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# THE ROOT OPERATOR ON INVARIANT SUBSPACES OF THE BERGMAN SPACE

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ABSTRACT. For an invariant subspace $I$ of the Bergman space we study an integral operator on $I$ defined in terms of the reproducing kernel of $I$. Such an operator will be called the root operator of $I$ and its associated integral kernel will be called the root function of $I$. We obtain fundamental spectral properties of the root operator when the invariant subspace $I$ has finite index.

## 1. Introduction

The Bergman space $A^2$ consists of analytic functions $f$ in the unit disk $\mathbb{D}$ such that

$$\|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty,$$

where $dA$ is the normalized area measure on $\mathbb{D}$. It is easy to see that $A^2$ is closed in $L^2(\mathbb{D}, dA)$ and so is a Hilbert space with inner product

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z) \, \overline{g(z)} \, dA(z), \qquad f, g \in A^2.$$

For the general theory of Bergman spaces see [6] and [11].

The Bergman shift $B$ is multiplication by the coordinate function $z$ on $A^2$. A closed subspace of $A^2$ is called an invariant subspace if it is invariant for $B$. For any $f \in A^2$ we let $I_f$ denote the smallest invariant subspace containing $f$ and call it the invariant subspace generated by $f$. It is clear that $I_f$ is simply the closure in $A^2$ of the set of polynomial multiples of $f$.

Throughout the paper we fix an invariant subspace $I$ of $A^2$ and let $T$ denote the restriction of the Bergman shift $B$ to $I$. The operator $T$ contains much information about $I$ and has been a subject of many studies. In particular, we mention the hyponormality of $T$, that is, the self-commutator

$$[T^*, \ T] = T^*T - TT^*$$

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is a positive operator. It then follows from the Berger-Shaw theorem (see [4]) that $[T^*,\ T]$ is in the trace class when $I$ is finitely generated, or according to [1], when

$$\operatorname{index}(I) := \dim(I \ominus zI) < \infty.$$

In fact, it is proved in [13] and [14] that the trace of $[T^*, T]$ is equal to the index of $I$.

We denote the reproducing kernel of $I$ by $K^I(z,w)$. The superscript $I$ will remain throughout the paper, because we use $K(z,w)$ to denote the full Bergman kernel,

$$K(z,w) = \frac{1}{(1-z\overline{w})^2}.$$

Since the reproducing kernel completely determines the underlying invariant subspace, every piece of information about $I$ is encoded in $K^I(z,w)$, and it is therefore only a matter of how to decipher this information.

The reproducing kernel $K^I(z,w)$ is not easily computable, except in very special situations. Therefore, it has not played a very important role in the study of invariant subspaces until only recently, when a number of papers (see [2], [7], [8], and [9]) exhibited certain structural properties of $K^I(z,w)$ and showed how such structural properties can be applied to prove new theorems and reprove some old theorems about $I$.

Although the boundary behavior of $K^I(z,w)$ is complicated in general, many examples seem to suggest that $K^I(z,w)$ behaves like $K(z,w)$ near those boundary points where $I$ does not have a singularity. So the factor $(1-z\overline{w})^2$ can be used to tame $K^I(z,w)$ on the boundary. This observation led us to the following definitions.

The root function $R^I$ of $I$ is defined on $\mathbb{D}\times\mathbb{D}$ as

$$R^I(z,w) := \frac{K^I(z,w)}{K(z,w)},$$

and the associated root operator on $I$ is defined as

$$C^I(f)(z) := \int_{\mathbb{D}} \frac{K^I(z,w)}{K(z,w)} f(w)\, dA(w).$$

When no confusion is likely, we shall suppress the superscript $I$ and simply write $R(z,w)$ for the root function and $C$ for the root operator. One notes that $R(z,z)$ is the square of the so-called majorization function for $I$ studied in [2]. The origin of the root function and the root operator is in [4] where, in the setting of the Hardy space over the bidisk, they were called the core function and the core operator, respectively.

It will be shown in the next section that the root operator $C$ is a bounded linear operator on $I$. The main purpose of the paper is to study the basic operator theoretic properties of $C$. More specifically, we obtain complete results about when the operator $C$ is compact, in the trace class, in the

Hilbert-Schmidt class, or more generally, in the Schatten $p$-classes. We also obtain partial results about when the root operator has finite rank.

We wish to thank the referee for several thoughtful remarks and suggestions. In particular, the referee pointed out:

(1) The root operator can be written as

$$C^I = P^I - D^I,$$

where $P^I$ is the orthogonal projection from $I$ onto $I \ominus zI$ and $D^I$ is some positive operator on $I$ that leaves $zI$ invariant.

(2) If the index of $I$ is one, then the root function can be written in the form

$$R^I(z, w) = G(z)\overline{G(w)}(1 - l(z, w)),$$

where $l(z, w)$ is some positive-definite function and $G$ is the extremal function for $I$.

(3) The rank of $C^I$ in Proposition 13 has been precisely computed in the recent paper [3].

## 2. The Bergman shift and the extremal function

Recall that $T$ is the operator of multiplication by the coordinate function $z$ on the invariant subspace $I$. Most of our analysis is based on the following explicit relationship between the operators $C$ and $T$.

LEMMA 1. *For every invariant subspace $I$ we have*

$$C = 1 - 2TT^* + T^2 T^{*2},$$

*where* 1 *denotes the identity operator.*

*Proof.* Observe that if $\varphi$ is any bounded analytic function in $\mathbb{D}$, then the operator of multiplication by $\varphi$, denoted by $M_\varphi$, maps $I$ boundedly into $I$. In fact, it is easy to check that $\|M_\varphi\| = \|\varphi\|_\infty$. Furthermore, the adjoint of $M_\varphi : I \to I$ admits the following integral representation:

$$M_\varphi^* f(z) = \int_{\mathbb{D}} K^I(z, w)\overline{\varphi(w)} f(w)\, dA(w), \qquad f \in I.$$

Since

$$\begin{aligned} Cf(z) &= \int_{\mathbb{D}} (1 - z\overline{w})^2 K^I(z, w) f(w)\, dA(w) \\ &= \int_{\mathbb{D}} (1 - 2z\overline{w} + z^2\overline{w}^2) K^I(z, w) f(w)\, dA(w), \end{aligned}$$

the desired result then follows from the above observation. $\square$

A consequence of Lemma 1 is that $C$ is a bounded self-adjoint operator on every $I$. Also, setting $I = A^2$, we conclude that the operator

$$1 - 2BB^* + B^2 B^{*2},$$

where $B$ is the Bergman shift, is the rank 1 operator which maps every $f$ to $f(0)$.

In many situations it is more useful for us to rewrite the operator $C$ as

$$C = (1 - TT^*) - T(1 - TT^*)T^*. \tag{1}$$

In particular, this suggests that the size of $C$ is controlled by that of $1 - TT^*$.

If $\{e_n\}$ is any orthonormal basis for $I$, then

$$K^I(z, z) = \sum_n |e_n(z)|^2.$$

Expanding $\{e_n\}$ to an orthonormal basis for the full space $A^2$, we see that $K^I(z, z) \le K(z, z)$, so that

$$0 \le R(z, z) \le 1, \qquad z \in \mathbb{D}. \tag{2}$$

This simple observation will be useful later when we calculate the trace of the root operator $C$.

If $m$ is the smallest non-negative integer such that $f^{(m)}(0) \ne 0$ for some $f \in I$, then the following extremal problem has a unique solution:

$$\sup\{\operatorname{Re} f^{(m)}(0) : f \in I, \|f\| \le 1\}.$$

This solution will be called the extremal function for $I$ and will be denoted by $G(z)$. It is easy to see that if $m = 0$, then

$$G(z) = K^I(z, 0)/\sqrt{K^I(0, 0)}, \qquad z \in \mathbb{D}.$$

Furthermore, if $Q$ is the orthogonal projection from $A^2$ onto $I$ and 1 is the constant function, then

$$K^I(z, 0) = Q(1)(z), \qquad z \in \mathbb{D}.$$

See [6] for more information about the extremal function.

The following estimate compares the root function on the diagonal to the extremal function.

Lemma 2. *For every invariant subspace $I$ we have*

$$(1 - |z|^2)|G(z)|^2 \le R(z, z), \qquad z \in \mathbb{D}.$$

*If $I$ has index 1, then we also have*

$$R(z, z) \le |G(z)|^2, \qquad z \in \mathbb{D}.$$

*Proof.* For the first inequality see the proof of Theorem 4.2 in [12]. The second inequality follows from (5.2) of [2]. □

The following proposition provides more accurate information about the spectrum $\sigma(C)$ of the operator $C$. Here, for an eigenvalue $\lambda$ of $C$, its corresponding eigenspace is denoted by $E_\lambda$.

PROPOSITION 3. *The root operator $C$ always has $1$ as an eigenvalue. Furthermore, $\|C\| = 1$, $E_1 = I \ominus zI$, and $-1$ is not an eigenvalue of $C$.*

*Proof.* Since $1 - TT^*$ is a positive contraction, equation (1) implies that

$$-T(1 - TT^*)T^* \le C \le 1 - TT^*,$$

from which it follows that $\|C\| \le 1$. Since

$$\ker(T^*) = I \ominus zI,$$

Lemma 1 shows that $Cf = f$ on $I \ominus zI$. This shows that $\|C\| = 1$ and $I \ominus zI$ is contained in the eigenspace $E_1$.

If $Cf = f$ for some $f \in I$, then by Lemma 1,

$$T(2 - TT^*)T^* f = 0.$$

Since $T$ is one-to-one, this gives

$$(2 - TT^*)T^* f = 0.$$

But the operator $2 - TT^*$ is invertible, so $T^* f = 0$, or $f \in I \ominus zI$. This along with the last statement in the previous paragraph shows that $E_1 = I \ominus zI$.

If $Cf = -f$, then Lemma 1 gives

$$TT^* f + T(1 - TT^*)T^* f = 2f.$$

Taking the inner product with $f$ on both sides, we get

$$\|T^* f\|^2 + \|(1 - TT^*)^{1/2} T^* f\|^2 = 2\|f\|^2.$$

Since $T^*$ and $(1 - TT^*)^{1/2}$ are both contractions, we easily deduce that $\|T^* f\| = \|f\|$. But the hyponormality of $T$ implies $\|T^* f\| \le \|Tf\|$, so $\|f\| \le \|Tf\|$, which happens only if $f = 0$. □

## 3. Preliminaries on Schatten ideals

If $S$ is a self-adjoint compact operator on a Hilbert space $H$, then there exists an orthonormal set $\{e_n\}$ in $H$ such that

$$Sx = \sum_n \lambda_n e_n \langle x, e_n \rangle, \qquad x \in H,$$

where $\{\lambda_n\}$ is the eigenvalue sequence of $S$, counting multiplicity.

For any $0 < p < \infty$, the Schatten $p$-class, or the Schatten $p$-ideal, consists of all compact operators $S$ on $H$ such that the eigenvalue sequence $\{\lambda_n\}$ of $(S^*S)^{1/2}$ satisfies

$$\|S\|_p = \left(\sum_n |\lambda_n|^p\right)^{1/p} < \infty.$$

It is well known that for $1 \le p < \infty$ the Schatten $p$-class is a Banach space with the above norm. Furthermore, if $S$ belongs to the Schatten $p$-class, then

$$\|ASB\|_p \le \|A\|\|S\|_p\|B\|$$

for all bounded linear operators $A$ and $B$ on $H$, so that the Schatten $p$-class is actually a two-sided ideal in the full algebra of bounded linear operators on $H$.

If $S$ is a self-adjoint compact operator with eigenvalue sequence $\{\lambda_n\}$, then

$$\|S\|_p^p = \sum_n |\lambda_n|^p,$$

because the eigenvalue sequence of $(S^*S)^{1/2}$ is simply $\{|\lambda_n|\}$.

When $p = 1$, the Schatten $p$-class is called the trace class. If $S$ is in the trace class and $\{e_n\}$ is an orthonormal basis for $H$, then the series

$$\sum_n \langle Se_n, e_n\rangle$$

converges and the sum is independent of the choice of the orthonormal basis. This sum is called the trace of $S$ and is denoted by $\mathrm{tr}(S)$. It is easy to see that if $S$ is a self-adjoint operator in the trace class, and if the eigenvalue sequence of $S$ is $\{\lambda_n\}$, then

$$\mathrm{tr}(S) = \sum_n \lambda_n.$$

When $p = 2$, the Schatten $p$-class is called the Hilbert-Schmidt class. If $S$ is in the Hilbert-Schmidt class and if $\{e_n\}$ is an orthonormal basis for $H$, then the series

$$\sum_n \|Se_n\|^2$$

converges and the sum is independent of the choice of the orthonormal basis. The square root of this sum is equal to the Hilbert-Schmidt norm $\|S\|_2$, or the Schatten 2-norm, of $S$.

It is well known that if $S$ is an integral operator on $L^2(\mu)$,

$$Sf(x) = \int H(x,y)f(y)\,d\mu(y),$$

then $S$ is Hilbert-Schmidt if and only if

$$\iint |H(x,y)|^2\,d\mu(x)\,d\mu(y) < \infty.$$

Furthermore, the above double integral equals $\|S\|_2^2$.

More details and references about the Schatten classes can be found in [11]. We now specialize to the case of operators on the invariant subspace $I$.

Recall that the zero set of $I$, denoted by $Z_I$, consists of the common zeros of all functions in $I$. For $z \in \mathbb{D} - Z_I$ we define

$$k_z^I(w) = \frac{K^I(w,z)}{\sqrt{K^I(z,z)}}, \qquad w \in \mathbb{D},$$

and call it the normalized reproducing kernel of $I$ at $z$.

For any bounded linear operator $S$ on $I$ we can define a bounded function $\widetilde{S}$ in $L^\infty(\mathbb{D}, dA)$ by

$$\widetilde{S}(z) = \langle Sk_z^I, k_z^I \rangle, \qquad z \in \mathbb{D} - Z_I.$$

This function is called the Berezin transform of $S$. See [11] and [6] for more information about the Berezin transform.

Lemma 4. *Let $S$ be a bounded linear operator on $I$. If $S$ is either positive or in the trace class, then*

$$\mathrm{tr}(S) = \int_{\mathbb{D}} \widetilde{S}(z) K^I(z,z)\, dA(z).$$

*In particular, a positive operator $S$ on $I$ belongs to the trace class if and only if the above integral is finite.*

*Proof.* This is similar to the proof of Proposition 6.3.2 in [11]. □

Lemma 5. *A bounded linear operator $S$ on $I$ is Hilbert-Schmidt if and only if*

$$\int_{\mathbb{D}} \|Sk_z^I\|^2 K^I(z,z)\, dA(z) < \infty.$$

*Furthermore, the square root of the above integral is equal to the Hilbert-Schmidt norm of $S$.*

*Proof.* The operator $S$ is Hilbert-Schmidt if and only if the positive operator $S^*S$ is in the trace class. The desired result then follows from Lemma 4. □

Corollary 6. *If the root operator $C$ is in the trace class, then*

$$\mathrm{tr}(C) = \int_{\mathbb{D}} R(z,z)\, dA(z),$$

*and the Hilbert-Schmidt norm of $C$ is*

$$\|C\|_2 = \left[ \int_{\mathbb{D}} \int_{\mathbb{D}} |R(z,w)|^2\, dA(z)\, dA(w) \right]^{1/2}.$$

*In particular, if $C$ is in the trace class, then $0 < \mathrm{tr}(C) \le 1$, and $\mathrm{tr}(C) = 1$ holds only for $I = A^2$.*

*Proof.* For the root operator $C$ on $I$ one verifies by the reproducing property of $K^I(z,w)$ that

$$CK^I(-,z)(w) = R(w,z),$$

and hence

$$\langle CK^I(-,z),\ K^I(-,z)\rangle = R(z,z).$$

In particular, the Berezin transform of $C$ is given by

$$\widetilde{C}(z) = \frac{R(z,z)}{K^I(z,z)} = (1-|z|^2)^2.$$

The trace formula then follows from Lemma 4, the formula for the Hilbert-Schmidt norm of $C$ follows from Lemma 5, and the estimates for $\mathrm{tr}(C)$ follow from the inequalities in (2). □

Note that the above formula for the Hilbert-Schmidt norm of $C$ also follows from the general theory of integral operators. In fact, if

$$f \in L^2(\mathbb{D}, dA) \ominus I,$$

then

$$\int_{\mathbb{D}} \frac{K^I(z,w)}{K(z,w)} f(w)\, dA(w) = 0, \qquad z \in \mathbb{D}.$$

Therefore, the Hilbert-Schmidt norm of $C$ on $I$ is the same as the Hilbert-Schmidt norm of the integral operator

$$Sf(z) = \int_{\mathbb{D}} \frac{K^I(z,w)}{K(z,w)} f(w)\, dA(w)$$

on $L^2(\mathbb{D}, dA)$, which is the square root of the double integral of the integral kernel.

Also note that the proof of Corollary 6 shows the Berezin transform of the root operator $C$ is independent of the underlying space $I$, and is always vanishing on the boundary of the unit disk. Later in the paper we will see that $C$ is compact if and only if the index of $I$ is finite. Since there exist invariant subspaces of infinite index, this gives us examples of bounded operators that are not compact but whose Berezin transforms vanish on the boundary.

## 4. Membership of $C$ in Schatten classes

This section deals with the compactness and membership in Schatten classes for the root operator $C$. It turns out that the index of $I$ is the key to these issues.

Lemma 7. *The following conditions are equivalent.*

(i) *The operator $1 - TT^*$ is Hilbert-Schmidt.*

(ii) *The operator* $1 - TT^*$ *is compact.*
(iii) *The index of* $I$ *is finite.*

*Proof.* First observe that the operator $1 - T^*T$ is always Hilbert-Schmidt, regardless of the index of $I$. In fact, it is a Toeplitz-type operator, and its integral representation is given by

$$(1 - T^*T)f(z) = \int_{\mathbb{D}} (1 - |w|^2)K^I(z, w)f(w)\,dA(w), \quad f \in I.$$

Therefore, we have the following estimate for the Hilbert-Schmidt norm.

$$\begin{aligned}\|1 - T^*T\|_2^2 &\leq \int_{\mathbb{D}}\int_{\mathbb{D}} (1 - |z|^2)^2|K^I(z, w)|^2\,dA(z)\,dA(w)\\ &= \int_{\mathbb{D}} (1 - |z|^2)^2\,dA(z) \int_{\mathbb{D}} |K^I(z, w)|^2 dA(w)\\ &= \int_{\mathbb{D}} (1 - |z|^2)^2 K^I(z, z)\,dA(u)\\ &= \int_{\mathbb{D}} R(z, z)\,dA(z) \leq 1.\end{aligned}$$

Next observe that

$$1 - TT^* = 1 - T^*T + [T^*,\ T]. \tag{3}$$

If index$(I) < \infty$, then by [13] the self-commutator $[T^*,\ T]$ is in the trace class, and in particular, it is Hilbert-Schmidt. Thus equation (3) shows that $1 - TT^*$ is Hilbert-Schmidt whenever $I$ has finite index.

If $1 - TT^*$ is Hilbert-Schmidt, then it is of course compact.

If $1 - TT^*$ is compact, then the fact that

$$(1 - TT^*)f = f, \qquad f \in I \ominus zI,$$

implies that $I \ominus zI$ is finite dimensional, or that $I$ has finite index. □

The above proof actually gives us upper and lower bounds for the Hilbert-Schmidt norm of $1 - TT^*$.

Corollary 8. *If $n$ is the index of $I$, then*

$$\sqrt{n} \leq \|1 - TT^*\|_2 \leq 1 + \sqrt{n}.$$

*Proof.* Since $0 \leq [T^*,\ T] \leq I$, all eigenvalues of $[T^*,\ T]$ are between 0 and 1, we have

$$\|[T^*,\ T]\|_2^2 \leq \mathrm{tr}[T^*,\ T].$$

It is shown in [13] that $\mathrm{tr}[T^*, T] = n$, and it is shown in the proof of Lemma 7 that $\|1 - T^*T\|_2 \leq 1$. So

$$\|1 - TT^*\|_2 \leq \|1 - T^*T\|_2 + \|[T^*,\ T]\|_2 \leq 1 + \sqrt{n}.$$

On the other hand, the operator $1 - TT^*$ fixes every vector in $I \ominus zI$, so

$$\|1 - TT^*\|_2^2 \geq \dim(I \ominus zI) = n,$$

completing the proof of the corollary. $\square$

We mention in passing that the operators $1 - TT^*$ and $1 - T^*T$ are never in the trace class. In fact, by calculating the Berezin transform and using Lemma 4, we see that the positive Toeplitz-type operator $1 - T^*T$ is in the trace class if and only if

$$\int_{\mathbb{D}} (1 - |z|^2) K^I(z,z)\, dA(z) < \infty.$$

To see that this never happens, consider $I_n = z^n I$ for $n \geq 1$. By Lemma 2,

$$\int_{\mathbb{D}} (1 - |z|^2) K^{I_n}(z,z)\, dA(z) \geq \int_{\mathbb{D}} |G_n(z)|^2\, dA(z) = 1, \tag{4}$$

where $G_n$ is the extremal function for $I_n$. The inclusion $I_n \subset I$ implies

$$K^{I_n}(z,z) \leq K^I(z,z), \qquad z \in \mathbb{D}.$$

Also, the inclusion $I_n \subset z^n A^2$ implies

$$\lim_{n\to\infty} K^{I_n}(z,z) = 0, \qquad z \in \mathbb{D};$$

see the proof of Proposition 12. So, if

$$\int_{\mathbb{D}} (1 - |z|^2) K^I(z,z)\, dA(z) < \infty,$$

then we can let $n \to \infty$ in (4) and apply the dominated convergence theorem to take the limit inside the first integral, which clearly results in a contradiction. This proves that the operator $1 - T^*T$ is never in the trace class. Since

$$1 - TT^* \geq 1 - T^*T,$$

the same is true for $1 - TT^*$.

In view of equation (1), the operator $C$ is smaller than $1 - TT^*$ in some sense, and so may well belong to the trace class. The next result tells us exactly when this happens.

Theorem 9. *The following conditions are equivalent.*

(i) *The index of $I$ is finite.*
(ii) *The root operator $C$ is in the trace class.*
(iii) *The root operator $C$ is compact.*

*Proof.* It is clear that (ii) implies (iii). That (iii) implies (i) follows from the fact that $I \ominus zI$ is the eigenspace of $C$ corresponding to the eigenvalue 1; see Proposition 3.

To show that (i) implies (ii), we observe that

$$C - (1 - TT^*)^2 = T^2 T^{*2} - TT^*TT^* = -T[T^*,\ T]T^*.$$

If the index of $I$ is finite, then by [13] the self-commutator $[T^*,\ T]$ is in the trace class. Since the trace class is actually an ideal in the algebra of all bounded linear operators on $I$, the operator $T[T^*,\ T]T^*$ belongs to the trace class as well. Also, by Lemma 7, the operator $1 - TT^*$ is Hilbert-Schmidt, so the operator $(1 - TT^*)^2$ is in the trace class. This proves that $C$ is in the trace class whenever the index of $I$ is finite. □

Combining Theorem 9 and Corollary 6, we see that if $I$ has finite index, then $0 < \mathrm{tr}(C) \leq 1$, and equality holds only when $I = A^2$.

For $0 < p < \infty$, let $\mathcal{S}_p$ denote the Schatten $p$-class of $I$. If $1 < p < \infty$, then $\mathcal{S}_p$ contains the trace class and is contained in the set of all compact operators. It follows that for any $p \in [1, \infty)$ the root operator $C$ belongs to $\mathcal{S}_p$ if and only if the index of $I$ is finite. In particular, $C$ is Hilbert-Schmidt if and only if the index of $I$ is finite. Combining this with Corollary 6, we see that the double integral

$$\int_{\mathbb{D}} \int_{\mathbb{D}} |R(z,w)|^2 \, dA(z)\, dA(w)$$

is finite if and only if the index of $I$ is finite. This demonstrates that the root function $R(z,w)$, as opposed to $K^I(z,w)$, has a much better boundary behavior.

The next result gives lower and upper bounds for the Hilbert-Schmidt norm of $C$, and so it also gives lower and upper bounds for the double integral in the previous paragraph.

COROLLARY 10. *If $I$ has finite index $n$, then*

$$\sqrt{n} \leq \|C\|_2 \leq 2(1 + \sqrt{n}).$$

*Proof.* By equation (1),

$$\|C\|_2 \leq \|1 - TT^*\|_2 + \|T(1 - TT^*)T^*\|_2.$$

Since

$$\|T(1 - TT^*)T^*\|_2 \leq \|T\|\|1 - TT^*\|_2\|T^*\| = \|1 - TT^*\|_2,$$

we obtain

$$\|C\|_2 \leq 2\|1 - TT^*\|_2.$$

This along with Corollary 8 shows that

$$\|C\|_2 \leq 2(1 + \sqrt{n}).$$

The inequality $n \leq \|C\|_2^2$ follows from the fact that $C$ fixes every function in $I \ominus zI$; see Proposition 3. □

## 5. The rank of the root operator

In this section we look at some examples of the root operator. We pay particular attention to the rank of $C$, which is the dimension of the range of $C$. First note that when $I = A^2$, the root operator becomes evaluation at the origin and so has rank 1.

PROPOSITION 11. *If $I \neq A^2$, then the rank of $C$ is at least $2n$, where $n$ is the index of $I$.*

*Proof.* The desired result is obvious when the rank of $C$ is infinite.

If $I \neq A^2$ and $C$ has finite rank, then by Corollary 6,

$$\mathrm{tr}(C) = \sum_n \lambda_n$$

is strictly less than 1, where $\{\lambda_n\}$ is the eigenvalue sequence of $C$, with each distinct eigenvalue repeated according to multiplicity. Since the eigenspace corresponding to the eigenvalue 1 is $I \ominus zI$ (see Proposition 3), the positive terms in the above series add up to at least $n$. Therefore, the negative terms must add up to something less than $1 - n$. But Proposition 3 also tells us that each negative eigenvalue of $C$ has absolute value strictly less than 1, so the trace formula above must contain at least $n$ negative terms. This shows that

$$\mathrm{rank}(C) \geq 2\ \mathrm{index}(I)$$

whenever $I \neq A^2$. $\square$

PROPOSITION 12. *If $I$ is the invariant subspace generated by $(z-a)^N$, where $a$ is a point in $\mathbb{D}$ and $N$ is a positive integer, then the root function of $I$ is*

$$R(z,w) = (N+1)\varphi_a(z)^N \overline{\varphi_a(w)}^N - N\varphi_a(z)^{N+1}\overline{\varphi_a(w)}^{N+1},$$

*where*

$$\varphi_a(z) = \frac{a-z}{1-\overline{a}z}.$$

*In particular, the root operator $C$ has rank 2. Moreover, the two eigenvalues of $C$ are 1 and*

$$\lambda = -1 + \int_{\mathbb{D}} |\varphi_a(z)|^{2N}(N+1-N|\varphi_a(z)|^2)\, dA(z).$$

*Proof.* If $a = 0$, then the reproducing kernel of $I$ is given by

$$\begin{aligned} K^I(z,w) &= \frac{1}{(1-z\overline{w})^2} - \sum_{k=0}^{N-1}(k+1)z^k\overline{w}^k \\ &= \frac{z^N\overline{w}^N}{(1-z\overline{w})^2}(N+1-Nz\overline{w}). \end{aligned}$$

In general, we use the above formula and the Möbius invariance of the reproducing kernel to obtain

$$K^I(z,w) = \frac{\varphi_a(z)^N \overline{\varphi_a(w)}^N}{(1-z\overline{w})^2}\big(N+1-N\varphi_a(z)\overline{\varphi_a(w)}\big),$$

where

$$\varphi_a(z) = \frac{a-z}{1-\overline{a}z}, \qquad z \in \mathbb{D}.$$

It follows that the root function for $I$ is

$$R(z,w) = (N+1)\varphi_a(z)^N\overline{\varphi_a(w)}^N - N\varphi_a(z)^{N+1}\overline{\varphi_a(w)}^{N+1},$$

and hence the root operator $C$ has rank two.

Let $\lambda$ be the other eigenvalue of $C$. Since $C$ is self-adjoint, we have

$$1+\lambda = \operatorname{tr}(C) = \int_{\mathbb{D}} R(z,z)\, dA(z).$$

It follows that

$$\lambda = -1 + \int_{\mathbb{D}} |\varphi_a(z)|^{2N}(N+1-N|\varphi_a(z)|^2)\, dA(z). \qquad \square$$

If $a=0$, the negative eigenvalue of $C$ above can be simplified to

$$\lambda = -N/(N+2),$$

and the corresponding eigenfunctions are just constant multiples of $z^{N+1}$.

If $a \neq 0$, the space $I \ominus zI$ is spanned by the function $K^I(z,0)$, so the eigenvectors of $C$ corresponding to the eigenvalue 1 are of the form

$$f(z) = c\varphi_a(z)^N\left(1 + N\frac{1-|a|^2}{1-\overline{a}z}\right),$$

where $c$ is any nonzero constant. To obtain the eigenvectors of $C$ corresponding to the other eigenvalue $\lambda$, note that the range of $C$ consists of all linear combinations of $\varphi_a^N$ and $\varphi_a^{N+1}$. Also note that eigenvectors of a self-adjoint operator corresponding to distinct eigenvalues are perpendicular. So, if a function

$$f(z) = c_1\varphi_a^N(z) + c_2\varphi_a^{N+1}(z)$$

is an eigenvector of $C$ with respect to $\lambda$, then $f$ must be perpendicular to $K^I(z,0)$. It then follows easily that the eigenvectors of $C$ corresponding to the negative eigenvalue $\lambda$ are of the form

$$f(z) = c\frac{z}{1-\overline{a}z}\varphi_a^N(z),$$

where $c$ is any nonzero constant.

When $N=1$ and $a \neq 0$, the negative eigenvalue of $C$ is

$$\lambda = -(1-|a|^2)^2\int_{\mathbb{D}} \frac{(1-|z|^2)^2}{|1-\overline{a}z|^4}\, dA(z).$$

A calculation using Taylor expansion gives

$$\lambda = -\frac{2(1-|a|^2)^2}{|a|^6}\left[(2-|a|^2)\log\frac{1}{1-|a|^2} - 2|a|^2\right].$$

We also observe that if $b = e^{i\theta}a$, then the root operator for $(z-b)A^2$ is unitarily equivalent to the root operator for $(z-a)A^2$, although the restrictions of the Bergman shift to the respective invariant subspaces are not unitarily equivalent.

We now consider the case where $I$ has finite codimension in $A^2$, that is,

$$N = \dim(A^2 \ominus I) < \infty.$$

Let $\{e_1, \cdots, e_N\}$ be an orthonormal basis for $A^2 \ominus I$. Then

$$K^I(z,w) = K(z,w) - \sum_{k=1}^N e_k(z)\overline{e_k(w)},$$

and so the root function is given by

$$R(z,w) = 1 - (1 - z\overline{w})^2 \sum_{k=1}^N e_k(z)\overline{e_k(w)}.$$

It follows that the root operator is given by

$$Cf(z) = f(0) + \sum_{k=1}^N \left[2ze_k(z)\langle f, Be_k\rangle - z^2 e_k(z)\langle f, B^2 e_k\rangle\right],$$

where $B$ is the Bergman shift on $A^2$. If $Q$ is the orthogonal projection from $A^2$ onto $I$, then projecting the above result to $I$ leads to

$$Cf = K^I(0,0)\langle f, G\rangle G + 2\sum_{k=1}^N \langle f, \varphi_k\rangle\varphi_k + \sum_{k=1}^N \langle f, \psi_k\rangle\psi_k,$$

where $G$ is the extremal function for $I$ and

$$\varphi_k = Q(Be_k), \quad \psi_k = Q(B^2 e_k), \qquad 1 \le k \le N.$$

This shows that $C$ is a finite rank operator, and its rank is less than or equal to $2N+1$. The following result improves this upper bound a little bit.

Proposition 13. *Suppose*

$$N = \dim(A^2 \ominus I) < \infty.$$

*Then the rank of $C$ is at most $2N$.*

*Proof.* We prove the result by induction on $N$. It is well known that $I$ has finite codimension in $A^2$ if and only if it is generated by a finite Blaschke product $B(z)$, and the codimension of $I$ in $A^2$ is equal to the number of zeros of $B(z)$, counting multiplicity. So the case $N = 1$ follows from Proposition 12.

For any fixed $N$, we assume $I \subset J$ are two invariant subspaces with

$$\dim(A^2 \ominus J) = N, \quad \dim(A^2 \ominus I) = N + 1,$$

so that

$$\dim(J \ominus I) = 1.$$

If $e$ is a unit vector in $J \ominus I$, then we can write

$$K^I(z, w) = K^J(z, w) - e(z)\overline{e(w)},$$

and hence

$$R^I(z, w) = R^J(z, w) - (1 - 2z\overline{w} + z^2\overline{w}^2)e(z)\overline{e(w)}.$$

It then follows that for every $f \in I \subset J$

$$\begin{aligned} C^I f(z) &= C^J f(z) - \int_{\mathbb{D}} (1 - 2z\overline{w} + z^2\overline{w}^2)e(z)\overline{e(w)}f(w)\, dA(w) \\ &= C^J f(z) + 2ze(z)\langle f, Be\rangle - z^2 e(z)\langle f, B^2 e\rangle, \end{aligned}$$

where $B$ is the Bergman shift. If we write $\phi = Q(Be)$ and $\psi = Q(B^2 e)$, where $Q$ is the orthogonal projection from $A^2$ onto $I$, then

$$C^I f = QC^J f + 2\phi\langle f, \phi\rangle - \psi\langle f, \psi\rangle, \quad f \in I.$$

This shows that if the rank of $C^J$ is at most $2N$, then the rank of $C^I$ is at most $2N + 2 = 2(N + 1)$. The desired result is thus proved by induction. □

Two natural questions arise here. First, does the root operator have rank two whenever the invariant subspace $I$ is generated by a finite Blaschke product? We know the answer is yes when there is only one zero with any multiplicity; we do not know what happens in general, because the reproducing kernel is not readily computable. Second, it is tempting to conjecture that the operator $C$ has finite rank if and only if the invariant subspace $I$ is generated by a finite Blaschke product. It turns out that this is not the case. In fact, if

$$S(z) = \exp\left(-\frac{1+z}{1-z}\right)$$

and $I = I_S$, then by [10]

$$K^I(z, w) = \frac{S(z)\overline{S(w)}}{(1 - z\overline{w})^2}\left(1 + \frac{1+z}{1-z} + \frac{1+\overline{w}}{1-\overline{w}}\right),$$

and so

$$R(z, w) = \frac{2S(z)}{1-z}\overline{S(w)} + S(z)\overline{S(w)}\frac{1+\overline{w}}{1-\overline{w}}.$$

This clearly shows that the corresponding root operator $C$ has rank two.

It will be interesting to obtain a characterization of the invariant subspaces such that the associated root operator has finite rank. We do not even have a reasonable guess.

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