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WHEN DO MCSHANE AND PETTIS INTEGRALS COINCIDE?

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ABSTRACT. We give a partial answer to the question in the title by showing that the McShane and Pettis integrals coincide for functions with values in super-reflexive spaces as well as for functions with values in $c_0(\Gamma)$. We also improve an example of Fremlin and Mendoza, according to which these integrals do not coincide in general, by showing that, at least under the Continuum Hypothesis, there is a scalarly negligible function which is not McShane integrable.

1. Introduction

We continue the investigation of relations between McShane and Pettis integrals of vector-valued functions defined on $[0,1]$, started in [6]; this investigation was extended in [5] to functions defined on "outer regular quasi-Radon measure spaces". These authors show, among other results, that every McShane integrable function is Pettis integrable, but that there are Pettis integrable functions that are not McShane integrable.

Here we address the question whether the above difference of Pettis and McShane integrals depends on the target Banach space. More precisely, we show that the integrals coincide for functions with values in super-reflexive Banach spaces as well as for functions with values in $c_0(\Gamma)$, where Γ is any set. The proof is based on Edgar's [2] reduction to the case of scalarly negligible functions and the technique of projectional resolutions of identity of Lindenstrauss [9][10]. In this connection we note that the present examples do not exclude the possibility that Pettis and McShane integrals coincide for scalarly negligible functions with values in arbitrary Banach spaces. This problem was also mentioned in another connection by K. Musial at a workshop on measure theory in Gorizia (1999). We answer this question by showing that, at least under the Continuum Hypothesis, there is a scalarly negligible function with values in $l_\infty(\omega_1)$ which is not McShane integrable.

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The main new feature of this note is that the coincidence of the integrals is shown in situations when the range of the functions in question need not be essentially separable. Previous results, such as those of [11], attempted to give conditions on a Banach space implying that the range of every Pettis integrable function is essentially separable, which then gives its Bochner measurability and so McShane integrability. Such results cannot apply to spaces like $c_0([0,1])$ or $\ell_2([0,1])$ for which it is immediate to construct even a scalarly negligible function whose range is not essentially separable. (In fact, for reflexive spaces the additional condition imposed in [11] is equivalent to separability.)

Although our method is applicable to any of a large class of gauge integrals (see [8] or [5] for a general approach to such integrals), we work in the simpler case of functions defined on $[0,1]$ only. The general approach would present only non-interesting technical modifications and so obscure the main ideas.

We also note that in all our considerations the McShane integral may be replaced by the Henstock integral, since for Pettis integrable functions these integrals coincide (see [4]).

For the purposes of this note, super-reflexivity of a Banach space may be defined as existence of an equivalent uniformly convex norm. (See [1] for more on super-reflexive spaces.) Recall that the norm is uniformly convex if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\|x+y\| < 2-\delta$ whenever $\|x\|, \|y\| \leq 1$ and $\|x-y\| > \varepsilon$.

An important part of our argument uses the fact that super-reflexive spaces admit long sequences of projections (see below for the definition). Since this property is shared by all weakly compactly generated spaces, one may conjecture that our statement remains valid for all such spaces. However, another part of the argument needs uniform convexity and it is not clear how our proof could be adapted to a more general situation.

2. Notations and preliminaries

Throughout this note $[0,1]$ is the closed unit interval of the real line endowed with the Euclidean topology and the Lebesgue measure. Unless specified otherwise, the terms "measure", "measurable" and "almost everywhere" refer to the Lebesgue measure. For a set $E \subset [0,1]$, we denote by $\mu(E)$ its outer Lebesgue measure. A *McShane partition* (in $[0,1]$) is a finite set of pairs $P = \{(E_i, t_i), i \in I\}$, where $(E_i)_i$ is a family of non-overlapping subintervals of $[0,1]$ and $t_i \in [0,1]$ for each $i \in I$. If $\bigcup_{i \in I} E_i = [0,1]$, we say that P is a *McShane partition of* $[0,1]$.

A *gauge* on $[0,1]$ is a positive function δ on $[0,1]$. We say that a McShane partition $\{(E_i, t_i), i \in I\}$ is *subordinate to a gauge* δ if $E_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for $i \in I$.

From now on X is a real Banach space with dual X^*. A function $f : [0,1] \to X$ is said to be *Pettis integrable* on $[0,1]$ if $x^* \circ f$ is Lebesgue integrable on $[0,1]$ for each $x^* \in X^*$ and for every measurable set $E \subset [0,1]$ there is a vector $\nu(E) \in X$ such that $x^*(\nu(E)) = \int_E x^* \circ f(t)\, dt$ for all $x^* \in X^*$.

A function $f : [0,1] \to X$ is said to be *McShane integrable* (see [5, Definition 1A]) with McShane integral $z \in X$ if for each $\varepsilon > 0$ there exists a gauge δ in $[0,1]$ such that

$$\left\| \sum_{i \in I} \mu(E_i) f(t_i) - z \right\| < \varepsilon$$

for each McShane partition $\{(E_i, t_i) : i \in I\}$ of $[0,1]$ subordinate to δ.

A function $f : [0,1] \to X$ is said to be *scalarly negligible* if, for each $x^* \in X^*$, $x^* f(t) = 0$ almost everywhere on $[0,1]$. We observe that each scalarly negligible function is Pettis integrable and its Pettis integral over every measurable set is the null vector.

LEMMA 1. *A function $f : [0,1] \to X$ is scalarly negligible and McShane integrable if and only if for every $\varepsilon > 0$ there is a gauge δ such that*

$$\left\| \sum_{i \in I} f(t_i) \mu(E_i) \right\| < \varepsilon$$

for every McShane partition $\{(E_i, t_i) : i \in I\}$ subordinated to δ.

Proof. If f is scalarly negligible and McShane integrable, this is a variant of the Henstock lemma (see [5]). Conversely, if the condition holds, then the McShane integral and therefore the Lebesgue integral of every $x^* \circ f$ is zero over any interval, hence f is scalarly negligible. □

LEMMA 2. (1) *If f is scalarly negligible and McShane integrable, then for every set $T \subset [0,1]$ the function $f\chi_T$ (where χ_T denotes the characteristic function of the set T) is also scalarly negligible and McShane integrable.*

(2) *If $f = \lim_{j \to \infty} f_j$, where f_j are scalarly negligible McShane integrable functions, then f is scalarly negligible and McShane integrable.*

Proof. The first statement is an immediate consequence of Lemma 1. For the second statement, let $\varepsilon > 0$ and choose gauges δ_j such that

$$\left\| \sum_{i \in I} f_j(t_i) \mu(E_i) \right\| < 2^{-j-1} \varepsilon$$

for every McShane partition $\{(E_i, t_i) : i \in I\}$ subordinated to δ_j. For each t choose $j(t)$ so that $\|f(t) - f_{j(t)}(t)\| < \varepsilon/2$ and define $\delta(t) := \min\{\delta_j(t) : 1 \le j \le j(t)\}$. Then, whenever $\{(E_i, t_i) : i \in I\}$ is a McShane partition subordinated to δ, denote $I_j := \{i \in I : j(t_i) = j\}$ and estimate

$$\begin{aligned}\Big\|\sum_{i\in I} f(t_i)\mu(E_i)\Big\| &\le \sum_{j=1}^{\infty}\Big\|\sum_{i\in I_j} f_j(t_i)\mu(E_i)\Big\| \\ &\qquad + \Big\|\sum_{i\in I}(f(t_i) - f_{j(t_i)}(t_i))\mu(E_i)\Big\| \\ &< \varepsilon.\end{aligned}$$

Hence f is McShane integrable, and it is obviously also scalarly negligible. $\square$

We now recall several notions useful to study nonseparable Banach spaces. The *density* of a Banach space X is the least cardinality of a dense subset of X; it is denoted by $\operatorname{dens}(X)$.

A *long sequence of projections* on a Banach space X is a collection $\{P_\alpha\}$ of projections of X into X indexed by ordinals $\gamma \le \alpha \le \kappa$ (where $\gamma < \kappa$ are any given ordinals) satisfying the following four conditions:

(p_1) $\|P_\alpha\| = 1$ for every $\gamma \le \alpha \le \kappa$;
(p_2) P_κ is the identity on X;
(p_3) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$, if $\gamma \le \alpha \le \beta \le \kappa$;
(p_4) $\bigcup\{P_{\beta+1}(X) : \gamma \le \beta < \alpha\}$ is norm-dense in $P_\alpha(X)$, if $\gamma < \alpha \le \kappa$.

Finally, the long sequence of projections $\{P_\alpha : \omega_0 \le \alpha \le \kappa\}$ is said to be a *projectional resolution of identity* on X if $\kappa = \operatorname{dens}(X)$, where, as usual, we identify cardinals with initial ordinals; and

(p_5) $\operatorname{dens}(P_\alpha(X)) \le \operatorname{card}(\alpha)$ for every $\omega_0 \le \alpha \le \kappa$.

The consequence of the condition (p_1) that $\|P_{\alpha+1} - P_\alpha\| \le 2$ will be used throughout without any further reference. We should also remark that (p_1) could be replaced without any significant change to our results or their proofs by the condition $\sup_\alpha \|P_\alpha\| < \infty$. In the sequel we will need the following simple observation.

LEMMA 3. *Whenever $\{P_\alpha : \gamma \le \alpha \le \kappa\}$ is a long sequence of projections and $x \in X$ is such that $P_\gamma(x) = 0$, then for every $\gamma \le \xi \le \kappa$ and $\varepsilon > 0$ there is a (possibly empty) finite set $A \subset \{\alpha : \gamma \le \alpha < \xi\}$ such that $\|P_\xi(x) - \sum_{\alpha\in A}(P_{\alpha+1} - P_\alpha)(x)\| < \varepsilon$.*

Proof. This is easy to see by transfinite induction with respect to ξ: For $\xi = \gamma$ we take $A = \emptyset$ (and use the convention that an empty sum is zero). For $\xi = \zeta + 1$ we find $B \subset \{\alpha : \gamma \le \alpha < \zeta\}$ such that $\|P_\zeta(x) - \sum_{\alpha\in B}(P_{\alpha+1} - P_\alpha)(x)\| < \varepsilon$, let $A = B \cup \{\zeta\}$ and note that $\|P_\xi(x) - \sum_{\alpha\in A}(P_{\alpha+1} - P_\alpha)(x)\| = \|P_\zeta(x) - \sum_{\alpha\in B}(P_{\alpha+1} - P_\alpha)(x)\| < \varepsilon$. For a limit ordinal ξ we first use (p_4) to find $\zeta < \xi$ such that $\|P_\xi(x) - P_\zeta(x)\| < \varepsilon/2$ and then $A \subset \{\alpha :$

$\gamma \leq \alpha < \zeta\}$ such that $\|P_\zeta(x) - \sum_{\alpha\in A}(P_{\alpha+1} - P_\alpha)(x)\| < \varepsilon/2$; then clearly $\|P_\xi(x) - \sum_{\alpha\in A}(P_{\alpha+1} - P_\alpha)(x)\| < \varepsilon$. □

LEMMA 4. *Let $\{P_\alpha : \gamma \leq \alpha \leq \kappa\}$ be a long sequence of projections on a Banach space X satisfying the following condition.*

($\star$) *For every $\varepsilon > 0$ there is $\eta > 0$ with the following property: Whenever $x_\alpha \in X$ are vectors with $\|x_\alpha\| \leq 1$ and $0 \leq c_\alpha \leq \eta$ are such that $\sum_{\gamma\leq\alpha<\kappa} c_\alpha \leq 1$, then $\|\sum_{\gamma\leq\alpha<\kappa} c_\alpha\,(P_{\alpha+1} - P_\alpha)(x_\alpha)\| < \varepsilon$.*

Then every function $f : [0,1] \to X$ such that the functions $P_\alpha \circ f$ are scalarly negligible and McShane integrable for every $\gamma \leq \alpha < \kappa$ is scalarly negligible and McShane integrable.

Proof. Suppose first that $P_\gamma \circ f = 0$ and $\|f(t)\| \leq 1$ for every $t \in [0,1]$ and denote $h_\alpha(t) = (P_{\alpha+1} - P_\alpha)(f(t))$. Given $\varepsilon > 0$, find $\eta > 0$ with the property from ($\star$) and use Lemma 3 to choose for each $t \in [0,1]$ finite sets of ordinals A_t such that $\|f(t) - g(t)\| < \varepsilon$, where $g(t) = \sum_{\alpha\in A_t} h_\alpha(t)$. For each $\gamma \leq \alpha < \kappa$ use the McShane integrability and scalar negligibility of h_α together with Lemma 1 to find a gauge δ_α such that

$$\left\|\sum_{i\in I} h_\alpha(t_i)\mu(E_i)\right\| < \varepsilon\eta$$

for every McShane partition $\{(E_i, t_i) : i \in I\}$ subordinated to δ_α.

Let $\delta(t) := \min_{\alpha\in A_t} \delta_\alpha(t)$. (The value of $\delta(t)$ may be chosen arbitrarily if $A_t = \emptyset$.) Suppose that $\{(E_i, t_i) : i \in I\}$ is a McShane partition in $[0,1]$ subordinated to δ. Denote $I_\alpha := \{i \in I : \alpha \in A_{t_i}\}$ and let A be the set of those ordinals α for which $I_\alpha \neq \emptyset$. For $\alpha \in A$ denote $c_\alpha := \sum_{i\in I_\alpha} \mu(E_i)$ and $x_\alpha := (1/c_\alpha)\sum_{i\in I_\alpha} f(t_i)\mu(E_i)$. Denote also $A_0 := \{\alpha \in A : c_\alpha < \eta\}$ and $A_1 := A \setminus A_0$. Using this notation we write

$$\begin{aligned}\sum_{i\in I} g(t_i)\mu(E_i) &= \sum_{i\in I}\sum_{\alpha\in A_{t_i}} h_\alpha(t_i)\mu(E_i)\\ &= \sum_{\alpha\in A_0} c_\alpha\,(P_{\alpha+1} - P_\alpha)(x_\alpha) + \sum_{\alpha\in A_1}\sum_{i\in I_\alpha} h_\alpha(t_i)\mu(E_i).\end{aligned}$$

To estimate the first sum, we note that $\|x_\alpha\| \leq 1$, $\sum_{\alpha\in A_0} c_\alpha \leq 1$ and $c_\alpha < \eta$ for $\alpha \in A_0$. Hence by ($\star$), $\|\sum_{\alpha\in A_0} c_\alpha\,(P_{\alpha+1} - P_\alpha)(x_\alpha)\| < \varepsilon$.

To estimate the second sum, we use that $\sum_{\alpha\in A_1} c_\alpha \leq 1$ and $c_\alpha \geq \eta$ for $\alpha \in A_1$ to infer that the set A_1 has at most $1/\eta$ elements. Moreover, whenever $i \in I_\alpha$ then $\alpha \in A_{t_i}$ and hence $\delta(t_i) \leq \delta_\alpha(t_i)$. This shows that $\{(E_i, t_i) : i \in I_\alpha\}$ is a McShane partition in $[0,1]$ subordinated to δ_α. Consequently,

$\| \sum_{i \in I_\alpha} h_\alpha(t_i)\mu(E_i)\| < \varepsilon\eta$ and so

$$\left\| \sum_{\alpha \in A_1} \sum_{i \in I_\alpha} h_\alpha(t_i)\mu(E_i) \right\| < \varepsilon.$$

Combining these estimates we get $\| \sum_{i \in I} g(t_i)\mu(E_i)\| < 2\varepsilon$, and finally

$$\left\| \sum_{i \in I} f(t_i)\mu(E_i) \right\| \le \left\| \sum_{i \in I} (f-g)(t_i)\mu(E_i) \right\| + \left\| \sum_{i \in I} g(t_i)\mu(E_i) \right\| < 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we infer from Lemma 1 that f is McShane integrable and scalarly negligible.

To prove the general case, we define $f_j(t) = (f(t) - P_\gamma(f(t)))/j$ if $\|f(t) - P_\gamma(f(t))\| \le j$ and $f_j(t) = 0$ for all other t and observe that by the first statement of Lemma 2 the functions f_j satisfy the assumptions of the above special case, and hence are scalarly negligible and McShane integrable. By the second statement of Lemma 2, the function $f - P_\gamma \circ f = \lim_{j\to\infty} j f_j$ is also McShane integrable. Since $P_\gamma \circ f$ is assumed to be McShane integrable, f can be written as a sum of two McShane integrable functions and so it is McShane integrable. A simpler version of the same argument shows that f is also scalarly negligible. □

Corollary 1. *Let $\mathcal{P}$ be the class of Banach spaces X having the property that every scalarly negligible X-valued function is McShane integrable. Then*

(a) *all separable spaces belong to $\mathcal{P}$;*
(b) *every Banach space X admitting a long sequence of projections $\{P_\alpha : \gamma \le \alpha \le \kappa\}$ satisfying the condition $(\star)$ of Lemma 4 and such that $P_\alpha(X)$ belong to $\mathcal{P}$ for all $\gamma \le \alpha < \kappa$ belongs to $\mathcal{P}$.*

Proof. The statement (a) is obvious since Pettis and McShane integrals of functions with values in separable spaces coincide by Corollary 4C of [5]. The statement (b) follows from Lemma 4 and the observation that $P \circ f$ is scalarly negligible whenever $f : [0,1] \to X$ is scalarly negligible and $P : X \to X$ is a projection. □

Lemma 5. *Let $\{P_\alpha : \gamma \le \alpha \le \kappa\}$ be a long sequence of projections on X and let d_n be the least constant such that, whenever $\gamma \le \alpha_0 \le \alpha_1 \le \cdots \le \alpha_n \le \kappa$, $x \in (P_{\alpha_n} - P_{\alpha_0})(X)$ and $\|(P_{\alpha_k} - P_{\alpha_{k-1}})(x)\| \le 1$ for each $k = 1, \ldots, n$, then $\|x\| \le d_n$. If $d_2 < 2$ then*

(1) $\lim_n d_n/n = 0$ *and*
(2) *the condition $(\star)$ of Lemma 4 holds.*

Proof. (1) Let $\gamma \le \alpha_0 \le \alpha_1 \le \cdots \le \alpha_{2n} \le \kappa$, $x \in (P_{\alpha_n} - P_{\alpha_0})(X)$ and $\|(P_{\alpha_k} - P_{\alpha_{k-1}})(x)\| \le 1$. We denote $\tilde{\alpha}_k = \alpha_{2k}$ and observe that, by the definition of d_2, $\|(P_{\tilde{\alpha}_k} - P_{\tilde{\alpha}_{k-1}})(x)\| \le d_2$. So, letting $\tilde{x} = x/d_2$, we have

$\|(P_{\tilde{\alpha}_k} - P_{\tilde{\alpha}_{k-1}})(\tilde{x})\| \le 1$ and hence $\|x\| = d_2\|\tilde{x}\| \le d_2 d_n$. Hence $d_{2n} \le d_2 d_n$. So, for any $2^{m-1} < n \le 2^m$, $d_n \le d_{2^m} \le d_2^m \le 2(d_2/2)^m n$.

(2) Let $\varepsilon > 0$ and choose $n > 1 + 6/\varepsilon$ so that $6d_n/(n-1) < \varepsilon$. We show that the condition $(\star)$ holds with $\eta = 1/(n-1)$. Suppose that $x_\alpha \in X$ are vectors with $\|x_\alpha\| \le 1$ and $0 \le c_\alpha \le \eta$ are such that $\sum_{\gamma \le \alpha < \kappa} c_\alpha \le 1$. We will also assume that $\sum_{\gamma \le \alpha < \kappa} c_\alpha > 3\eta$, since otherwise

$$\left\| \sum_{\gamma \le \alpha < \kappa} c_\alpha \, (P_{\alpha+1} - P_\alpha)(x_\alpha) \right\| \le 2 \sum_{\gamma \le \alpha < \kappa} c_\alpha \le 6\eta < \varepsilon.$$

Denote $\alpha_0 = \gamma$ and define α_k recursively as follows: If $\sum_{\alpha_{k-1} \le \alpha < \kappa} c_\alpha \le 3\eta$, let $\alpha_k = \kappa$ and stop the construction; otherwise choose the least ordinal α_k such that $\sum_{\alpha_{k-1} \le \alpha < \alpha_k} c_\alpha \ge \eta$. Then $(k-1)\eta \le \sum_{\alpha_0 \le \alpha < \alpha_k} c_\alpha \le 3\eta k$ for every k for which α_k is defined; consequently there is a last index, say m, for which α_m is defined and $2 \le m \le 1 + 1/\eta = n$. Using the definition of d_m with $x = (1/(6\eta)) \sum_{\gamma \le \alpha < \kappa} c_\alpha \, (P_{\alpha+1} - P_\alpha)(x_\alpha)$ and the above α_k, we get

$$\left\| \sum_{\gamma \le \alpha < \kappa} c_\alpha \, (P_{\alpha+1} - P_\alpha)(x_\alpha) \right\| \le 6\eta d_m \le 6\eta d_n = 6d_n/(n-1) < \varepsilon,$$

as required by the condition $(\star)$. □

Theorem 1. *If X is a super-reflexive Banach space or if $X = c_0(\Gamma)$ for some set Γ, then the Pettis and McShane integrals of X-valued functions coincide.*

Proof. We first show by transfinite induction with respect to $\operatorname{dens}(X)$ that super-reflexive spaces belong to the class $\mathcal{P}$ of Corollary 1. Hence we suppose that X is nonseparable, has uniformly convex norm and every super-reflexive space with density less than the density of X belongs to $\mathcal{P}$ and we show that X belongs to $\mathcal{P}$ as well. Choose any projectional resolution of identity $\{P_\alpha : \omega_0 \le \alpha \le \kappa\}$, where κ is the least ordinal of cardinality $\operatorname{dens}(X)$. (For its existence see [9], [10].) Then, for $\alpha < \kappa$, $P_\alpha(X)$ are super-reflexive spaces of smaller density, so they belong to $\mathcal{P}$. To prove the condition $(\star)$ of Lemma 4, we first estimate d_2 from Lemma 5. Let $\omega_0 \le \alpha_0 \le \alpha_1 \le \alpha_2 \le \kappa$ and $x \in (P_{\alpha_2} - P_{\alpha_0})(X)$ be such that the vectors $u = (P_{\alpha_2} - P_{\alpha_1})(x)$ and $v = (P_{\alpha_1} - P_{\alpha_0})(x)$ satisfy $\|u\|, \|v\| \le 1$. By the uniform convexity of the norm of X there is $3/2 < d < 2$ (independent of x) such that either $\|u - v\| \le 1/2$ or $\|u + v\| < d$. Hence $\|u + v\| < d$ since otherwise

$$\|u + v\| \le \|u\| + \|v\| = \|P_{\alpha_1}(u - v)\| + \|v\| \le \|u - v\| + \|v\| \le 3/2 < d.$$

Since $x = u + v$, $\|x\| < d$ and consequently $d_2 \le d$. Now we get condition $(\star)$ of Lemma 4 from Lemma 5. Hence, by Corollary 1, X belongs to $\mathcal{P}$.

We now prove by transfinite induction with respect to cardinal numbers m that the spaces $c_0(m)$ belong to $\mathcal{P}$. For this suppose that $c_0(\tilde{m})$ belongs to $\mathcal{P}$ for each $\tilde{m} < m$. Let κ be the least ordinal of cardinality m. Let P_α be the natural projections of $c_0(\kappa)$ onto $\{x \in c_0(\kappa) : x_\beta = 0 \text{ for } \beta \geq \alpha\}$. Then $\{P_\alpha : 1 \leq \alpha \leq \kappa\}$ is a long sequence of projections satisfying $(\star)$ of Lemma 4. Moreover, for $\alpha < \kappa$, $P_\alpha(X)$ belong to $\mathcal{P}$. By Corollary 1, X belongs to $\mathcal{P}$.

We now note that the spaces in question have the property that every Pettis integrable function is scalarly equivalent to a Bochner (or strongly) measurable function. (In fact, they are weakly compactly generated, so they are Lindelöf in their weak topology, and therefore measure compact. Hence scalarly measurable functions are scalarly equivalent to Bochner measurable functions. See [12, Theorem 2-7-2, Theorem 3-4-6] or [2], [3].) Since Pettis integrable Bochner measurable functions are McShane integrable (see [7]), we infer that Pettis integrable functions with values in these spaces are McShane integrable.

The converse that every McShane integrable function is Pettis integrable holds in every Banach space; see [5, Theorem 1Q]. □

Example (CH). There exists a function $f : [0,1] \to X$, where X is a Banach space, satisfying the following properties:

(i_1) f is scalarly negligible;
(i_2) f is Pettis-integrable;
(i_3) f is not McShane-integrable.

We construct the example with $X = l_\infty(\omega_1)$, where ω_1 is the first uncountable ordinal. Let $\{N_\alpha\}_{\alpha\in\omega_1}$ and $\{C_\alpha\}_{\alpha\in\omega_1}$ be two collections of subsets of $[0,1]$ satisfying the following properties:

(j_1) for each $\alpha \in \omega_1$, N_α is a set of zero Lebesgue measure;
(j_2) $N_\alpha \subset N_\beta$, if $\alpha < \beta$;
(j_3) every subset of $[0,1]$ of zero Lebesgue measure is contained in some set N_α;
(j_4) for each $\alpha \in \omega_1$, C_α is a countable set;
(j_5) $C_\alpha \subset C_\beta$, if $\alpha < \beta$;
(j_6) every countable subset of $[0,1]$ is contained in some set C_α.

Now define $f : [0,1] \to l_\infty(\omega_1)$ by

$$f(t)(\alpha) = \begin{cases} 1 & \text{if } t \in N_\alpha \setminus C_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Set $g(t) := \{\alpha \in \omega_1 : f(t)(\alpha) = 1\}$ and note that for each $t \in [0,1]$ there is $\beta < \omega_1$ such that $t \in C_\beta$. Then for each $\alpha \geq \beta$, $t \in C_\alpha$, hence $t \notin N_\alpha \setminus C_\alpha$, and consequently $f(t)(\alpha) = 0$. Therefore $g(t) \subset \{\alpha : \alpha < \beta\}$, which shows:

Fact. *For each $t \in [0,1]$, the set $g(t)$ is countable.*

We now prove property (i_1). For this, it is sufficient to fix an arbitrary $x^* \in (l_\infty(\omega_1))^*$ and $\varepsilon > 0$ and to prove that the set

$$T := \{t \in [0,1] : x^* f(t) > \varepsilon\}$$

has zero measure.

We assume that $\mu(T) > 0$ and recursively choose points $t_1, t_2, \dots$ of T. We take an arbitrary $t_1 \in T$ and, whenever $t_1, \dots, t_j$ have been defined, we use that the sets $g(t_i)$ are countable and that the sets N_α have measure zero to infer that $\mu\left(T \setminus \bigcup_{\alpha \in g(t_1)\cup\cdots\cup g(t_j)} N_\alpha\right) > 0$. Hence we may choose a point $t_{j+1} \in T \setminus \bigcup_{\alpha \in g(t_1)\cup\cdots\cup g(t_j)} N_\alpha$.

For $i < j$ the sets $g(t_i)$ and $g(t_j)$ are disjoint. In fact, assume that $\alpha \in g(t_i)$. Then $t_i \in N_\alpha \setminus C_\alpha$, hence by the choice of t_j we have that $t_j \notin N_\alpha$, which, by the definition of $g(t_j)$, shows that $\alpha \notin g(t_j)$.

We use this pairwise disjointness of the sets $g(t_j)$ to infer that $\|\chi_{g(t_1)} + \cdots + \chi_{g(t_n)}\| \le 1$ for every n, where the symbol χ_A denotes the characteristic function of the set A. In particular, with $n > \|x^*\|/\varepsilon$ we get

$$\|x^*\| \ge x^*(\chi_{g(t_1)} + \cdots + \chi_{g(t_n)}) = x^*(f(t_1) + \cdots + f(t_n)) > n\varepsilon > \|x^*\|,$$

which is the desired contradiction proving property (i_1).

From (i_1) it follows that f is Pettis integrable and its indefinite Pettis integral is identically equal to the null vector in $l_\infty(\omega_1)$ and we have (i_2).

It remains to prove (i_3). We say that a set $S \subset [0,1]$ satisfies property (P) if for each open subinterval $J \subset [0,1]$, either $S \cap J = \emptyset$ or $S \cap J$ is uncountable.

Now for $A \subset [0,1]$, we set

$$A^* := \bigcup \{S \subset A : \mu(S) = 0 \text{ and } S \text{ satisfies property } (P)\}.$$

Lemma 6. *Let $\{A_n\}$ be a sequence of subsets of $[0,1]$. If $A_n \uparrow [0,1]$, then $\mu(\overline{A_n^*}) \to 1$.*

Proof. If not, then $\mu(\bigcup_n \overline{A_n^*}) < 1$. Therefore there is an uncountable null set $N \subset [0,1] \setminus (\bigcup_n \overline{A_n^*})$. If for some n the set $A_n \bigcap N$ were uncountable, we would infer from the Cantor-Bendixson Theorem that it contains a nonempty subset with property (P), which is impossible since $A_n^* \cap N = \emptyset$. Hence the uncountable set N is contained in a countable union $\bigcup_n (A_n \cap N)$ of countable sets, which is a contradiction. $\square$

We show that f is not McShane integrable. Fix $1 > \varepsilon > 0$ and let δ be any gauge on $[0,1]$. For a fixed positive integer n we set $E_n = \{t \in [0,1] : \delta(t) > 1/n\}$, and according to the previous lemma, let n_o be an integer such that $\mu(\overline{E_{n_o}^*}) > 1 - \varepsilon$. For every open rational interval J for which $J \cap E_{n_o}^* \neq \emptyset$ choose a non-empty set $S_J \subset J \cap E_{n_o}$ having property (P). Then the set S defined as the union of the sets S_J has measure zero, satisfies property (P) and $\overline{S} = \overline{E_{n_o}^*}$. By property (j_3) there exists $\alpha_o \in \omega_1$ such that $S \subset N_{\alpha_o}$, and by property (P), $\overline{S \setminus C_{\alpha_o}} = \overline{E^*_{n_o}}$. Hence $\mu(\overline{S \setminus C_{\alpha_o}}) > 1 - \varepsilon$ and we can find a finite set $\{y_j : j = 1, \dots, q\} \subset S \setminus C_{\alpha_o}$ and a finite collection $\{I_j : j = 1, \dots, q\}$ of non-overlapping subintervals of $[0,1]$, such that, for each $j = 1, \dots, q$, $y_j \in I_j$, $\mu(I_j) < 1/n_o$ and $\mu(\bigcup_{j=1}^q I_j) > 1 - \varepsilon$.

Now add to the partition $\{(I_j, y_j) : j = 1, \dots, q\}$ any δ-fine McShane partition $\{(I_j, y_j) : j = q+1, \dots, s\}$ of $\overline{[0,1] \setminus \bigcup_{j=1}^q I_j}$ so that

$$\{(I_j, y_j) : j = 1, \dots, s\}$$

is a δ-fine McShane partition of $[0,1]$.

Then we have

$$\left\| \sum_{j=1}^s f(y_j)\mu(I_j) \right\| = \sup_{\alpha \in \omega_1} \left| \sum_{j=1}^s f(y_j)(\alpha)\mu(I_j) \right| \geq \sum_{j=1}^q f(y_j)(\alpha_o)\mu(I_j)$$
$$> 1 - \varepsilon.$$

Since δ is arbitrary, f cannot be McShane integrable.

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