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# APPROXIMATIONS OF GENERALIZED COHEN-MACAULAY MODULES

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Abstract. It is shown that any generalized Cohen-Macaulay module $M$ can be approximated by a maximal generalized Cohen-Macaulay module $X$ up to a module of finite projective dimension, and such that the local cohomology modules of $M$ and $X$ coincide for all cohomological degrees different from the dimensions of the two modules. By a theorem of Migliore there exist graded generalized Cohen-Macaulay rings which, up to a shift, have predescribed local cohomology modules. Bounds for this shift are given in terms of homological data.

## Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $n$, and $M$ a finitely generated $R$-module. $M$ is called a generalized CM-module if the $i$th local cohomology $H^i_{\mathfrak{m}}(M)$ of $M$ is of finite length for all $i < \dim M$. We call $M$ a maximal generalized CM-module if $\dim M = n$.

In this paper we show that if $R$ is Gorenstein, then any generalized CM-module $R$-module $M$ of dimension $d < n$ can be approximated by a maximal generalized CM-module in way similar to the CM-approximations introduced by Auslander and Buchweitz [3]. To be precise, we show in Theorem 1.1 that there exists an exact sequence $0 \to Y \to X \to M \to 0$, where $X$ is a maximal generalized CM-module with $H^d_{\mathfrak{m}}(X) = 0$, $Y$ is of projective dimension $n-d-1$, and where the epimorphism $X \to M$ induces isomorphisms $H^i_{\mathfrak{m}}(X) \to H^i_{\mathfrak{m}}(M)$ for $i < n$, $i \neq d$.

The result implies, in particular, that for any ideal $I \subset R$ of codimension 2 for which $R/I$ is a generalized CM-ring there exists an exact sequence $0 \to Y \to X \to I \to 0$, where $Y$ is free, $H^{n-1}_{\mathfrak{m}}(X) = 0$, and the induced homomorphisms $H^i_{\mathfrak{m}}(X) \to H^{i-1}_{\mathfrak{m}}(R/I)$ are isomorphisms for $i < n-1$. Such a sequence for $I$ is called a Bourbaki sequence. Conversely, given a generalized maximal CM-module $X$, constructions of Bourbaki sequences yield codimension 2 ideals $I$ for which $H^i_{\mathfrak{m}}(X) \cong H^{i-1}_{\mathfrak{m}}(R/I)$ for $i < n-1$. This technique

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has first been used by Evans and Griffith [6] to show that there exist codimension 2 ideals $I \subset S$ in a polynomial ring $S$ such that $S/I$ is generalized CM with prescribed local cohomology.

For these constructions there is a graded analogue. In the graded situation the following basic question arises: Let $K$ be a field, $S = K[x_1, \dots, x_n]$ a polynomial ring, $\mathfrak{m} = (x_1, \dots, x_n)$, $0 < d < n-1$ an integer and $M_0, \dots, M_{d-1}$ graded $S$-modules of finite length. Does there exist a graded ideal $I \subset S$ of codimension $n-d$ such that there exist graded isomorphisms $H^i_{\mathfrak{m}}(S/I) \cong M_i$ for $i = 0, \dots, d-1$?

It has been shown [11] that there exists an integer $c_0$ such that for all $c \geq c_0$ there exists a graded ideal $I$ with $H^i_{\mathfrak{m}}(S/I) \cong M_i(-c)$. In Section 2, under the assumption that $K$ is an infinite field, we use arguments similar to those of Migliore, Nagel and Peterson [12] and a general version of Bertini's theorem, due to Flenner [9], as well as some arguments on Hilbert functions, to give formulas for this bound $c_0$ in terms of numerical data of the graded resolutions of the $M_i$. This bound is not sharp, but as we show in the last section, it is sharp in codimension 2, if all but one $M_i$ vanish.

## 1. The approximation theorem

Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring of dimension $n$, and let $M$ be a finitely generated $R$-module of dimension $d$. The module $M$ is called a *generalized CM-module* if $H^i_{\mathfrak{m}}(M)$ has finite length for all $i \neq d$. We say that $M$ has maximal dimension if $\dim M = n$. The main purpose of this section is to prove that if $R$ is Gorenstein, then any generalized CM-module of dimension $d < n$ can be approximated by a maximal generalized CM-module as described in the next theorem. A related result has been proved by Amasaki in [2, Lemma 1.3].

THEOREM 1.1. *Let $(R, \mathfrak{m})$ be a local Gorenstein ring, $M$ a generalized CM-module over $R$ of dimension $d < n$. Then there exists an exact sequence*

$$0 \longrightarrow Y \longrightarrow X \xrightarrow{\varphi} M \longrightarrow 0$$

*with the following properties:*

(a) (i) $H^i_{\mathfrak{m}}(\varphi) \colon H^i_{\mathfrak{m}}(X) \to H^i_{\mathfrak{m}}(M)$ *is an isomorphism for* $i < n$ *and* $i \neq d$;
 (ii) $H^d_{\mathfrak{m}}(X) = 0$.

(b) $Y$ *is a module of projective dimension* $n - d - 1$.

*Moreover, for any epimorphism* $\varphi : X \to M$ *satisfying* (a), $X$ *is a maximal generalized CM-module.*

The existence of a sequence $0 \to Y \to X \to M \to 0$, where $X$ is a generalized CM-module, and $Y$ has finite projective dimension, is guaranteed by the approximation theorem [3] of Auslander and Buchweitz. For example, one could get such an exact sequence with $X$ a maximal CM-module. However our condition (a) excludes this case. Thus our approximation of $M$ is not a Cohen-Macaulay approximation in the sense of Auslander and Buchweitz.

In the course of the proof of Theorem 1.1 we shall need the following simple lemma.

Lemma 1.2. *Let*

$$\begin{aligned} &G_r \xrightarrow{d_r} G_{r-1} \xrightarrow{d_{r-1}} \cdots \\ &\cdots \longrightarrow G_2 \xrightarrow{d_2} G_1 \xrightarrow{d_1} G_0 \longrightarrow 0 \end{aligned}$$

*be a complex of maximal CM-modules and with finite length homology. Then for* $C_i = \operatorname{Coker}(d_{i+1})$ *we have* $H^j_{\mathfrak m}(C_i) = 0$ *for all* $j$ *with* $i < j < n$.

*Proof.* Let $C_i = \operatorname{Coker}(d_{i+1})$, $D_i = \operatorname{Im}(d_i)$, $K_i = \operatorname{Ker}(d_i)$ and $H_i = K_i/D_{i+1}$. Then for $i = 1, \dots, r-1$ we have the following exact sequences:

$$0 \longrightarrow H_i \longrightarrow C_i \longrightarrow D_i \longrightarrow 0, \tag{1}$$

$$0 \longrightarrow D_{i+1} \longrightarrow K_i \longrightarrow H_i \longrightarrow 0, \tag{2}$$

$$0 \longrightarrow K_i \longrightarrow G_i \longrightarrow D_i \longrightarrow 0. \tag{3}$$

Now (1) yields the isomorphisms $H^j_{\mathfrak m}(C_i) \cong H^j_{\mathfrak m}(D_i)$ for $j > 0$, (2) the isomorphisms $H^j_{\mathfrak m}(D_{i+1}) \cong H^j_{\mathfrak m}(K_i)$ for $j > 1$, and (3) the isomorphisms $H^{j-1}_{\mathfrak m}(D_i) \cong H^j_{\mathfrak m}(K_i)$ for $j < n$. The isomorphisms arising from (2) and (3) imply $H^j_{\mathfrak m}(D_{i+1}) \cong H^{j-1}_{\mathfrak m}(D_i)$ for $1 < j < n$, and thus the isomorphisms arising from (1) imply $H^j_{\mathfrak m}(C_{i+1}) \cong H^{j-1}_{\mathfrak m}(C_i)$ for $1 < j < n$. Hence induction on $i$ proves the assertion. □

*Proof of Theorem 1.1.* Let $t = \operatorname{depth} M$, set $p = n - t$ and let $F_{p-1} \to F_{p-2} \to \dots \to F_0 \to M$ be the beginning of the minimal free resolution of $M$. We put $F_p = \operatorname{Ker}(F_{p-1} \to F_{p-2})$, and obtain the exact sequence

$$0 \longrightarrow F_p \longrightarrow F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0, \tag{4}$$

where $F_i$ is free for $i = 0, \dots, p-1$ and $F_p$ is a maximal CM-module.

For an $R$-module $W$ we set $W^* = \operatorname{Hom}_R(W, R)$ and $W^\vee = \operatorname{Hom}_R(W, E)$, where $E$ is the injective hull of $R/\mathfrak m$. Then, dualizing (4) with respect to $R$, we get the co-complex $F^\bullet$ with $F^i = F_i^*$ for $i = 0, \dots, p$.

Let $s = \max\{i \colon i < d, \quad H^i_{\mathfrak m}(M) \neq 0\}$, and set $q = n - s$ and $g = n - d$.

Since $F_p$ is a maximal CM-module and $R$ is a Gorenstein ring, it follows that $\operatorname{Ext}^i(F_p, R) = 0$ for $i > 0$, and hence $H^i(F^\bullet) = \operatorname{Ext}^i_R(M, R)$ for all $i$. Thus by local duality we have $H^i(F^\bullet) = H^{n-i}_{\mathfrak m}(M)^\vee$, so that

$$\begin{aligned} F^\bullet : 0 \longrightarrow F_0^* \longrightarrow \cdots \longrightarrow F_g^* \longrightarrow \cdots \\ \cdots \longrightarrow F_q^* \longrightarrow \cdots \longrightarrow F_p^* \longrightarrow 0, \end{aligned} \tag{5}$$

has finite length homology in the range from $q$ to $p$, and is otherwise exact except at cohomological degree $g$.

Now let $C = \operatorname{Ker}(F_g^* \to F_{g+1}^*)$, and let $H_\bullet$ be a minimal free resolution of $C$. Then we modify $F^\bullet$ to obtain the co-complex

$$G^\bullet : 0 \longrightarrow H_{g-1} \longrightarrow \cdots \longrightarrow H_0 \longrightarrow F_g^* \longrightarrow \cdots \longrightarrow F_p^* \longrightarrow 0,$$

that is, we have $G^i = H_{g-1-i}$ for $i = 0, \ldots, g-1$, and $G^i = F_i^*$ for $i \geq g$. There is a homomorphism of co-complexes $\varphi^\bullet \colon F^\bullet \to G^\bullet$,

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_0^* & \longrightarrow & \cdots & \longrightarrow & F_{g-1}^* \\ & & \Big\downarrow{\scriptstyle \varphi^0} & & & & \Big\downarrow{\scriptstyle \varphi^{g-1}} \\ 0 & \longrightarrow & H_{g-1} & \longrightarrow & \ldots & \longrightarrow & H_0 \end{array}$$

(6)

$$\begin{array}{cccccccc} \longrightarrow & F_g^* & \longrightarrow & \cdots & \longrightarrow & F_p^* & \longrightarrow & 0 \\ & \Big\downarrow{\scriptstyle \varphi^g} & & & & \Big\downarrow{\scriptstyle \varphi^p} & & \\ \longrightarrow & F_g^* & \longrightarrow & \ldots & \longrightarrow & F_p^* & \longrightarrow & 0, \end{array}$$

where $\varphi^i = \mathrm{id}$ for $i = g, \ldots, p$. Dualizing this diagram with respect to $R$ we obtain the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_p & \longrightarrow & \cdots & \longrightarrow & F_g \\ & & \Big\uparrow{\scriptstyle \varphi_p} & & & & \Big\uparrow{\scriptstyle \varphi_g} \\ 0 & \longrightarrow & F_p & \longrightarrow & \ldots & \longrightarrow & F_g \end{array}$$

(7)

$$\begin{array}{cccccccc} \longrightarrow & F_{g-1} & \longrightarrow & \cdots & \longrightarrow & F_0 & \longrightarrow & 0 \\ & \Big\uparrow{\scriptstyle \varphi_{g-1}} & & & & \Big\uparrow{\scriptstyle \varphi_0} & & \\ \longrightarrow & H_0^* & \longrightarrow & \ldots & \longrightarrow & H_{g-1}^* & \longrightarrow & 0, \end{array}$$

with $\varphi_i = (\varphi^i)^*$ for $i = 0, \ldots, p$, and where the lower row of the diagram is the complex $L_\bullet = (G^\bullet)^*$. We claim that $L_\bullet$ is acyclic. In fact, $H_i(L_\bullet) = 0$ for $i > g$, because in homological degree $i > g$ the complexes $L_\bullet$ and $F_\bullet$ coincide. Let $D = \operatorname{Coker}(F_{q-1}^* \to F_q^*)$. Then

$$H_{g-1} \longrightarrow \cdots \longrightarrow H_0 \longrightarrow F_g^* \longrightarrow \cdots \longrightarrow F_{q-1}^* \longrightarrow F_q^* \longrightarrow D \to 0$$

is the beginning of a free resolution of $D$. It follows that $H_i(L_\bullet) = \operatorname{Ext}_R^{q-i}(D, R)$ $= H_{\mathfrak{m}}^{n-q+i}(D)^\vee$ for $0 < i \leq g$. By 1.2 we have $H_{\mathfrak{m}}^j(D) = 0$ for all $j$ with $p - q < j < n$, and therefore $H_i(L_\bullet) = 0$ for all $0 < i \leq g$.

We now define a slightly modified complex $L'_\bullet$ with $L'_0 = L_0 \oplus F_0 = H^*_{g-1} \oplus F_0$, and $L'_i = L_i$ for $i > 0$. The chain map $H^*_{g-2} = L'_1 \to L'_0 = H^*_{g-1} \oplus F_0$ is given by $a \mapsto (\partial_1(a), 0)$, where $\partial_\bullet$ is the chain map of $L_\bullet$. It is clear that $L'_\bullet$ is again acyclic with $H_0(L'_\bullet) = H_0(L_\bullet) \oplus F_0$.

We define the complex homomorphism $\varphi'_\bullet \colon L'_\bullet \to F_\bullet$ as follows: $(\varphi'_0)_{|H^*_{g-1}} = \varphi_0$, $(\varphi'_0)_{|F_0} = \mathrm{id}$, and $\varphi'_i = \varphi_i$ for all $i > 0$. Then $\varphi'_0 \colon L'_0 \to F_0$ is surjective (and this is why we modified $L_\bullet$).

Set $X = H_0(L'_\bullet)$, and let $\varphi \colon X \to M$ be the homomorphism induced by $\varphi'_\bullet$. We claim that $\varphi \colon X \to M$ has all the desired properties. In fact, since $L'_\bullet$ is acyclic with $L'_i$ free for $i = 0, \ldots, p-1$ and $L'_p = F_p$ is a maximal CM-module, it follows that $\mathrm{Ext}^i_R(X, R) = H^i((L'_\bullet)^*) = H^i((L_\bullet)^*) = H^i(G^\bullet)$ for $i > 0$. Hence $\mathrm{Ext}^i_R(X, R) = 0$ for $0 < i < q$; equivalently, $H^i_{\mathfrak{m}}(X) = 0$ for $s < i < n$. Therefore (a)(ii) is satisfied, and (a)(i) is also satisfied for $s < i < n$ since $H^i_{\mathfrak{m}}(M) = 0$ for $s < i < n$ and $i \neq d$. Moreover, we deduce from diagram (6) that $\mathrm{Ext}^i_R(\varphi, R) \colon \mathrm{Ext}^i_R(M, R) \to \mathrm{Ext}^i_R(X, R)$ is an isomorphism for all $i \geq q$, which is equivalent to saying that $H^i_{\mathfrak{m}}(\varphi) \colon H^i_{\mathfrak{m}}(X) \to H^i_{\mathfrak{m}}(M)$ is an isomorphism for $i \leq s$. This proves assertion (a).

In order to prove (b) we form the mapping cone of $C_\bullet(\varphi'_\bullet)$ of $\varphi'_\bullet$ and get the short exact sequence of complexes

$$0 \longrightarrow F_\bullet \longrightarrow C_\bullet(\varphi'_\bullet) \longrightarrow L'_\bullet[-1] \longrightarrow 0,$$

where $L'_\bullet[-1]$ is the complex $L'_\bullet$ shifted in homological degree by $-1$, that is, $(L'_\bullet[-1])_i = L'_{i-1}$ for all $i$. Since $F_\bullet$ and $L'_\bullet$ are acyclic complexes it follows that $H_i(C_\bullet(\varphi'_\bullet)) = 0$ for $i > 1$. Moreover, we get the exact sequence

$$0 \longrightarrow H_1(C_\bullet(\varphi'_\bullet)) \longrightarrow H_0(L'_\bullet) \longrightarrow H_0(F_\bullet) \longrightarrow H_0(C_\bullet(\varphi'_\bullet)) \longrightarrow 0.$$

Notice that $H_0(L'_\bullet) \to H_0(F_\bullet) = \varphi \colon X \to M$. Hence $H_1(C(\varphi'_\bullet)) = Y$, and $H_0(C_\bullet(\varphi'_\bullet)) = 0$, since $\varphi$ is surjective. Let $U_\bullet$ be the subcomplex of $C_\bullet(\varphi'_\bullet)$ with $U_i = F_0$ for $i = 0, 1$, $U_i = 0$ for $i > 1$ and the identity as chain map $U_1 \to U_0$. Let $C_\bullet = C_\bullet(\varphi'_\bullet)/U_\bullet$. Then $C_i = L_{i-1} \oplus F_i$ for $i = 1, \ldots p$, $C_{p+1} = L_p$, and $C_\bullet$ is acyclic with $H_1(C_\bullet) = Y$. Recall that $\varphi_i = \mathrm{id}$ for $i = g, \ldots, p$, which implies that the subcomplex

$$V_\bullet : 0 \longrightarrow L_p \longrightarrow L_{p-1} \oplus F_p \longrightarrow \cdots \longrightarrow L_g \oplus F_{g+1} \longrightarrow F_g \longrightarrow 0$$

is exact. It follows that the quotient complex $\tilde{C}_\bullet = C_\bullet/V_\bullet$ is an acyclic complex of free modules of length $g-1$ with $H_1(\tilde{C}_\bullet) = Y$. This implies that the projective dimension of $Y$ is $\leq g-1$.

It remains to show that for an exact sequence $0 \to Y \to X \to M \to 0$ satisfying condition (a) one has $\dim X = n$. Suppose $\dim X < n$. Then $\dim X < d$ because $H^i_{\mathfrak{m}}(X) = 0$ for $i \geq d$ by (a)(i) and (ii). Then by the short exact sequence we know that $d = \dim M \leq \dim X < d$, a contradiction. $\square$

We denote by $\Omega_i(M)$ the $i$th syzygy module of $M$.

Corollary 1.3. *Let $(R,\mathfrak{m})$ be a local Gorenstein ring of dimension $n$, and $M$ a generalized CM-module over $R$ of dimension $d < n$. Then for all $j < n - d$ there exists an exact sequence*

$$0 \longrightarrow Y_j \longrightarrow X_j \xrightarrow{\varphi_j} \Omega_j(M) \longrightarrow 0$$

*with the following properties:*

(a) (i) $H^i_{\mathfrak{m}}(\varphi_j)\colon H^i_{\mathfrak{m}}(X_j) \to H^i_{\mathfrak{m}}(\Omega_j(M))$ *is an isomorphism for* $i < n-j$ *and* $i \neq d+j$;

(ii) $H^{d+j}_{\mathfrak{m}}(X_j) = 0$.

(b) $Y_j$ *is a module of projective dimension* $\leq n - d - j - 1$.

*Proof.* We refer to the notation of Theorem 1.1. Let $F_\bullet$ be the minimal free resolution of $M$, and $G_\bullet$ the minimal free resolution of $Y$. Then there is a free resolution $L_\bullet$ of $X$, not necessarily minimal, and an exact sequence of complexes

$$0 \longrightarrow G_\bullet \longrightarrow L_\bullet \longrightarrow F_\bullet \longrightarrow 0$$

such that $0 \to H_0(G_\bullet) \to H_0(L_\bullet) \to H_0(F_\bullet) \to 0$ is the exact sequence of 1.1. For each $j < n - d$, this sequence of complexes induces an exact sequence

$$0 \longrightarrow \Omega_j(Y) \longrightarrow \Omega_j(X) \oplus H_j \longrightarrow \Omega_j(M) \longrightarrow 0,$$

where $H_j$ is a suitable free $R$-module. This is our desired exact sequence with $Y_j = \Omega_j(Y)$, and $X_j = \Omega_j(X) \oplus H_j$. $\square$

If we let $M = R/I$, where $I$ is of codimension 2, then 1.3 implies:

Corollary 1.4. *Let $(R,\mathfrak{m})$ be a local Gorenstein ring of dimension $n$, $I \subset R$ an ideal of codimension 2 such that $R/I$ is a generalized Cohen-Macaulay ring. Then there exists an exact sequence*

$$0 \longrightarrow Y \longrightarrow X \longrightarrow I \longrightarrow 0,$$

*where $Y$ is free, $H^{n-1}_{\mathfrak{m}}(X) = 0$, and the induced homomorphisms $H^i_{\mathfrak{m}}(X) \to H^i_{\mathfrak{m}}(I) \cong H^{i-1}_{\mathfrak{m}}(R/I)$ are isomorphisms for $i < n - 1$.*

There is also a converse to 1.4.

Proposition 1.5. *Let $(R,\mathfrak{m})$ be a local Gorenstein ring, and let*

$$0 \longrightarrow Y \longrightarrow X \longrightarrow I \longrightarrow 0$$

*be a non-split exact sequence of $R$-modules, where $Y$ is free, $X$ is a generalized maximal CM-module and $I$ is an ideal in $R$ of codimension $> 1$. Then $I$ is of codimension 2 and $R/I$ is a generalized CM.*

*Proof.* Let $\dim R = n$. Since $Y$ is free it follows that $H^i_{\mathfrak{m}}(X) \cong H^i_{\mathfrak{m}}(I) \cong H^{i-1}_{\mathfrak{m}}(R/I)$ for $i \leq n-2$. It follows that $R/I$ is a generalized CM ring, because $\dim(R/I) \leq n - 2$, by assumption. Suppose $\dim(R/I) = d < n - 2$. Then

$\mathrm{Ext}^1_R(I,R) = \mathrm{Ext}^2_R(R/I,R) = 0$, and hence $\mathrm{Ext}^1_R(I,Y) = 0$ for any finitely generated free $R$-module $Y$. This implies that there exists no non-split exact sequence $0 \to Y \to X \to I \to 0$, a contradiction. □

## 2. Graded generalized CM rings

Let $K$ be an infinite field, $S = K[x_1,\ldots,x_n]$ the polynomial ring over $K$ and $\mathfrak{m} = (x_1,\ldots,x_n)$. For a graded module $N = \bigoplus_{i\in\mathbb{Z}} N_i$ and $a \in \mathbb{Z}$, we define the graded module $N(a)$ to be given by $N(a)_i = N_{a+i}$ for $i \in \mathbb{Z}$. In [11, Proposition 1.2.8] the following is shown:

(∗) Let $d$ be an integer with $1 < d < n-1$, and let $M_0,\ldots,M_{d-1}$ be graded $S$-modules of finite length. Then there exists an integer $c_0$ such that for all $c \geq c_0$, there exists an ideal $I_c \subset S$ of codimension $n-d$ such that $H^i_{\mathfrak{m}}(S/I_c) \cong M_i(-c)$ for $i = 0,\ldots,d-1$.

Actually Proposition 1.2.8 in [11] is somewhat stronger than quoted in (∗); it also shows that there exists a smallest number $c_0$ satisfying the conditions of (∗).

The proof of (∗) is based on a result of Migliore, Nagel and Peterson [12], where it is shown that for a suitable integer c there exists $I_c \subset S$ of codimension $n-d$ with $H^i_{\mathfrak{m}}(S/I_c) \cong M_i(-c)$ for $i = 0,\ldots,d-1$, and on results by Bolondi and Migliore [4]. The proof of Migliore, Nagel and Peterson in [12] generalizes a construction by Evans and Griffith [6], using Bourbaki sequences, where a similar result was shown in codimension 2.

The purpose of this section is to describe a bound $c_0$ as described in (∗) in terms of the modules $M_i$. We may assume that $\bigoplus_i M_i \neq 0$. There exists an integer $m$ such that $(x_1,\ldots,x_n)^m M_i = 0$ for $i = 0,\ldots,d-1$. By a strong version of Bertini's theorem as it is proved by Flenner [9, Satz 5.4] we can find a regular sequence $f_1,\ldots,f_{n-d-2} \in S_{m+1}$ such that $R = S/(f_1,\ldots,f_{n-d-2})$ is regular on the punctured spectrum. By Grothendieck's theorem [8, Exposé XI, Corollary 3.14], $R$ is a factorial Gorenstein domain. In Flenner's theorem, which is much more general, it is required that $\operatorname{char} K = 0$. But as Flenner pointed out to us, for the existence of a regular sequence as above one only needs that $K$ is an infinite field. The argument is similar to that in the proof of [9, Satz 4.1].

The $M_i$ may be viewed as $R$-modules since they are annihilated by $f_1,\ldots,f_{n-d-2}$. We set $M = \bigoplus_{i=0}^{d-1} \Omega^R_{i+1}(M_i)$.

Note that for any finitely generated graded $S$-module $N$ of Krull dimension $d$, the Hilbert series $H_N(t)$ of $N$ is of the form

$$H_N(t) = \frac{Q_N(t)}{(1-t)^d},$$

where $Q_N(t) \in \mathbb{Z}[t,t^{-1}]$ and $Q_N(1) \neq 0$.

Let $a$ be the highest degree of a generator in a minimal set of generators of $M$, and let $r$ be the rank of $M$. With the notation introduced we then have:

THEOREM 2.1. *For a bound $c_0$ as in $(*)$ we can choose*

$$c_0 = \frac{rQ'_R(1) - Q'_M(1)}{Q_R(1)} + (r-1)a.$$

*with a ring $R$ and a module $M$ constructed as above.*

The proof of the theorem will depend on a slight refinement of arguments used in [12] as well as on an observation on Hilbert functions.

*Proof of* (2.1). Note that

$$H^j_{\mathfrak{m}}(\Omega^R_{i+1}(M_i)) \cong \begin{cases} 0, & \text{if } j < d+2 \text{ and } j \neq i+1, \\ M_i, & \text{if } j = i+1. \end{cases}$$

Thus it follows that $H^{i+1}_{\mathfrak{m}}(M) \cong M_i$ for $i = 0, \ldots, d-1$. Let $a_1, \ldots, a_{r-1}$ be any integers $\geq a$. Since $M$ is torsion free and $R$ is normal, it follows by a theorem of Flenner [9, Satz 1.5] that there exists a free submodule $Y = \bigoplus_{i=1}^{r-1} R(-a_i)$ of $M$ such that $M/Y$ is a rank 1 torsion free module. In other words, for all integers $a_i \geq a$ we can find a Bourbaki-sequence

$$0 \longrightarrow \bigoplus_i R(-a_i) \longrightarrow M \longrightarrow J_c(c) \longrightarrow 0,$$

where $J_c \subset R$ is a graded ideal. Note that $J_c \neq 0$, since by assumption at least one of the $M_i$ is non-trivial. We may assume that $\operatorname{codim} J_c > 1$. In fact, since $R$ is factorial, $J_c = fJ$, for some $f \in R$ and $J$ of codimension 2, and we may replace $J_c$ by $J$, if $f$ is not a unit.

According to Proposition 2.2(a) below the shift $c$ can be any number greater than or equal to

$$c_0 = \frac{rQ'_R(1) - Q'_M(1)}{Q_R(1)} + (r-1)a,$$

since the $a_i \geq a$ can be chosen arbitrarily.

Now let $I_c \subset S$ be the preimage of $J_c$. Then $S/I_c \cong R/J_c$, and hence $\dim S/I_c = \dim R/J_c = d$, and the Bourbaki sequence implies that

$$H^i_{\mathfrak{m}}(S/I_c) = H^i_{\mathfrak{m}}(R/J_c) = M_i(-c). \qquad \square$$

PROPOSITION 2.2. *Let $R = S/J$ be a graded $K$-algebra. Consider the exact sequence of graded $R$-modules*

$$0 \longrightarrow F \longrightarrow M \longrightarrow I(c) \longrightarrow 0,$$

*where $F$ is free, $M = \Omega_{j+1}(N)$ is the $(j+1)$th syzygy-module of a module $N$ of finite length, say,*

$$0 \longrightarrow M \longrightarrow F_j \longrightarrow \cdots \longrightarrow F_0 \longrightarrow N \longrightarrow 0$$

*with* $F_i = \bigoplus_k R(-b_{ik})$, *and where* $I \subset R$ *is a graded ideal of codimension* 2. *Suppose* $r = \operatorname{rank} M$. *Then:*

(a) *Let* $F = \bigoplus_i R(-a_i)$. *Then we have*

$$c = \frac{rQ'_R(1) - Q'_M(1)}{Q_R(1)} + \sum_{i=1}^{r-1} a_i.$$

(b) $$\frac{rQ'_R(1) - Q'_M(1)}{Q_R(1)} = (-1)^{j+1} \sum_{i=0}^{j} (-1)^i \sum_k b_{ik}.$$

*Proof.* Let $d = \dim R$. Since

$$0 \longrightarrow F(-c) \longrightarrow M(-c) \longrightarrow I \longrightarrow 0$$

is an exact sequence of modules which are all of dimension $d$, it follows that

$$Q_I(t) = (Q_M(t) - Q_F(t))t^c. \tag{8}$$

Since $I$ is of codimension 2, the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

yields

$$(1-t)^2 Q_{R/I}(t) = Q_R(t) - Q_I(t).$$

This implies $Q_R(1) = Q_I(1)$ and $Q'_R(1) = Q'_I(1)$. Thus, by (8),

$$Q_R(1) = Q_I(1) = Q_M(1) - Q_F(1),$$

and

$$\begin{aligned} Q'_R(1) &= Q'_I(1) = Q'_M(1) - Q'_F(1) + (Q_M(1) - Q_F(1))c \\ &= Q'_M(1) - Q'_F(1) + Q_R(1)c. \end{aligned}$$

Therefore

$$c = \frac{Q'_R(1) + Q'_F(1) - Q'_M(1)}{Q_R(1)}.$$

Since $Q_F(t) = Q_R(t)\left(\sum_{i=1}^{r-1} t^{a_i}\right)$, we see that

$$Q'_F(1) = Q'_R(1)(r-1) + Q_R(1)\left(\sum_{i=1}^{r-1} a_i\right).$$

Hence

$$c = \frac{rQ'_R(1) - Q'_M(1)}{Q_R(1)} + \sum_{i=1}^{r-1} a_i.$$

It remains to prove (b). Since $\dim N = 0$, the exact sequence

$$0 \longrightarrow M \longrightarrow F_j \longrightarrow \cdots \longrightarrow F_0 \longrightarrow N \longrightarrow 0$$

yields

$$Q_M(t) = Q_{F_j}(t) - Q_{F_{j-1}}(t) + \cdots \\ \cdots + (-1)^j Q_{F_0}(t) + (-1)^{j+1}(1-t)^d Q_N(t).$$

Therefore

$$Q'_M(1) = (-1)^j \sum_{i=0}^{j} (-1)^i Q'_{F_i}(1).$$

Since

$$Q_{F_i}(t) = Q_R(t)\left(\sum_k t^{b_{ik}}\right),$$

we have

$$Q'_{F_i}(1) = Q'_R(1)\operatorname{rank} F_i + Q_R(1)\left(\sum_k b_{ik}\right).$$

Hence

$$\begin{aligned} Q'_M(1) &= Q'_R(1)(-1)^j \sum_{i=0}^{j}(-1)^i \operatorname{rank} F_i \\ &\qquad + Q_R(1)(-1)^j \sum_{i=0}^{j}(-1)^i \sum_k b_{ik} \\ &= Q'_R(1) r + Q_R(1)(-1)^j \sum_{i=0}^{j}(-1)^i \sum_k b_{ik}. \end{aligned}$$

We conclude that

$$\frac{rQ'_R(1) - Q'_M(1)}{Q_R(1)} = (-1)^{j+1} \sum_{i=0}^{j}(-1)^i \sum_k b_{ik},$$

as desired. □

We conclude this section by computing the number $c$ in Proposition 2.2 in some cases explicitly.

EXAMPLES 2.3. (a) We let $R = S = K[x_1, \ldots, x_n]$, $N = K$, and $M = \Omega_{j+1}(K)$. All generators of $M$ are of degree $j+1$ because the Koszul complex $K_\bullet$ of the sequence $x_1, \ldots, x_n$ provides a minimal graded free $S$-resolution of $K$. We also take a (general enough) free submodule $F = \bigoplus_{i=1}^{r-1} R(-j-1)$ of $M$, so that $M/F$ is isomorphic to a codimension 2 ideal. Then we see that

$$Q_M(t) = (-1)^j \sum_{i=0}^{j}(-1)^i \binom{n}{i} t^i.$$

Therefore

$$r = \operatorname{rank} M = Q_M(1) = (-1)^j \sum_{i=0}^{j} (-1)^i \binom{n}{i} = \binom{n-1}{j}.$$

We set

$$\alpha(n,j) = Q'_M(1) = (-1)^j \sum_{i=0}^{j} (-1)^i \binom{n}{i} i.$$

An easy calculation shows that

$$\begin{aligned} \alpha(n,j) \quad &+ \quad \alpha(n-1,j-1) \\ &= \quad \sum_{i=1}^{j} (-1)^{i+j} \left[ \binom{n}{i} - \binom{n-1}{i} \right] + \binom{n-1}{j} j \\ &= \quad \alpha(n-1,j-1) + \binom{n-2}{j-1} + \binom{n-1}{j} j, \end{aligned}$$

so that

$$\alpha(n,j) = \binom{n-2}{j-1} + \binom{n-1}{j} j.$$

Therefore

$$\begin{aligned} c \quad &= \quad -\binom{n-1}{j} j - \binom{n-2}{j-1} + \left( \binom{n-1}{j} - 1 \right) (j+1) \\ &= \quad \binom{n-2}{j} - j - 1. \end{aligned}$$

(b) We let $R = S/(f)$, where $f$ is a quadratic form, $N = K$ and $M = \Omega_{j+1}(K)$. Since $R$ is a complete intersection, the Tate resolution provides a minimal graded free resolution of $K$ over $R$. It follows that $K$ has linear $R$-resolution and that the Poincaré series of $K$ is given by

$$P(t) = \frac{(1+t)^n}{1-t^2} = \frac{(1+t)^{n-1}}{1-t} = \left( \sum_{i=0}^{n-1} \binom{n-1}{i} t^i \right) \left( \sum_{i=0}^{\infty} t^i \right).$$

Thus if we set $a_i = \binom{n-1}{i}$ and denote by $b_i$ the $i$th Betti number of $K$, we get $b_i = \sum_{k=0}^{i} a_k$ for all $i \geq 0$.

The formulas in 2.2 (with $F = \bigoplus_{i=1}^{r-1} R(-j-1)$ a general enough graded free submodule of $M$) imply that

$$r = (-1)^j \sum_{i=0}^{j} (-1)^i b_i \quad \text{and} \quad c = -(-1)^j \sum_{i=0}^{j} (-1)^i i b_i + (r-1)(j+1).$$

From this it follows that

$$c = \begin{cases} \sum_{k=0}^{j/2} \frac{j-2k+2}{2} \left( \binom{n-1}{2k} - \binom{n-1}{2k-1} \right) - j - 1, & \text{if } j \text{ is even,} \\ \sum_{k=0}^{(j-1)/2} \frac{j-2k+1}{2} \left( \binom{n-1}{2k+1} - \binom{n-1}{2k} \right) - j - 1, & \text{if } j \text{ is odd.} \end{cases}$$

According to Theorem 2.1 for any number $c'$ greater than or equal to the number computed in Examples (a), respectively (b), there exists a graded codimension 2, respectively codimension 3, ideal $I \subset S$ such that $H^j_{\mathfrak{m}}(S/I) = K(-c')$, and $H^i_{\mathfrak{m}}(S/I) = 0$ for $i < \dim S/I$, $i \neq j$. The bounds given by 2.1 are not sharp in general, but as we shall see in the next section, the bounds are sharp in codimension 2.

Other constructions of ideals $I \subset S$ such that $H^j_{\mathfrak{m}}(S/I) = K(-c')$, and $H^i_{\mathfrak{m}}(S/I) = 0$ for $i < \dim S/I$, $i \neq j$, are given in [7] and [10].

## 3. Graded generalized CM rings of codimension 2

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring over an infinite field $K$ and $\mathfrak{m} = (x_1, \dots, x_n)$. In this section we study the special case of generalized CM-rings $R = S/I$ of dimension $n-2$ and depth $t < n-2$ for which $H^i_{\mathfrak{m}}(R) = 0$ for $i < n-2$ and $i \neq t$.

Proposition 3.1. *Let $I \subset S$ be a graded ideal such that $R = S/I$ is a generalized CM-ring of codimension 2 and depth $t$. The following conditions are equivalent:*

(a) $H^i_{\mathfrak{m}}(R) \cong \begin{cases} 0, & \text{for } i < n-2 \text{ and } i \neq t, \\ M, & \text{for } i = t. \end{cases}$

(b) *There exists an exact sequence of graded modules*

$$0 \longrightarrow F \longrightarrow \Omega_{t+1}(M) \oplus G \longrightarrow I \longrightarrow 0,$$

*where $F$ and $G$ are graded free $S$-modules and $\Omega_{t+1}(M)$ is the $(t+1)$th syzygy-module of $M$ over $S$.*

*Proof.* It is obvious that (b)⇒(a). For the converse implication we use 1.4, and thus it remains to show that if $X$ is a graded $S$-module of maximal dimension with the property that for some $s$ with $0 < s < n-1$ one has

$$H^i_{\mathfrak{m}}(X) \cong \begin{cases} 0, & \text{for } i < n \text{ and } i \neq s, \\ M, & \text{for } i = s, \end{cases}$$

then $X \cong \Omega_s(M) \oplus G$, where $G$ is free.

Let

$$0 \longrightarrow F_{n-s} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow X \longrightarrow 0$$

be the minimal graded free $S$-resolution of $X$.

Let $N$ be an $S$-module. We set $N^* = \operatorname{Hom}_S(N, S(-n))$ and $N^\vee = \operatorname{Hom}_S(N, E)$, where $E$ denotes the injective hull of $K$. Then we get the exact sequence

$$0 \longrightarrow X^* \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow \cdots \longrightarrow F_{n-s}^* \longrightarrow M^\vee \longrightarrow 0, \tag{9}$$

since by local duality, $M^\vee = H_{\mathfrak{m}}^s(X)^\vee = \operatorname{Ext}_S^{n-s}(X, S(-n))$.

If $s = 1$, then $X^*$ is free. Taking again the dual with respect to $S(-n)$ we obtain the exact sequence

$$0 \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow X^{**} \longrightarrow M \longrightarrow 0,$$

from which we deduce the short exact sequence $0 \to X \to X^{**} \to M \to 0$. This concludes the proof in the case $s = 1$.

If $s > 1$, let $0 \to G_{s-1} \to \cdots \to G_0 \to X^* \to 0$ be the graded minimal free resolution of $X^*$. Composing it with (9) we obtain a graded minimal free resolution of $M^\vee$. Dualizing this resolution with respect to $S(-n)$ we conclude that $0 \to X^{**} \to G_0^* \to \cdots \to G_{s-1}^* \to M \to 0$ is exact, and so $X^{**} \cong \Omega_s(M) \oplus G$ for some free $S$-module $G$. Since $X$ is free on the punctured spectrum and has depth $s \geq 2$, it follows that $X \cong X^{**}$; see, for example, [5, Proposition 1.4.1]. □

Notice that Proposition 3.1 also holds when we replace $S$ by a Gorenstein local ring $(R, \mathfrak{m})$.

Now we come to the main result of this section, which is a refinement of Theorem 2.1 in the special case of codimension 2, where all but one $M_i$ vanish.

THEOREM 3.2. *Let $0 \leq t < n-2$ be an integer, and let $M$ be a graded $S$-module of finite length. Let $g_1, \cdots, g_m$ be a minimal set of homogeneous generators of $\Omega_{t+1}(M)$ with* $\deg g_i = a_i$ *and $a_1 \leq a_2 \leq \cdots \leq a_m$. Then the following holds:*

(a) *If $I \subset S$ is an ideal of codimension 2 such that*

$$H_{\mathfrak{m}}^i(S/I) \cong \begin{cases} 0, & \text{for } i < n-2 \text{ and } i \neq t, \\ M(-c), & \text{for } i = t, \end{cases}$$

*then $c \geq -Q'_{\Omega_{t+1}(M)}(1) + \sum_{i=1}^{r-1} a_i$, where $r = \operatorname{rank}\Omega_{t+1}(M)$.*

(b) *If all generators of $\Omega_{t+1}(M)$ have the same degree $a$, then an ideal of codimension 2 with local cohomology as in* (a) *exists, if and only if*

$$c \geq -Q'_{\Omega_{t+1}(M)}(1) + (r-1)a.$$

*Proof.* For the proof of (a) we use 3.1 according to which there exists a short exact sequence $0 \to F_0 \to \Omega_{t+1}(M(-c)) \oplus G_0 \to I \to 0$ of graded modules with $F_0$ and $G_0$ free. Twisting this exact sequence with $c$, we get the exact sequence

$$0 \longrightarrow F \xrightarrow{\ j\ } \Omega_{t+1}(M) \oplus G \xrightarrow{\ p\ } I(c) \longrightarrow 0$$

with the graded free modules $F = F_0(c)$ and $G = G_0(c)$. Now 2.2(a) yields that $c = -Q'_{\Omega_{t+1}(M)}(1) - Q'_G(1) + Q'_F(1)$. Let $g_{m+1}, \dots, g_{m+s}$ be a homogeneous basis of $G$ with $\deg g_i = a_i$ for $i = m+1, \dots, m+s$, and $h_1, \dots, h_{r+s-1}$ a homogeneous basis of $F$ with $\deg h_i = b_i$ for $i = 1, \dots, r+s-1$. Then

$$c = -Q'_{\Omega_{t+1}(M)}(1) - \sum_{i=m+1}^{m+s} a_i + \sum_{i=1}^{r+s-1} b_i. \tag{10}$$

Let $\pi_1 \colon \Omega_{t+1}(M) \oplus G \to \Omega_{t+1}(M)$ and $\pi_2 \colon \Omega_{t+1}(M) \oplus G \to G$ be the natural projection maps, and $\iota_1 \colon \Omega_{t+1}(M) \to \Omega_{t+1}(M) \oplus G$ and $\iota_2 \colon G \to \Omega_{t+1}(M) \oplus G$ the natural inclusion maps.

We may assume that $(p \circ \iota_1)(\Omega_{t+1}(M)) \neq 0$ and $(p \circ \iota_2)(G) \neq 0$. In fact, suppose that $(p \circ \iota_1)(\Omega_{t+1}(M)) = 0$. Then $\iota_1(\Omega_{t+1}(M)) \subset \operatorname{Ker} p = \operatorname{Im} j$. Let $\Omega' = j^{-1}(\iota_1(\Omega_{t+1}(M))$. Then $\Omega' \subset F$ and the composition $\Omega' \subset F \to \Omega_{t+1}(M)$ of the inclusion map with $\pi_1 \circ j$ is an isomorphism. This implies that $\Omega_{t+1}(M)$ is isomorphic to a direct summand of $F$, and hence free since it is a graded module. This however is a contradiction, since $\operatorname{depth} \Omega_{t+1}(M) = t + 1 < n$.

Similarly, if $(p \circ \iota_2)(G) = 0$, we conclude that $F = H \oplus G$ for some free module $H$, and we may replace the exact sequence $0 \to F \to \Omega_{t+1}(M) \oplus G \to I(c) \to 0$ with the exact sequence $0 \to H \to \Omega_{t+1}(M) \to I(c) \to 0$.

Now since $(p \circ \iota_1)(\Omega_{t+1}(M)) \neq 0$ and $I(c)$ is a rank 1 module, it follows that $I(c)/(p \circ \iota_1)(\Omega_{t+1}(M))$ is a torsion module. Hence, since $I(c)/(p \circ \iota_1)(\Omega_{t+1}(M)) \cong G/\operatorname{Im}(\pi_2 \circ j)$, it follows that $\operatorname{rank}(\operatorname{Im}(\pi_2 \circ j)) = \operatorname{rank} G = s$. Similarly it follows that $\operatorname{rank}(\operatorname{Im}(\pi_1 \circ j)) = \operatorname{rank} \Omega_{t+1}(M) = r$.

Next consider the map

$$\bigwedge^{r+s-1} F \longrightarrow \bigwedge^{r+s-1} (\Omega_{t+1}(M) \oplus G) \cong \bigoplus_{i=0}^{r+s-1} \bigwedge^{i} (\Omega_{t+1}(M)) \otimes \bigwedge^{r+s-1-i} (G).$$

We claim that $\bigwedge^{r+s-1}(j)$ composed with the natural projection map

$$\bigwedge^{r+s-1} (\Omega_{t+1}(M) \oplus G) \to \bigwedge^{r-1} (\Omega_{t+1}(M)) \otimes \bigwedge^{s} (G)$$

is not trivial. The claim will imply that the component of $j(h_1) \wedge j(h_2) \wedge \cdots \wedge j(h_{r+s-1})$ in $\bigwedge^{r-1}(\Omega_{t+1}(M)) \otimes \bigwedge^{s}(G)$ is of the form $\sum_I a_I g_I \wedge g_{m+1} \wedge \cdots \wedge g_{m+s}$, where the sum is taken over all subsets $I \subset \{1, \dots, m\}$ with $r-1$ elements, where $g_I = g_{i_1} \wedge \cdots \wedge g_{i_{r-1}}$ for $I = \{i_1, \dots, i_{r-1}\}$, $i_1 < i_2 < \cdots < i_{r-1}$, and where at least one $a_I \neq 0$.

It follows that

$$\sum_{i=1}^{r+s-1} b_i \geq \sum_{j=1}^{r-1} a_{i_j} + \sum_{i=m+1}^{m+s} a_i$$

for some subset $\{i_1, \dots, i_{r-1}\} \subset \{1, \dots, m\}$. By (10) this implies that

$$c \geq -Q'_{\Omega_{t+1}(M)}(1) + \sum_{j=1}^{r-1} a_{i_j} \geq -Q'_{\Omega_{t+1}(M)}(1) + \sum_{i=1}^{r-1} a_i,$$

as desired.

In order to prove the claim, it suffices to show that the map in question is not trivial after tensorizing it with the quotient field $Q$ of $S$. Notice that $\bigwedge^{r+s-1}(\Omega_{t+1}(M) \oplus G) \otimes Q$ is isomorphic to

$$\begin{aligned}\left(\bigwedge^{r}(\Omega_{t+1}(M)) \otimes Q\right) \otimes \left(\bigwedge^{s-1}(G) \otimes Q\right)& \\ \oplus \left(\bigwedge^{r-1}(\Omega_{t+1}(M)) \otimes Q\right) \otimes \left(\bigwedge^{s}(G) \otimes Q\right)&,\end{aligned}$$

since all other direct summands $\bigwedge^{i}(\Omega_{t+1}(M)) \otimes \bigwedge^{r+s-1-i}(G)$ are torsion modules. Thus the claim will follow from Lemma 3.3 below.

(b) The 'only if' part of statement (b) follows from (a). On the other hand, the converse follows from Theorem 2.1. $\square$

LEMMA 3.3. *Let $K$ be field, and $U$, $V$ and $W$ be $K$-vector spaces with $\dim V = r$, $\dim W = s$ and $\dim U = r + s - 1$, and let $j \colon U \to V \oplus W$ be an injective map such that $\pi_1 \circ j \colon U \to V$ and $\pi_2 \circ j \colon U \to W$ are surjective, where $\pi_1$ and $\pi_2$ are the natural projections. Then for the induced map*

$$\bigwedge^{r+s-1}(j) \colon \bigwedge^{r+s-1}(U) \longrightarrow \bigwedge^{r}(V) \otimes \bigwedge^{s-1}(W) \oplus \bigwedge^{r-1}(V) \otimes \bigwedge^{s}(W)$$

*we have that $p_1 \circ \bigwedge^{r+s-1}(j) \neq 0$ and $p_2 \circ \bigwedge^{r+s-1}(j) \neq 0$, where $p_1$ and $p_2$ are the natural projections of $\bigwedge^{r}(V) \otimes \bigwedge^{s-1}(W) \oplus \bigwedge^{r-1}(V) \otimes \bigwedge^{s}(W)$ onto its direct summands.*

*Proof.* Suppose, for example, that $p_2 \circ \bigwedge^{r+s-1}(j) = 0$. Since $\pi_2 \circ j \colon U \to W$ is surjective, we may choose bases $u_1, \dots, u_{r+s-1}$ of $U$ and $w_r, \dots, w_{r+s-1}$ of $W$, such that $j(u_i) = v_i \in V$ for $i = 1, \dots, r-1$ and $j(u_i) = v_i + w_i$ with $v_i \in V$ for $i = r, \dots, r+s-1$. Then

$$\begin{aligned} 0 &= (p_2 \circ \bigwedge^{r+s-1}(j))(u_1 \wedge \cdots \wedge u_{r+s-1}) \\ &= (v_1 \wedge \cdots \wedge v_{r-1}) \otimes (w_r \wedge \cdots \wedge w_{r+s-1}). \end{aligned}$$

This implies that $v_1 \wedge \cdots \wedge v_{r-1} = 0$. So the vectors $v_1, \dots, v_{r-1}$ are linearly dependent, a contradiction, since $j$ is injective. $\square$

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