let X be a Banach space. A bounded linear operator $T : L_1[0, 1] \to X$ is said to be near representable if for each Dunford-Pettis operator $D : L_1[0, 1] \to L_1[0, 1]$, the composition operator $T \circ D : L_1[0, 1] \to X$ is Bochner representable; that is, there exists $g \in L_\infty([0, 1], X)$ such that $T \circ D(f) = \int_{[0,1]} fg\, dm$ for all $f \in L_1[0, 1]$. A Banach space X is said to have the near Radon-Nikodym property (NRNP) if every near representable operator from $L_1[0, 1]$ to X is Bochner representable.

For comparison, let us recall that a Banach space X has the Radon-Nikodym property if and only if every bounded linear operator $T : L_1[0,1] \to X$ is Bochner representable [12, page 63].

For any Banach space X, we will denote its topological dual by X^* and its closed unit ball by B_X. For two Banach spaces X and Y, let $\mathcal{L}(X,Y)$ denote the space of all continuous linear operators from X to Y with its operator norm $\|\cdot\|$, and let $X\hat{\otimes}Y$ denote the completion of the tensor product $X \otimes Y$ with respect to the projective tensor norm. It is known that the dual of $X\hat{\otimes}Y$ is isometrically isomorphic to $\mathcal{L}(X,Y^*)$ (see [12, page 230]).

2. Radon-Nikodym properties and boundedly complete Schauder decompositions

Let X be a Banach space. A Schauder decomposition of X is a sequence $(X_n)_{n=1}^{\infty}$ of non-trivial closed subspaces of X such that every $x \in X$ can be expressed uniquely in the form $x = \sum_{n=1}^{\infty} x_n$, where $x_n \in X_n$ for every $n \in \mathbb{N}$. Clearly, a sequence $(e_n)_{n=1}^{\infty}$ in X is a basis of X if and only if the one-dimensional subspaces $X_n = \operatorname{span}\{e_n\}$ form a Schauder decomposition of X.

A Schauder decomposition $(X_n)_{n=1}^{\infty}$ is boundedly complete if, whenever $(\sum_{n=1}^{m} x_n)_{m=1}^{\infty}$ is a bounded sequence with $x_n \in X_n$ for every $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} x_n$ converges.

The following theorem, which is the main result of this paper, shows that the Radon-Nikodym properties, considered in Section 1, are inherited by Banach spaces having a boundedly complete Schauder decomposition.

Recall that Dunford showed that a Banach space with a boundedly complete Schauder basis has the Radon-Nikodym property [12, page 64, Theorem 6]. The proof of the following theorem is similar to Dunford's proof.

THEOREM 4. *Let G be a compact metrizable abelian group and let Λ be a subset of Γ. Let X be a Banach space having a boundedly complete Schauder decomposition $(X_n)_{n=1}^{\infty}$. Then X has I-Λ-RNP, II-Λ-RNP or, respectively, the NRNP if each X_n, $n \in \mathbb{N}$, has the same property.*

Proof. We will first give the proof for II-Λ-RNP. The almost identical proof for I-Λ-RNP will be omitted.

Let $P_i : X \longrightarrow X_i$ be the coordinate projections defined by $P_i(\sum_n x_n) = x_i$. It is well known that these projections are bounded linear operators. Since II-Λ-RNP is invariant under equivalent renormings, we may assume, without loss of generality, that the Schauder decomposition is monotone. This means that for each $n \in \mathbb{N}$

$$\left\|\sum_{i=1}^{n} x_i\right\| \le \left\|\sum_{i=1}^{n+1} x_i\right\|$$

whenever $x_i \in X_i$, for $i \in \mathbb{N}$.

Let $\mu : \mathcal{B}(G) \longrightarrow X$ be a Λ-measure of bounded variation which is absolutely continuous with respect to λ. For each $i \in \mathbb{N}$, define

$$\begin{array}{rrcl} \mu_i : & \mathcal{B}(G) & \longrightarrow & X_i \\ & E & \longmapsto & P_i(\mu(E)). \end{array}$$

It is easy to show that μ_i is a Λ-measure of bounded variation which is absolutely continuous with respect to λ, for each $i \in \mathbb{N}$. Since each X_i has II-Λ-RNP, there exists $f_i \in L_1(G, X_i)$ such that

$$\mu_i(E) = \int_E f_i \, d\lambda \,, \quad E \in \mathcal{B}(G), \quad i = 1, 2, \ldots .$$

For each $n \in \mathbb{N}$, define

$$\begin{array}{rrcl} \tilde{f}_n : & G & \longrightarrow & X \\ & t & \longmapsto & \sum_{i=1}^n f_i(t). \end{array}$$

Since each $f_i \in L_1(G, X_i)$ and each X_i is a subspace of X, each $f_i \in L_1(G, X)$, and hence $\tilde{f}_n \in L_1(G, X)$ for each $n \in \mathbb{N}$. Now define

$$\begin{array}{rrcl} \tilde{\mu}_n : & \mathcal{B}(G) & \longrightarrow & X \\ & E & \longmapsto & \sum_{i=1}^n \mu_i(E). \end{array}$$

Furthermore, since $(X_n)_{n=1}^\infty$ is monotone,

$$\|\tilde{\mu}_n(E)\| = \left\| \sum_{i=1}^n \mu_i(E) \right\| \leq \left\| \sum_{i=1}^\infty \mu_i(E) \right\| = \|\mu(E)\|.$$

Therefore,

$$|\tilde{\mu}_n|(E) \leq |\mu|(E), \quad E \in \mathcal{B}(G), \quad n = 1, 2, \ldots .$$

Now for each $E \in \mathcal{B}(G)$ and each $i, n \in \mathbb{N}$ with $i \leq n$,

$$\begin{aligned} P_i(\tilde{\mu}_n(E)) = \mu_i(E) &= \int_E f_i(t) \, d\lambda(t) \\ &= \int_E P_i(\tilde{f}_n(t)) \, d\lambda(t) \\ &= P_i \left(\int_E \tilde{f}_n(t) \, d\lambda(t) \right), \end{aligned}$$

and hence

$$\tilde{\mu}_n(E) = \int_E \tilde{f}_n(t) \, d\lambda(t), \quad E \in \mathcal{B}(G), \quad n = 1, 2, \ldots .$$

Thus for each $E \in \mathcal{B}(G)$ and each $n \in \mathbb{N}$,

$$\begin{aligned} \int_E \left\| \sum_{i=1}^n f_i(t) \right\| d\lambda(t) &= \int_E \|\tilde{f}_n\| \, d\lambda = |\tilde{\mu}_n|(E) \\ &\leq |\mu|(E) \leq |\mu|(G) < \infty. \end{aligned}$$

Note that

$$\left\| \sum_{i=1}^{n} f_i(t) \right\| \leq \left\| \sum_{i=1}^{n+1} f_i(t) \right\|, \quad n = 1, 2, \dots .$$

By the Monotone Convergence Theorem, for each $E \in \mathcal{B}(G)$,

$$\begin{aligned} \int_E \sup_n \left\| \sum_{i=1}^{n} f_i(t) \right\| \, d\lambda(t) &= \int_E \lim_n \left\| \sum_{i=1}^{n} f_i(t) \right\| \, d\lambda(t) \\ &= \lim_n \int_E \left\| \sum_{i=1}^{n} f_i(t) \right\| \, d\lambda(t) \\ &\leq |\mu|(G) < \infty. \end{aligned}$$

Hence

$$\sup_n \left\| \sum_{i=1}^{n} f_i(t) \right\| < \infty, \quad \lambda\text{-a.e.}.$$

Since $(X_n)_{n=1}^{\infty}$ is also boundedly complete, the series $\sum_i f_i(t)$ converges in X, λ-a.e.. Now define

$$\begin{aligned} \tilde{f} : \quad G &\longrightarrow X \\ t &\longmapsto \textstyle\sum_{i=1}^{\infty} f_i(t), \quad \lambda\text{-a.e.}. \end{aligned}$$

Note that $\lim_n \tilde{f}_n(t) = \tilde{f}(t)$, λ-a.e. in X. Thus $\tilde{f}$ is λ-measurable. Furthermore,

$$\int_G \|\tilde{f}(t)\| \, d\lambda(t) = \int_G \left\| \sum_{i=1}^{\infty} f_i(t) \right\| \, d\lambda(t) \leq |\mu|(G) < \infty.$$

Therefore,

$$\tilde{f} \in L_1(G, X).$$

Now for each $E \in \mathcal{B}(G)$ and each $i \in \mathbb{N}$,

$$\begin{aligned} P_i \left(\int_E \tilde{f}(t) \, d\lambda(t) \right) &= \int_E P_i \tilde{f}(t) \, d\lambda(t) = \int_E f_i(t) \, d\lambda(t) \\ &= \mu_i(E) = P_i(\mu(E)), \end{aligned}$$

and so

$$\mu(E) = \int_E \tilde{f}(t) \, d\lambda(t), \quad E \in \mathcal{B}(G).$$

It follows that $\tilde{f}$ is a Radon-Nikodym derivative of μ, and hence X has II-Λ-RNP. This completes the proof for II-Λ-RNP.

We will now give the proof for the NRNP. Let $T : L_1[0,1] \to X$ be a nearly representable operator. As in the first part of the proof of this theorem, it is easy to show that the operators $P_i \circ T : L_1[0,1] \to X_i$ are also nearly representable for each i, and hence, for each i, $P_i \circ T$ is Bochner representable since each X_i has the NRNP. Now, just as in the first part of the proof, we can show that T is Bochner representable. Consequently, X has the NRNP and the proof is complete. □

REMARK 3. A special case of Theorem 4 asserts (see Remarks 1 and 2) that X does not contain a subspace isomorphic to c_0 if each of the X_n do not contain a subspace isomorphic to c_0. This result was established in [34, Lemma 3].

3. Applications to vector-valued sequence spaces and projective tensor products

Let U be a Banach space with a boundedly complete 1-unconditional normalized basis $(e_i)_{i=1}^\infty$; the 1-unconditionality means that, for all $n \in \mathbb{N}$, and scalars $a_1, a_2, \dots, a_n$ and $s_1, s_2, \dots, s_n$ with $|s_i| = 1$ for each $1 \le i \le n$, $\| \sum_{i=1}^n s_i a_i e_i \| \le \| \sum_{i=1}^n a_i e_i \|$.

It is well known and easy to verify (using the Hahn-Banach Theorem) that for each $n \in \mathbb{N}$, $\| \sum_{i=1}^n a_i e_i \| \le \| \sum_{i=1}^n b_i e_i \|$ whenever $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are scalars with $|a_i| \le |b_i|$ for each $1 \le i \le n$.

For a sequence $(X_i)_{i=1}^\infty$ of Banach spaces, define

$$U(X_i) = \left\{ \bar{x} = (x_i)_i : x_i \in X_i, \sum_i \|x_i\| e_i \text{ converges in } U \right\},$$

and define the norm on $U(X_i)$ to be

$$\|\bar{x}\|_{U(X_i)} = \left\| \sum_{i=1}^\infty \|x_i\| e_i \right\|_U .$$

PROPOSITION 5. *The space $U(X_i)$ is a Banach space and the subspaces $\{(0, \dots, 0, x_i, 0, \dots) : x_i \in X_i\}$, $i \in \mathbb{N}$, form its boundedly complete Schauder decomposition.*

Proof. Let us observe that for each $\bar{x} = (x_i)_i \in U(X_i)$,

$$\sup_m \left\| \sum_{i=1}^m \|x_i\| e_i \right\|_U \le \|\bar{x}\|_{U(X_i)}$$

and, for each $i \in \mathbb{N}$,

$$\|x_i\| = \left\| \|x_i\| e_i \right\|_U \le \|\bar{x}\|_{U(X_i)}.$$

The last inequality shows that the coordinate projections from $U(X_i)$ to X_i are continuous.

To show that $U(X_i)$ is a Banach space, consider $\bar{x}^{(n)} = (x_i^{(n)})_i \in U(X_i)$ such that $(\bar{x}^{(n)})_{n=1}^\infty$ is a Cauchy sequence in $U(X_i)$. Then

$c = \sup_n \|\bar{x}^{(n)}\|_{U(X_i)} < \infty$ and for each $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for $n, k > n_0$,

$$\|\bar{x}^{(n)} - \bar{x}^{(k)}\|_{U(X_i)} < \varepsilon/2\,. \tag{1}$$

By the continuity of coordinate projections from $U(X_i)$ to X_i, $(x_i^{(n)})_{n=1}^{\infty}$ is a Cauchy sequence in X_i for each $i \in \mathbb{N}$. Hence there is $x_i \in X_i$ such that

$$\lim_n x_i^{(n)} = x_i\,, \qquad i = 1, 2, \ldots.$$

Thus for each fixed $m \in \mathbb{N}$, there exists an $m_0 \in \mathbb{N}$ with $m_0 > n_0$ such that

$$\|x_i^{(m_0)} - x_i\| < \varepsilon/2m\,, \qquad i = 1, 2, \ldots, m. \tag{2}$$

Note that

$$\begin{aligned}\left\|\sum_{i=1}^{m} \|x_i\|\, e_i\right\|_U &\le \left\|\sum_{i=1}^{m} \|x_i - x_i^{(m_0)}\|\, e_i\right\|_U + \left\|\sum_{i=1}^{m} \|x_i^{(m_0)}\|\, e_i\right\|_U \\ &\le \varepsilon/2 + \|\bar{x}^{(m_0)}\|_{U(X_i)} \le \varepsilon/2 + c.\end{aligned}$$

So

$$\sup_m \left\|\sum_{i=1}^{m} \|x_i\|\, e_i\right\|_U \le \varepsilon/2 + c < \infty.$$

Since the basis $(e_i)_{i=1}^{\infty}$ is boundedly complete, $\sum_i \|x_i\|\, e_i$ converges in U, and hence $\bar{x} = (x_i)_i \in U(X_i)$. Furthermore, by (1) and (2), for each $n > n_0$,

$$\begin{aligned}\left\|\sum_{i=1}^{m} \|x_i^{(n)} - x_i\|\, e_i\right\|_U &\le \left\|\sum_{i=1}^{m} \|x_i^{(n)} - x_i^{(m_0)}\|\, e_i\right\|_U + \left\|\sum_{i=1}^{m} \|x_i^{(m_0)} - x_i\|\, e_i\right\|_U \\ &\le \|\bar{x}^{(n)} - \bar{x}^{(m_0)}\|_{U(X_i)} + \varepsilon/2 \le \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Thus for each $n > n_0$,

$$\|\bar{x}^{(n)} - \bar{x}\|_{U(X_i)} = \sup_m \left\|\sum_{i=1}^{m} \|x_i^{(n)} - x_i\|\, e_i\right\|_U \le \varepsilon.$$

Therefore, $(\bar{x}^{(n)})_{n=1}^{\infty}$ converges to $\bar{x}$ in $U(X_i)$. This proves that $U(X_i)$ is a Banach space.

To see that the subspaces $\{(0, \ldots, 0, x_i, 0, \ldots) : x_i \in X_i\}$, $i \in \mathbb{N}$, form a Schauder decomposition for $U(X_i)$, we denote by $\bar{x}_i$ the element $(0, \ldots, 0, x_i, 0, \ldots)$ in $U(X_i)$, where $x_i \in X_i$, and observe that, for any $\bar{x} = (x_i)_i \in U(X_i)$,

$$\left\|\bar{x} - \sum_{i=1}^{m} \bar{x}_i\right\|_{U(X_i)} = \left\|\sum_{i=m+1}^{\infty} \|x_i\| e_i\right\|_U \to 0 \quad \text{as} \quad m \to \infty. \tag{3}$$

The Schauder decomposition is boundedly complete because

$$\sup_m \left\| \sum_{i=1}^{m} \bar{x}_i \right\|_{U(X_i)} = \sup_m \left\| \sum_{i=1}^{m} \|x_i\| e_i \right\|_U < \infty$$

implies that $\sum_{i=1}^{\infty} \|x_i\| e_i$ converges in U. Hence, $\bar{x} = (x_i)_i \in U(X_i)$ and, by (3), $\sum_{i=1}^{\infty} \bar{x}_i = \bar{x}$. □

Remark 4. The last part of the above proof shows that the Schauder decomposition is a complete Schauder decomposition for the normed linear space $U(X_i)$. Therefore, $U(X_i)$ is a Banach space by [25].

Theorem 4 and Proposition 5 immediately yield:

Theorem 6. *The space $U(X_i)$ has I-Λ-RNP, II-Λ-RNP or, respectively, the NRNP if all of the Banach spaces X_i have the same property.*

Remark 5. If $U = \ell_p$, $1 \le p < \infty$, and $(e_i)_{i=1}^{\infty}$ is the unit vector basis of U, then $U(X_i) = \ell_p(X_i)$ is clearly the usual ℓ_p-direct sum of Banach spaces X_i. It is well known (see [12, page 219]) that $\ell_p(X_i)$ has the Radon-Nikodym property if all the X_i have the Radon-Nikodym property. The particular case of Theorem 6 for $U(X_i)$, where each X_i is equal to a Banach space X and U is an equivalent renorming of $L_p[0,1]$, $1 < p < \infty$, with its normalized Haar basis, was established in [5].

Let X be a Banach space with a boundedly complete Schauder decomposition $(X_n)_{n=1}^{\infty}$, where each of the spaces X_n are finite dimensional; such a decomposition is called a boundedly complete FDD. Let $P_i : X \to X_i$ be the coordinate projection defined by $P_i(\sum_n x_n) = x_i$. Let Y be a Banach space and let I_Y denote the identity operator on Y. Consider the natural tensor product of the operators P_i and I_Y; $\pi_i = P_i \otimes I_Y : X\hat{\otimes}Y \to X\hat{\otimes}Y$. It is easily verified (see [21]) that $(\pi_i(X\hat{\otimes}Y))_{i=1}^{\infty}$ is a Schauder decomposition of $X\hat{\otimes}Y$. Also note that since each X_i is finite dimensional, $\pi_i(X\hat{\otimes}Y)$ is isomorphic to $\ell_1^{\dim(X_i)}(Y)$. Consequently, each subspace $\pi_i(X\hat{\otimes}Y)$ of $X\hat{\otimes}Y$ has I-Λ-RNP, II-Λ-RNP or, respectively, the NRNP if Y has the same property. Moreover, in [33, Proposition 1] it is proved that if X has a boundedly complete FDD, then $(\pi_i(X\hat{\otimes}Y))_{i=1}^{\infty}$ is a boundedly complete Schauder decomposition of $X\hat{\otimes}Y$. Therefore we immediately get from Theorem 4:

Theorem 7. *Let X be a Banach space with a boundedly complete FDD and let Y be a Banach space. Then $X\hat{\otimes}Y$, the projective tensor product of X and Y, has I-Λ-RNP, II-Λ-RNP or, respectively, the NRNP if Y has the same property.*

A specific case of Theorem 7 is when one of the spaces has a boundedly complete basis. We explicitly state this result so we can refer back to it in later sections.

THEOREM 8. *Let X be a Banach space with a boundedly complete basis and let Y be a Banach space. Then $X\hat{\otimes}Y$, the projective tensor product of X and Y, has I-Λ-RNP, II-Λ-RNP or, respectively, the NRNP if Y has the same property.*

Let us recall that a Banach space with a boundedly complete basis has the Radon-Nikodym property.

REMARK 6. The following result, giving a particular case of Theorem 8, was proved by Holub [23] (see also [42, Proposition 4.28]): if X and Y are Banach spaces with boundedly complete bases, then $X\hat{\otimes}Y$ has a boundedly complete basis.

The particular case of Theorem 8 with $X = L_p[0,1]$, $1 < p < \infty$, was proved in [7] using a different method which, in fact, will be developed further in the next section of this paper. This method was first used in [6] and then in [5] to show, respectively, that $\ell_p\hat{\otimes}X$ and $L_p[0,1]\hat{\otimes}X$, $1 < p < \infty$, have the Radon-Nikodym property whenever X has the Radon-Nikodym property.

REMARK 7. A particular case of Theorem 8 (see Remarks 1 and 2) asserts that $X\hat{\otimes}Y$ contains no copy of c_0 whenever X has a boundedly complete basis and Y contains no copy of c_0. A similar result is true for complemented copies of c_0 (see [35, Theorem 3]). Moreover (see [33, Theorem 3] and [36, Theorem 2]), if $1 \le p < q < \infty$, then $\ell_p\hat{\otimes}X$ contains no (complemented) copy of ℓ_q, whenever X contains no (complemented) copy of ℓ_q. These results were proved, like Theorem 7, using the natural Schauder decomposition of $X\hat{\otimes}Y$ associated to the basis of X.

James [24] (see [29, Theorem 1.c.10]) showed that an unconditional basis for a Banach space is boundedly complete if the space contains no subspace isomorphic to c_0. This is the case when the space has the (analytic) Radon-Nikodym property or near Radon-Nikodym property. Therefore, from Theorem 8 and Remarks 1 and 2, we immediately obtain:

THEOREM 9. *Let X and Y be Banach spaces such that one of them has an unconditional basis. Then $X\hat{\otimes}Y$, the projective tensor product of X and Y, has the Radon-Nikodym property, the analytic Radon-Nikodym property, the near Radon-Nikodym property or, respectively, contains no subspace isomorphic to c_0 if both X and Y have the same property.*

REMARK 8. It is well known that the reflexive Banach spaces have the Radon-Nikodym property. However, Theorem 9 does not remain valid for reflexivity. In [33, Theorem 2], it is proved that if X and Y are reflexive Banach spaces such that one of them has an unconditional basis, then $X\hat{\otimes}Y$ is reflexive if and only if it contains no complemented subspace isomorphic to ℓ_1. (Notice, for instance, that $\ell_2\hat{\otimes}\ell_2$ contains a complemented subspace isomorphic to ℓ_1, but $\ell_2\hat{\otimes}\ell_3$ does not (see, for example, [42, Example 2.10 and Corollary 4.24] or [33, Theorems 4 and 5]).)

Remark 9. In general, the Radon-Nikodym property and the property of not containing c_0 isomorphically are not stable under projective tensor products: the Banach space X constructed by Bourgain and Pisier [3, Corollary 2.4] has the Radon-Nikodym property (and hence X contains no subspace isomorphic to c_0), but the projective tensor product $X\hat{\otimes}X$ contains c_0 isomorphically.

4. Semi-embeddings of $U\hat{\otimes}X$ into $U(X)$

If X and Y are Banach spaces, then a mapping $T : X \to Y$ is called a semi-embedding if T is injective and $T(B_X)$ is closed in Y. An important result in the theory of semi-embeddings, appearing in a paper of Bourgain and Rosenthal [4], which they attribute to F. Delbaen, is: if X is a separable Banach space, if Y is a Banach space with the Radon-Nikodym property and if there is a semi-embedding $T : X \to Y$ of X into Y, then X has the Radon-Nikodym property. This result of Delbaen has been extended to other types of Radon-Nikodym properties; to the near Radon-Nikodym property in [26], to the type-I-Radon-Nikodym property in [15], and to the type type-II-Radon-Nikodym property in [38].

The main result of this section is that the projective tensor product, $U\hat{\otimes}X$, of the Banach spaces U and X semi-embeds in the sequence space $U(X)$, when U has a boundedly complete unconditional basis. Of course, the space $U(X)$ is the Banach space $U(X_i)$, where all the Banach spaces X_i are equal to X. We will then use this result to obtain an alternate proof of Theorem 9.

Throughout this section, unless otherwise stated, U will denote a Banach space with a normalized boundedly complete 1-unconditional basis $(e_i)_{i=1}^\infty$ and X will denote an arbitrary Banach space. Then the basis $(e_i)_{i=1}^\infty$ will also have normalized biorthogonal functionals, $(e_i^*)_{i=1}^\infty$; that is, $\|e_i\| = \|e_i^*\| = 1$ for all $i \in \mathbb{N}$ and

$$e_i^*(e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

It is well known that $(e_i^*)_{i=1}^\infty$ is an unconditional basic sequence in U^* and (see, for example, [29, Proposition 1.b.4]) U is isometrically isomorphic to the dual space of $V = \overline{\text{span}}\{e_i^* : i \in \mathbb{N}\}$; that is, $U = V^*$. Since the basis $(e_i)_{i=1}^\infty$ is 1-unconditional, we immediately have the following result:

Proposition 10. *Let* $u = \sum_{i=1}^\infty e_i^*(u)e_i \in U$. *Then:*

(i) *For each subset σ of $\mathbb{N}$*, $\|\sum_{i\in\sigma} e_i^*(u)e_i\| \le \|u\|$.
(ii) *For each choice of signs* $\theta = (\theta_i)_{i=1}^\infty$, $\|\sum_{i=1}^\infty \theta_i e_i^*(u)e_i\| \le \|u\|$.
(iii) *For each* $\lambda = (\lambda_i)_i \in \ell_\infty$, $\|\sum_{i=1}^\infty \lambda_i e_i^*(u)e_i\| \le \|\lambda\|_{\ell_\infty} \cdot \|u\|$.

THEOREM 11. *$U\hat{\otimes}X$ semi-embeds in $U(X)$.*

Proof. Throughout the proof, let $\varepsilon > 0$ be arbitrary. Define

$$\begin{array}{rccl} \psi: & U\hat{\otimes}X & \longrightarrow & U(X) \\ & z & \longmapsto & \left(\sum_{k=1}^\infty e_i^*(u_k)x_k\right)_i, \end{array}$$

where $\sum_{k=1}^\infty u_k \otimes x_k$ is a representation of z.

Step 1. ψ is a continuous linear one-to-one map from $U\hat{\otimes}X$ into $U(X)$.
In fact, $z \in U\hat{\otimes}X$ admits a representation $z = \sum_{k=1}^\infty u_k \otimes x_k$ such that

$$\sum_{k=1}^\infty \|u_k\| \cdot \|x_k\| \le \|z\|_{U\hat{\otimes}X} + \varepsilon.$$

For each $i \in \mathbb{N}$, choose $x_i^* \in B_{X^*}$ such that

$$\|\psi(z)_i\| \le \langle \psi(z)_i, x_i^*\rangle + \varepsilon/2^i, \qquad i = 1, 2, \ldots.$$

By Proposition 10, for each $m \in \mathbb{N}$,

$$\begin{aligned} \left\|\sum_{i=1}^m \|\psi(z)_i\| e_i\right\|_U &\le \left\|\sum_{i=1}^m (\langle \psi(z)_i, x_i^*\rangle + \varepsilon/2^i)e_i\right\|_U \\ &\le \left\|\sum_{i=1}^m \langle \sum_{k=1}^\infty e_i^*(u_k)x_k, x_i^*\rangle e_i\right\|_U + \sum_{i=1}^m \varepsilon/2^i \\ &\le \sum_{k=1}^\infty \left\|\sum_{i=1}^m e_i^*(u_k)x_i^*(x_k)e_i\right\|_U + \varepsilon \\ &\le \sum_{k=1}^\infty \|x_k\| \cdot \left\|\sum_{i=1}^\infty e_i^*(u_k)e_i\right\|_U + \varepsilon \\ &= \sum_{k=1}^\infty \|x_k\| \cdot \|u_k\| + \varepsilon \\ &\le (\|z\|_{U\hat{\otimes}X} + \varepsilon) + \varepsilon. \end{aligned}$$

Therefore,

$$\sup_m \left\|\sum_{i=1}^m \|\psi(z)_i\| e_i\right\|_U \le \|z\|_{U\hat{\otimes}X}.$$

Since $(e_i)_{i=1}^\infty$ is a boundedly complete basis of U, the series $\sum_i \|\psi(z)_i\| e_i$ converges in U, and hence $\psi(z) \in U(X)$ with $\|\psi(z)\|_{U(X)} \le \|z\|_{U\hat{\otimes}X}$. Therefore, ψ is a well-defined continuous linear map.

To show ψ is one-to-one, suppose that $\psi(z) = 0$. Then z admits a representation $z = \sum_{k=1}^{\infty} u_k \otimes x_k$ such that

$$\psi(z)_i = \sum_{k=1}^{\infty} e_i^*(u_k)x_k = 0\,, \qquad i = 1, 2, \dots .$$

Now for each $T \in \mathcal{L}(U, X^*) = (U\hat{\otimes}X)^*$,

$$\begin{aligned}\langle z, T\rangle &= \sum_{k=1}^{\infty}\langle Tu_k, x_k\rangle = \sum_{k=1}^{\infty}\langle\sum_{i=1}^{\infty} e_i^*(u_k)Te_i, x_k\rangle \\ &= \sum_{i=1}^{\infty}\langle Te_i, \sum_{k=1}^{\infty} e_i^*(u_k)x_k\rangle = 0.\end{aligned}$$

So $z = 0$, and hence ψ is one-to-one. Step 1 is complete.

Next we want to show ψ is a semi-embedding, that is, for a sequence $z_n \in B_{U\hat{\otimes}X}$ and an element $(y_i)_i \in U(X)$ such that $\lim_n \psi(z_n) = (y_i)_i$ in $U(X)$, there exists a $z \in B_{U\hat{\otimes}X}$ such that $\psi(z) = (y_i)_i$.

Step 2. If ϕ is defined by $\langle T, \phi\rangle = \sum_{i=1}^{\infty}\langle y_i, Te_i\rangle$ for each $T \in \mathcal{L}(U, X^*)$, then $\phi \in \mathcal{L}(U, X^*)^*$ with $\|\phi\| \le 1$.

In fact, for each $n \in \mathbb{N}$, $z_n \in U\hat{\otimes}X$ admits a representation

$$z_n = \sum_{k=1}^{\infty} u_{k,n} \otimes x_{k,n}\,, \qquad n = 1, 2, \dots$$

such that

$$\sum_{k=1}^{\infty}\|u_{k,n}\| \cdot \|x_{k,n}\| \le \|z_n\|_{U\hat{\otimes}X} + \varepsilon, \qquad n = 1, 2, \dots .$$

Since $\lim_n \psi(z_n) = \lim_n(\sum_{k=1}^{\infty} e_i^*(u_{k,n})x_{k,n})_i = (y_i)_i$ in $U(X)$,

$$\lim_n \sum_{k=1}^{\infty} e_i^*(u_{k,n})x_{k,n} = y_i\,, \qquad i = 1, 2, \dots .$$

Fix $m \in \mathbb{N}$. Then there exists an $n_0 \in \mathbb{N}$ such that

$$\left\|\sum_{k=1}^{\infty} e_i^*(u_{k,n_0})x_{k,n_0} - y_i\right\| \le \varepsilon/m\,, \qquad i = 1, 2, \dots, m.$$

For any $T \in \mathcal{L}(U, X^*)$, define S by $Su = \sum_{i=1}^{m}\theta_i e_i^*(u)Te_i$ for each $u \in U$, where $\theta_i = \operatorname{sign}(\langle\sum_{k=1}^{\infty} e_i^*(u_{k,n_0})x_{k,n_0}, Te_i\rangle)$. Then by Proposition 10, $S \in$

$\mathcal{L}(U, X^*)$ with $\|S\| \leq \|T\|$. So

$$\begin{aligned}\sum_{i=1}^{m} |\langle y_i, Te_i\rangle| &\leq \sum_{i=1}^{m} \left|\langle y_i - \sum_{k=1}^{\infty} e_i^*(u_{k,n_0})x_{k,n_0}, Te_i\rangle\right| \\ &\quad + \sum_{i=1}^{m} \left|\langle \sum_{k=1}^{\infty} e_i^*(u_{k,n_0})x_{k,n_0}, Te_i\rangle\right| \\ &\leq \sum_{i=1}^{m} \varepsilon/m \cdot \|Te_i\| + \left|\sum_{i=1}^{m} \theta_i \langle \sum_{k=1}^{\infty} e_i^*(u_{k,n_0})x_{k,n_0}, Te_i\rangle\right| \\ &\leq \varepsilon\|T\| + \left|\sum_{k=1}^{\infty} \langle \sum_{i=1}^{m} \theta_i e_i^*(u_{k,n_0})Te_i, x_{k,n_0}\rangle\right| \\ &= \varepsilon\|T\| + \left|\sum_{k=1}^{\infty} \langle Su_{k,n_0}, x_{k,n_0}\rangle\right| = \varepsilon\|T\| + |\langle S, z_{n_0}\rangle| \\ &\leq \varepsilon\|T\| + \|S\| \cdot \|z_{n_0}\| \leq \varepsilon\|T\| + \|T\|.\end{aligned}$$

Letting $m \longrightarrow \infty$ and $\varepsilon \longrightarrow 0$,

$$\sum_{i=1}^{\infty} |\langle y_i, Te_i\rangle| \leq \|T\|.$$

Therefore, ϕ is a well-defined continuous linear functional with $\|\phi\| \leq 1$. Step 2 is complete.

Step 3. There exists a $z \in B_{U\hat{\otimes}X^{**}}$ *such that* $\psi(z) = (y_i)_i$.

In fact, note that $U = V^*$. So $K = (B_U, \text{weak}^*) \times (B_{X^{**}}, \text{weak}^*)$ is a compact Hausdorff space. Define $J : \mathcal{L}(U, X^*) \longrightarrow C(K)$ by $JT(u, x^{**}) = \langle Tu, x^{**}\rangle$ for each $u \in B_U$ and each $x^{**} \in B_{X^{**}}$. Then $\|JT\|_{C(K)} = \|T\|$. So $J(\mathcal{L}(U, X^*))$ is a closed subspace of $C(K)$. Define $F_\phi : J(\mathcal{L}(U, X^*)) \longrightarrow \mathbb{K}$ by $F_\phi(JT) = \langle T, \phi\rangle$ for each $T \in \mathcal{L}(U, X^*)$. Then $\|F_\phi\| = \|\phi\|$. By the Hahn-Banach Theorem, F_ϕ can be norm-preservingly extended to $C(K)$, and moreover, by the Riesz Representation Theorem, there exists a regular Borel measure μ on K such that

$$F_\phi(JT) = \int_K JT(u, x^{**})\, d\mu(u, x^{**}), \qquad T \in \mathcal{L}(U, X^*), \tag{4}$$

and

$$|\mu|(K) = \|F_\phi\| = \|\phi\|. \tag{5}$$

Define

$$\begin{aligned} g: \quad K &\longrightarrow X^{**} \\ (u, x^{**}) &\longmapsto x^{**}. \end{aligned}$$

Then g is weak* continuous and hence weak* μ-measurable. Furthermore, for each $x^* \in X^*$,

$$\int_K |x^* g| \, d|\mu| = \int_K |x^{**}(x^*)| \, d|\mu| \leq \int_K \|x^*\| \, d|\mu| \leq \|x^*\| \cdot |\mu|(K) < \infty.$$

So g is Gel'fand integrable (see [12, page 53]). Define

$$\begin{array}{rrcl} h: & K & \longrightarrow & U \\ & (u, x^{**}) & \longmapsto & u. \end{array}$$

Then h is weak* continuous and hence weak* μ-measurable. Note that U is separable. By [12, page 42, Corollary 4], h is strongly μ-measurable. Moreover,

$$\int_K \|h(u, x^{**})\| \, d|\mu| = \int_K \|u\| \, d|\mu| \leq |\mu|(K) < \infty.$$

So h is Bochner $|\mu|$-integrable. It follows from [12, page 172, Lemma 3] that there exist a sequence $(u_k)_{k=1}^\infty$ of U and a sequence $(E_k)_{k=1}^\infty$ of Borel measurable subsets of K such that

$$h = \sum_{k=1}^\infty u_k \chi_{E_k}\,, \ |\mu|\text{-a.e.}$$

and

$$\sum_{k=1}^\infty \|u_k\| \cdot |\mu|(E_k) \leq \int_K \|h\| \, d|\mu| + \varepsilon \leq |\mu|(K) + \varepsilon.$$

Now for each $T \in \mathcal{L}(U, X^*)$, by (4),

$$\langle T, \phi \rangle = F_\phi(JT) = \int_K JT(u, x^{**}) \, d\mu(u, x^{**}) = \int_K \langle Tu, x^{**} \rangle \, d\mu(u, x^{**}).$$

For each $i \in \mathbb{N}$ and each $x^* \in X^*$, plugging $T_i = e_i^* \otimes x^*$ in the above equality,

$$\begin{aligned} \langle y_i, x^* \rangle &= \langle T_i, \phi \rangle \\ &= \int_K \langle T_i u, x^{**} \rangle \, d\mu(u, x^{**}) \\ &= \int_K \langle e_i^*(u) x^*, x^{**} \rangle, d\mu(u, x^{**}) \\ &= \int_K x^*(g) e_i^*(h) \, d\mu(u, x^{**}) \\ &= \int_K x^*(g) \langle e_i^*, \sum_{k=1}^\infty u_k \chi_{E_k} \rangle \, d\mu(u, x^{**}) \\ &= \int_K \sum_{k=1}^\infty x^*(g) e_i^*(u_k) \chi_{E_k} \, d\mu(u, x^{**}) \end{aligned} \tag{6}$$

$$= \sum_{k=1}^{\infty} e_i^*(u_k) \int_{E_k} x^*(g)\, d\mu(u, x^{**})$$
$$= \sum_{k=1}^{\infty} e_i^*(u_k) x^*(w_k^{**}),$$

where

$$w_k^{**} = \text{Gel'fand--}\int_{E_k} g\, d\mu(u, x^{**}), \qquad k = 1, 2, \dots .$$

Notice that for each $x^* \in X^*$ and each $k \in \mathbb{N}$,

$$|w_k^{**}(x^*)| = \left| \int_{E_k} x^*(g)\, d\mu \right| \leq \int_{E_k} |x^*(g)|\, d|\mu|$$
$$\leq \int_{E_k} \|x^*\| \cdot \|g\|\, d|\mu| \leq \|x^*\| \cdot |\mu|(E_k).$$

So

$$\|w_k^{**}\| \leq |\mu|(E_k), \qquad k = 1, 2, \dots .$$

Thus for each $i \in \mathbb{N}$,

$$\sum_{k=1}^{\infty} \|e_i^*(u_k) w_k^{**}\| = \sum_{k=1}^{\infty} |e_i^*(u_k)| \cdot \|w_k^{**}\|$$
$$\leq \sum_{k=1}^{\infty} \|u_k\| \cdot |\mu|(E_k) \leq |\mu|(K) + \varepsilon.$$

It follows that the series $\sum_k e_i^*(u_k) w_k^{**}$ converges absolutely in X^{**} for each $i \in \mathbb{N}$. Therefore, by (6),

$$y_i = \sum_{k=1}^{\infty} e_i^*(u_k) w_k^{**}, \qquad i = 1, 2, \dots . \tag{7}$$

Now let $z = \sum_{k=1}^{\infty} u_k \otimes w_k^{**}$. Then $z \in U \hat{\otimes} X^{**}$ and $\psi(z) = (y_i)_i$. Furthermore,

$$\|z\|_{U \hat{\otimes} X^{**}} \leq \sum_{k=1}^{\infty} \|u_k\| \cdot \|w_k^{**}\| \leq \sum_{k=1}^{\infty} \|u_k\| \cdot |\mu|(E_k) \leq |\mu|(K) + \varepsilon. \tag{8}$$

Letting $\varepsilon \longrightarrow 0$,

$$\|z\|_{U \hat{\otimes} X^{**}} \leq |\mu|(K) = \|\phi\| \leq 1. \tag{9}$$

Step 3 is complete.

Step 4. $z \in B_{U\hat{\otimes}X}$.

In fact, for each $n \in \mathbb{N}$, define $z_n' = \sum_{i=1}^n e_i \otimes y_i \in U\hat{\otimes}X$. Then for each $T \in \mathcal{L}(U, X^*)$,

$$\begin{aligned}\langle z_n' - z, T\rangle &= \sum_{i=1}^{n}\langle Te_i, y_i\rangle - \sum_{k=1}^{\infty}\langle Tu_k, w_k^{**}\rangle \\ &= \sum_{i=1}^{n}\langle Te_i, \sum_{k=1}^{\infty} e_i^*(u_k)w_k^{**}\rangle - \sum_{k=1}^{\infty}\langle Tu_k, w_k^{**}\rangle \\ &= \sum_{k=1}^{\infty}\langle \sum_{i=1}^{n} e_i^*(u_k)Te_i, w_k^{**}\rangle - \sum_{k=1}^{\infty}\langle Tu_k, w_k^{**}\rangle \\ &= \sum_{k=1}^{\infty}\langle T\left(\sum_{i=1}^{n} e_i^*(u_k)e_i - u_k\right), w_k^{**}\rangle \\ &= \sum_{k=1}^{\infty}\langle \sum_{i=1}^{n} e_i^*(u_k)e_i - u_k, T^*w_k^{**}\rangle\,.\end{aligned}$$

Since $\sum_{k=1}^{\infty} \|u_k\| \cdot \|w_k^{**}\| < \infty$, there exists a $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0}^{\infty} \|u_k\| \cdot \|w_k^{**}\| \le \varepsilon.$$

Since $\lim_n \| \sum_{i=1}^n e_i^*(u_k)e_i - u_k\| = 0$ for each $k \in \mathbb{N}$, there exists an $n_0 \in \mathbb{N}$ such that for each $n > n_0$,

$$\left\|\sum_{i=1}^{n} e_i^*(u_k)e_i - u_k\right\| \le \varepsilon\|u_k\|\,, \qquad k = 1, 2, \ldots, k_0.$$

Thus for each $n > n_0$,

$$\begin{aligned}|\langle z_n' - z, T\rangle| &\le \sum_{k=1}^{k_0}\left|\langle \sum_{i=1}^{n} e_i^*(u_k)e_i - u_k, T^*w_k^{**}\rangle\right| \\ &\quad + \sum_{k=k_0}^{\infty}\left|\langle \sum_{i=1}^{n} e_i^*(u_k)e_i - u_k, T^*w_k^{**}\rangle\right| \\ &\le \sum_{k=1}^{k_0}\left\|\sum_{i=1}^{n} e_i^*(u_k)e_i - u_k\right\| \cdot \|T^*w_k^{**}\| \\ &\quad + \sum_{k=k_0}^{\infty}\left\|\sum_{i=1}^{n} e_i^*(u_k)e_i - u_k\right\| \cdot \|T^*w_k^{**}\| \\ &\le \sum_{k=1}^{k_0}\varepsilon\|u_k\| \cdot \|T^*\| \cdot \|w_k^{**}\| + \sum_{k=k_0}^{\infty} \|u_k\| \cdot \|T^*\| \cdot \|w_k^{**}\|\end{aligned}$$

$$\begin{aligned} &\le \varepsilon\|T\| \sum_{k=1}^{\infty} \|u_k\| \cdot \|w_k^{**}\| + \varepsilon\|T\| \\ &\le \varepsilon\|T\|(|\mu|(K)+\varepsilon) + \varepsilon\|T\| \qquad \text{(from (8) and (9))} \\ &\le \varepsilon\|T\|(1+\varepsilon) + \varepsilon\|T\|. \end{aligned}$$

So for each $n > n_0$,

$$\|z_n' - z\|_{U\hat{\otimes}X^{**}} \le \varepsilon(1+\varepsilon)+\varepsilon.$$

By [12, page 238, Corollary 14], $U\hat{\otimes}X$ is a subspace of $U\hat{\otimes}X^{**}$. So $z = \lim_n z_n' \in U\hat{\otimes}X$ and $\|z\|_{U\hat{\otimes}X} = \|z\|_{U\hat{\otimes}X^{**}} \le 1$. Step 4 is complete.

Steps 1–4 complete the proof of Theorem. □

Lemma 12. *Let S be a closed separable subspace of $U\hat{\otimes}X$. Then there is a closed separable subspace Y of X such that S is a closed subspace of $U\hat{\otimes}Y$.*

Proof. Let S be a closed separable subspace of $U\hat{\otimes}X$, and let $D = (d_n)_{n=1}^{\infty}$ be a countably dense subset of S. Then for each fixed $m \in \mathbb{N}$, d_n has a representation

$$d_n = \sum_{k=1}^{\infty} u_k^{(n,m)} \otimes x_k^{(n,m)}, \qquad n = 1,2,\ldots \tag{10}$$

such that

$$\sum_{k=1}^{\infty} \|u_k^{(n,m)}\| \cdot \|x_k^{(n,m)}\| \le \|d_n\|_{U\hat{\otimes}X} + 1/m, \qquad n = 1,2,\ldots. \tag{11}$$

Let

$$Y = \overline{\text{span}}\{x_k^{(n,m)} : n,m,k = 1,2,\ldots\}.$$

Then Y is a closed separable subspace of X. Moreover, from (10) and (11), $d_n \in U\hat{\otimes}Y$ for each $n \in \mathbb{N}$ and

$$\|d_n\|_{U\hat{\otimes}Y} \le \|d_n\|_{U\hat{\otimes}X} + 1/m, \qquad n = 1,2,\ldots.$$

Letting $m \longrightarrow \infty$,

$$\|d_n\|_{U\hat{\otimes}Y} \le \|d_n\|_{U\hat{\otimes}X}, \qquad n = 1,2,\ldots.$$

Obviously,

$$\|d_n\|_{U\hat{\otimes}Y} \ge \|d_n\|_{U\hat{\otimes}X}, \qquad n = 1,2,\ldots.$$

So

$$\|d_n\|_{U\hat{\otimes}Y} = \|d_n\|_{U\hat{\otimes}X}, \qquad n = 1,2,\ldots.$$

Thus $(S, \|\cdot\|_{U\hat{\otimes}X}) =$ closure of $(D, \|\cdot\|_{U\hat{\otimes}X}) =$ closure of $(D, \|\cdot\|_{U\hat{\otimes}Y}) \subseteq U\hat{\otimes}Y$. Therefore, S is a closed subspace of $U\hat{\otimes}Y$. The proof is complete. □

REMARK 10. Notice that Y in Lemma 12 may be chosen so that $U\hat{\otimes}Y$ is a subspace of $U\hat{\otimes}X$. In fact (see [44]), any separable subspace of X is contained in a separable closed subspace Y of X such that there exists a linear Hahn-Banach extension operator from Y^* to X^*. But, in this case (see [40, Theorem 1]), $U\hat{\otimes}Y$ is a subspace of $U\hat{\otimes}X$.

Using the "semi-embeddings method" (that is, relying on Theorem 11), we now give an alternate proof for the following important special case of Theorem 8.

THEOREM 13. *Let G be a compact metrizable abelian group and let Λ be a subset of Γ. Then $U\hat{\otimes}X$, the projective tensor product of U and X, has I-Λ-RNP, II-Λ-RNP or, respectively, the NRNP if X has the same property.*

Proof. From [15] and [26], we know that a Banach space has I-Λ-RNP, II-Λ-RNP or, respectively, the NRNP if all its separable closed linear subspaces have the same property. Also, from [15], [38] and [26] we know that if a separable Banach space Z semi-embeds in a Banach space which has I-Λ-RNP, II-Λ-RNP or, respectively, the NRNP then Z has the same property.

Now suppose that X has I-Λ-RNP (respectively, II-Λ-RNP or NRNP). Take a closed separable subspace S of $U\hat{\otimes}X$. By Lemma 12, there is a separable subspace Y of X such that S is a subspace of $U\hat{\otimes}Y$. As a subspace of X, Y has I-Λ-RNP (respectively, II-Λ-RNP or NRNP). By Theorem 6, $U(Y)$ has I-Λ-RNP (respectively, II-Λ-RNP or NRNP). Since U and Y are separable, $U\hat{\otimes}Y$ is separable, too. By Theorem 11, $U\hat{\otimes}Y$ semi-embeds in $U(Y)$. Thus, $U\hat{\otimes}Y$ has I-Λ-RNP (respectively, II-Λ-RNP or NRNP). Hence, S, as a subspace of $U\hat{\otimes}Y$, has I-Λ-RNP (respectively, II-Λ-RNP or NRNP), too. Therefore, we have shown that each closed separable subspace of $U\hat{\otimes}X$ has I-Λ-RNP (respectively, II-Λ-RNP or NRNP), which shows that $U\hat{\otimes}X$ has I-Λ-RNP (respectively, II-Λ-RNP or NRNP), also. The proof is complete. □

Finally, we give an alternate

Proof of Theorem 9. Suppose that X has an unconditional basis $(x_n)_{n=1}^\infty$. By scaling if necessary, we can assume that $(x_n)_{n=1}^\infty$ is a normalized basis. Let $(x_n^*)_{n=1}^\infty$ denote the sequence of biorthogonal functionals associated with $(x_n)_{n=1}^\infty$.

If X has the Radon-Nikodym property, the analytic Radon-Nikodym property, the near Radon-Nikodym property, or does not contain a copy of c_0, then X does not contain a copy of c_0. By James's Theorem (see Section 3), the basis $(x_n)_{n=1}^\infty$ is also boundedly complete. We can equivalently renorm X by letting

$$\|x\|_{new} = \sup\left\{\left\|\sum_{i=1}^m \beta_i x_i^*(x) x_i\right\| : m \in \mathbb{N} \text{ and } |\beta_i| \le 1, i \in \mathbb{N}\right\}, \qquad x \in X$$

(see [45, page 463, Theorem II.16.1]). It is straightforward that $\|x_n\|_{new} = \|x_n\| = 1$ and $(x_n)_{n=1}^{\infty}$ is a 1-unconditional basis for $(X, \|\cdot\|_{new})$. Consequentially, X is isomorphic to $(X, \|\cdot\|_{new})$ which has a normalized boundedly complete, 1-unconditional basis with normalized biorthogonal functionals. Note that $X\hat{\otimes}Y$ is isomorphic to $(X, \|\cdot\|_{new})\hat{\otimes}Y$. Therefore, by Theorem 13, $(X, \|\cdot\|_{new})\hat{\otimes}Y$, and hence $X\hat{\otimes}Y$, has the Radon-Nikodym property, the analytic Radon-Nikodym property, the near Radon-Nikodym property or, respectively, contains no copy of c_0 if Y has the same property. This completes the proof. □

5. Applications to concrete Banach spaces

It is well known and easy to verify that the unit vectors form a boundedly complete unconditional basis in ℓ_p, for $1 \leq p < \infty$. So we have:

FACT 1. *The classical sequence space ℓ_p $(1 \leq p < \infty)$ has a boundedly complete unconditional basis.*

From [30, Theorem 2.c.5] we know that the Haar system forms an unconditional basis of $L_p[0,1]$ for $1 < p < \infty$. By a classical result of James [24] (see [29, Theorem 1.b.4]) every basis in a reflexive Banach space is boundedly complete. So we have:

FACT 2. *The classical Lebesgue function space $L_p[0,1]$ $(1 < p < \infty)$ has a boundedly complete unconditional basis.*

From [29, Proposition 4.a.4] we know that if $M \in \Delta_2$, then the unit vectors form a boundedly complete symmetric basis of ℓ_M. Also from [29, page 113] we know that every symmetric basis is an unconditional basis. Thus we have:

FACT 3. *The Orlicz sequence space ℓ_M $(M \in \Delta_2)$ has a boundedly complete unconditional basis.*

From [11, Corollary 1.46 and Theorem 1.98] we know that if $M \in \Delta_2$ and $M^* \in \Delta_2$, then the Orlicz function space $L_M[0,1]$ is a reflexive space with the Haar system as its an unconditional basis. Thus we have:

FACT 4. *The Orlicz function space $L_M[0,1]$ $(M, M^* \in \Delta_2)$ has a boundedly complete unconditional basis.*

Let $1 \leq p < \infty$ and let $w = (w_i)_{i=1}^{\infty}$ be a non-increasing sequence of positive numbers such that $w_1 = 1$, $\lim_i w_i = 0$ and $\sum_{i=1}^{\infty} w_i = \infty$. The Banach space of all sequences of scalars $x = (a_1, a_2, \ldots)$ for which

$$\|x\| = \sup_{\pi} \left(\sum_{i=1}^{\infty} |a_{\pi(i)}|^p w_i \right)^{1/p} < \infty,$$

where π ranges over all the permutations of integers, is denoted by $d(w,p)$ and is called a *Lorentz sequence space*. By [8], the unit vectors form a boundedly complete unconditional basis of $d(w,1)$. By [1], [19], [20], $d(w,p)$, $1 < p < \infty$, is a reflexive Banach space and the unit vectors form a symmetric basis. Thus we have:

FACT 5. *The Lorentz sequence space $d(w,p)$ $(1 \le p < \infty)$ has a boundedly complete unconditional basis.*

Let m denote the Lebesgue measure on [0,1]. For a real-valued Lebesgue measurable function f on [0,1] we denote the distribution function of $|f|$ by d_f, that is,

$$d_f(t) = m(\{x : |f(x)| > t\});$$

and we denote by f^* the decreasing rearrangement of $|f|$, that is,

$$f^*(t) = \inf\{x > 0 : d_f(x) \le t\}.$$

A function w on [0,1] will be called a *Lorentz weight* on [0,1] if w is non-negative, non-increasing, $w(1) > 0$, and $\int_0^1 w(t)\,dt = 1$. Given a Lorentz weight w and $1 \le p < \infty$, the *Lorentz function space* $L_{w,p}[0,1]$ is defined to be the set of all equivalence classes of measurable functions f on [0,1] for which $\|f\|_{w,p} < \infty$, where

$$\|f\|_{w,p} = \left(\int_0^1 f^*(t)^p w(t)\,dt\right)^{1/p}.$$

If $w(x) \equiv 1$, then $L_{w,p}[0,1] \equiv L_p[0,1]$. If $w(x) = \frac{q}{p}x^{(q/p)-1}$, $1 \le q \le p < \infty$, then $L_{w,p}[0,1]$ is the classical Lorentz space $L_{p,q}[0,1]$. If $w(x) = c(p,q,\alpha)x^{(q/p)-1}(1+|\log x|)^{\alpha q}$, $1 \le q \le p < \infty$, $0 \le \alpha < \infty$, where $c(p,q,\alpha)$ is a constant chosen to satisfy $\int_0^1 w(t)\,dt = 1$, then $L_{w,p}[0,1]$ is the so-called *Lorentz-Zygmund space* $L_{p,q,\alpha}[0,1]$ (see [2]).

Associated to a Lorentz weight w is the function $S(x) = \int_0^x w(t)\,dt$. The weight w is called *regular* if there is a constant $K > 1$ such that $S(2x)/S(x) \ge K$ for all x with $2x \in [0,1]$. Note that in each of the Lorentz spaces $L_{p,q}[0,1]$ and $L_{p,q,\alpha}[0,1]$ mentioned above, the weight is regular (see [10, page 8]).

From [10, page 25] we know that for $1 < p < \infty$, the Haar system forms an unconditional basis for $L_{w,p}[0,1]$ exactly when w is regular. Also from [31], $L_{w,p}[0,1]$, $1 < p < \infty$, is reflexive. Thus we have:

FACT 6. *The Lorentz function space $L_{w,p}[0,1]$ $(1 < p < \infty$, w is regular$)$ has a boundedly complete unconditional basis.*

From [9], [32], [47] we know that the classical Hardy space on the unit disk in the complex plane, $H_1(D)$, has an unconditional basis. Since $H_1(D)$ is a subspace of $L_1(\mathbb{T})$ and $L_1(\mathbb{T})$ does not contain a copy of c_0, $H_1(D)$ does not contain c_0. Thus an application of James's Theorem (see Section 3) yields:

FACT 7. *The Hardy space $H_1(D)$ has a boundedly complete unconditional basis.*

Now from Facts 1–7, and Theorem 9 or Theorem 8 together with Remarks 1 and 2, we have:

COROLLARY 14. *Let X be any Banach space and U be ℓ_p $(1 \leq p < \infty)$, $L_p[0,1]$ $(1 < p < \infty)$, ℓ_M $(M \in \Delta_2)$, $L_M[0,1]$ $(M, M^* \in \Delta_2)$, $d(w,p)$ $(1 \leq p < \infty)$, $L_{w,p}[0,1]$ $(1 < p < \infty$, w is regular$)$, or $H_1(D)$. Then $U\hat{\otimes}X$, the projective tensor product of U and X, has the Radon-Nikodym property (respectively, the analytic Radon-Nikodym property, the near Radon-Nikodym property, contains no copy of c_0) if and only if X has the same property.*

REMARK 11. It is shown in [5], [6], [7] that for $1 < p < \infty$, $L_p[0,1]\hat{\otimes}X$ has the Radon-Nikodym property (respectively, the analytic Radon-Nikodym property, the near Radon-Nikodym property, contains no copy of c_0) whenever X has the same property. For $p = 1$, it is known that $L_1[0,1]\hat{\otimes}X$ is isometrically isomorphic to the Bochner integrable function space $L_1([0,1], X)$ which is known to have the analytic Radon-Nikodym property (respectively, the near Radon-Nikodym property, contain no copy of c_0) whenever X has the same property [13], [28], [39].

It follows from [14] that $H_1(D)\hat{\otimes}X$ has the Radon-Nikodym property whenever X has the Radon-Nikodym property. It can also be seen that $H_1(D)\hat{\otimes}X$ has the analytic Radon-Nikodym property (respectively, the near Radon-Nikodym property, contains no copy of c_0) whenever X has the same property, by noting that $H_1(D)\hat{\otimes}X$ is a subspace of $L_1(\mathbb{T}, X)$ and using the results of the last paragraph. It should be noted that, unlike the case of $L_1(\mathbb{T})\hat{\otimes}X$, $H_1(D)\hat{\otimes}X$ is not necessarily isomorphic to the function space $H_1(D, X)$ (see [22], [27]).

Let $\mathcal{M}$ be a semifinite von Neumann algebra acting on a separable Hilbert space and let τ be a normal faithful semifinite trace on $\mathcal{M}$. For $1 \leq p < \infty$, let $L_p(\mathcal{M}, \tau)$ be the vector space of all τ-measurable operators A, such that $\tau(|A|^p) < \infty$, where $|A| = (A^*A)^{1/2}$. The space $L_p(\mathcal{M}, \tau)$ is a Banach space when equipped with the norm $\|A\|_p = (\tau(|A|^p))^{1/p}$ [18]. A von Neumann algebra $\mathcal{M}$ is called hyperfinite if $\mathcal{M}$ is the weak closure of the union of an increasing sequence of finite dimensional von Neumann algebras. It follows from [37], [46] that if $\mathcal{M}$ is hyperfinite and $1 < p < \infty$, then $L_p(\mathcal{M}, \tau)$ has an unconditional finite dimensional decomposition. Since $L_p(\mathcal{M}, \tau)$ is reflexive for $1 < p < \infty$, by an extension of James's result due to Sanders [43], it follows that the FDD of $L_p(\mathcal{M}, \tau)$ is boundedly complete. In particular, when $\mathcal{M} = B(\ell^2)$, the space of bounded linear operators on ℓ^2, then $L_p(\mathcal{M}, \tau) = C_p$, the Schatten p-classes. Since $B(\ell^2)$ is hyperfinite, we have that the Schatten

p-classes C_p have a boundedly complete FDD when $1 < p < \infty$. Therefore from Theorem 7 and Remarks 1 and 2 we have:

Corollary 15. *Let $1 < p < \infty$ and let X be C_p or $L_p(\mathcal{M}, \tau)$, where $\mathcal{M}$ is a hyperfinite von Neumann algebra acting on a separable Hilbert space and τ is a normal faithful semifinite trace on $\mathcal{M}$, and let Y be any Banach space. Then $X \hat{\otimes} Y$, the projective tensor product of X and Y, has the Radon-Nikodym property (respectively, the analytic Radon-Nikodym property, the near Radon-Nikodym property, contains no copy of c_0) if and only if Y has the same property.*

Acknowledgement. The authors thank Narcisse Randrianantoanina for his helpful comments related to Corollary 15.

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