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A COMPARISON THEOREM ON SIMPLY CONNECTED COMPLETE RIEMANNIAN MANIFOLDS

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Abstract. We consider simply connected complete Riemannian manifolds with sectional curvature bounded above by $-C^2 < 0$, and curves on such manifolds with geodesic curvature at most $C > 0$ in absolute value. We give an estimate for the rate at which such curves approach the boundary of the manifold.

1. Introduction and main results

The Theorem of Hayman and Wu [HaW] establishes that there is a universal constant which bounds from above the Euclidean length of level curves for conformal mappings from the disk onto a simply connected domain. This theorem is in fact a statement of hyperbolic geometry in the unit disk, and it can be interpreted as an estimate for the rate at which curves of controlled hyperbolic curvature approach the boundary of the disk. Namely, from [FG] it follows that

$$\int_{-\infty}^{\infty} \frac{1}{2\cosh^2(\frac{d(\sigma(0),\sigma(s))}{2})} ds \leq \pi \tag{1}$$

for curves σ on the unit disk with hyperbolic curvature bounded above in absolute value by 1. Here ds is the hyperbolic arclength element.

Throughout, M will be a simply connected complete Riemannian manifold of sectional curvature bounded above by -1 (i.e., the upper half space in $\mathbb{R}^n$ endowed with a metric of negative variable curvature).

We generalize the estimate (1) to such manifolds and curves of geodesic curvature bounded above by 1, thus showing that these curves approach the boundary of the manifold "fast".

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All results stated in this work also hold, with obvious modifications, when the upper curvature bounds are $-C^2 < 0$ for the manifold and $C > 0$ for the curve.

Dekster, as an application of his main result in [D], showed that an infinitely long curve in a simply connected complete space of curvature $\leq K \leq 0$ goes to infinity if its *total* curvature does not grow "too fast". He also established the rate of escape.

When considering points a, b of M, $d(a, b)$ denotes the distance between a and b. Our main result is the following geometric version of the Hayman-Wu Theorem:

THEOREM 1. *Let M be as above, and let σ be a curve in M parameterized by arclength whose geodesic curvature is, in absolute value, at most* 1*. Then*

$$d(\sigma(0), \sigma(s)) \longrightarrow \infty \quad \text{as } s \to \infty.$$

Moreover,

$$\int_{-\infty}^{\infty} \frac{1}{2\cosh^2\left(\frac{d(\sigma(0),\sigma(s))}{2}\right)} ds \leq \pi. \tag{2}$$

In the hyperbolic disk, a similar result also holds for any point, not necessarily on the curve, with the integral being strictly less than 2π (see [FG]).

Roughly speaking, if the curve σ does not curve too much, the distance between the fixed point $\sigma(0)$ and $\sigma(s)$ grows fast enough for the given integral to converge (see [C, p. 36] and [FG] for surfaces of zero and negative constant curvature, respectively). The main idea in the proof of Theorem 1 comes from the Aleksandrov-Toponogov Triangle Comparison Theorem (see, e.g., [CE, Ch. 2], [K, p. 197]). A suitable version for our purposes is:

THEOREM 2 (Theorem AT). *Let T be a triangle in M and let a_k, $k = 1, 2, 3$, be its angles. Then, in the simply connected surface of constant curvature -1, there is a triangle, T^{alek}, whose sides have the same length as those of T and whose internal angles, a_k^{alek}, satisfy*

$$a_k \leq a_k^{\text{alek}}, \quad k = 1, 2, 3.$$

For the particular group of manifolds under consideration, Theorem AT can be generalized to curves: If the curvature of the manifold M is increased and the curvature of the curve σ is kept fixed, the distance between the endpoints of the arc will decrease. That is, we have:

THEOREM 3. *Let M be as above and let $\sigma : \mathbb{R} \longrightarrow M$ be a curve parameterized by arclength whose geodesic curvature is, in absolute value, bounded above by $|b|$. If $b > 1$ assume also that its length is less than $\pi/\sqrt{b^2 - 1}$. Then*

$$d_M(\sigma(0), \sigma(s)) \geq N(s, b), \tag{3}$$

where the bound $N(s,b)$ is attained on a curve in $\mathbb{D}$ of constant curvature b. Moreover, if $b=1$ then

$$d_M\left(\sigma(0),\sigma(s)\right) \geq d_{\mathbb{D}}\left(0,H(s)\right) \ \left(= \log\frac{1+c(s)}{1-c(s)}\right),$$

where $c(s)=\sqrt{1-4/(s^2+4)}$. Here H is any curve in $\mathbb{D}$ of constant curvature $+1$ with $0=H(0)$.

Here and in the sequel, $\mathbb{D}$ denotes the surface of constant sectional curvature -1, i.e., the unit disk endowed with the metric

$$ds = \frac{2|dz|}{1-|z|^2}. \tag{4}$$

(That is, $\mathbb{D}$ is the Poincaré disk. For relations in hyperbolic geometry, see [B].) Curves of constant curvature b in $\mathbb{D}$ are arcs of Euclidean circles intersecting the unit circle at an angle whose cosine is b; if $|b|<1$, (3) holds for all s.

The main tool in the proof of Theorem 3 will be an iterative triangle comparison process using both the Aleksandrov-Toponogov Theorem and the cosine law.

The outline of the paper is as follows: Section 2 contains some tools and known facts. In Section 3 we prove some technical lemmas. Section 5 is devoted to Theorem 3, a basic tool in the proof of Theorem 1, whose proof is given in Section 4.

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2. Tools and known facts

2.1. Angles will play an important role throughout the paper. We will be considering two kinds: angles *between curves* and angles *of triangles*. The former are oriented angles and can be positive or negative. The latter are always positive and less than π. Some notation is needed:

The angle between two curves σ_1,σ_2 at a point $p=\sigma_1(t_1)=\sigma_2(t_2)$ is

$$\alpha = \angle(\sigma_1'(t_1),\sigma_2'(t_2)) \quad \in [-\pi,\pi).$$

T is a geodesic triangle in M with sides given by the geodesic arcs $(\gamma_1,\gamma_2,\gamma_3)$, parameterized by arclength in such a way that, if their respective lengths are l_j, $j=1,2,3$, then $\gamma_j(l_j)=\gamma_{j+1}(0)$, $j=1,2$, $\gamma_3(l_3)=\gamma_1(0)$. The interior angles of T, a_1,a_2 and a_3 are

$$a_j = \left|\angle\left((\gamma_{j+1})'(0),-(\gamma_j)'(l_j)\right)\right|, \quad j=1,2; \quad a_3 = \left|\angle\left((\gamma_1)'(0),-(\gamma_3)'(l_3)\right)\right|.$$

We shall specify a triangle by giving either its sides or its vertices (ordered with respect to the orientation of the triangle). Observe that $a_j \in [0, \pi)$.

2.2 Aleksandrov triangles. Let T be a geodesic triangle in M of sides $(\gamma_1, \gamma_2, \gamma_3)$ and interior angles a_i, $i = 1, 2, 3$. Denote by T^{alek} a corresponding *Aleksandrov triangle*, that is, a geodesic triangle in $\mathbb{D}$ with sides (γ_1^{alek}, γ_2^{alek}, γ_3^{alek}) and interior angles a_i^{alek}, such that $\text{length}(\gamma_i^{\text{alek}}) = \text{length}(\gamma_i)$. (Observe that they are all the same via Möbius transformations.)

2.3. Let σ be a curve in M. The following observation will be useful in Section 5. Let P, Q, R be three consecutive points on σ such that

$$d(P, Q) = d(Q, R) = L.$$

Let γ, $\tilde{\gamma} : [0, L] \longrightarrow M$ be two geodesic arcs parametrized by arclength so that $\gamma(0) = P$, $\gamma(L) = \tilde{\gamma}(0) = Q$ and $\tilde{\gamma}(L) = R$. If we denote by α the angle between γ and $\tilde{\gamma}$ at Q, and if Q is kept fixed, then

$$\lim_{L \to 0} \frac{\alpha}{L} = \text{ geodesic curvature of } \sigma \text{ at } Q.$$

The *curvature of a triangle* $T = (\gamma_1, \gamma_2, \gamma_3)$ at the vertex $\gamma_2(0)$ is $\pi - a_1$ (with the above notation).

In particular, if the curvature of σ is constant and equal to 1 or -1, the value of the angle α is

$$\beta = 2 \arctan \left(\sinh \frac{L}{2} \right).$$

Notice that if the curvature of σ at Q is bounded by 1, then $|\alpha| \leq \beta$.

3. Some lemmas for triangles

Let M be as in the introduction. Theorem AT gives an angle comparison between a triangle in M and a corresponding triangle in $\mathbb{D}$, both having the same side lengths. Here we consider triangles with different side lengths.

Given three points a, b, c, we let $\angle abc$ be the angle at vertex b of the triangle with vertices a, b and c. In what follows, let $0, z, w$ be three points in $\mathbb{D}$ satisfying

$$d(0, w) < d(0, z) \leq d(z, w). \tag{5}$$

The first two lemmas compare triangles in the surface $\mathbb{D}$.

Lemma 1. *Subdivide the triangle* $(0, z, w)$ *into triangles* $(0, \tilde{z}, w)$ *and* $(\tilde{z}, z, w)$*, with* $\tilde{z}$ *satisfying* $d(\tilde{z}, w) \geq d(0, \tilde{z}) \geq d(0, w)$*. Then,*

$$\angle 0\tilde{z}w \geq \angle 0zw \tag{6}$$

Proof. As a function of the point $\tilde{z}$, the angle $\angle 0\tilde{z}w$ is minimal when $\tilde{z} = z$. □

LEMMA 2. *With the same notation as above, let $\tilde{w}$ be so that*

(i) $$d(\tilde{w}, 0) = d(w, 0),$$

(ii) $$\angle w0z \leq \angle \tilde{w}0z \leq \pi.$$

Then,

(7) $$\angle 0z\tilde{w} \leq \angle 0\tilde{z}w.$$

Proof. By Lemma 1 it suffices to show that $\angle 0\tilde{z}\tilde{w} \leq 0\tilde{z}w$.

Given a triangle in $\mathbb{D}$ of side lengths A, B, C and opposite angles α, β, γ, we have the following trigonometric relations (see, e.g., [Bu, p. 33]):

(1) $$\cosh B = \frac{\cos\alpha\cos\gamma + \cos\beta}{\sin\alpha\sin\gamma};$$

(2) $$\frac{\sinh C}{\sin\gamma} = \frac{\sinh A}{\sin\alpha} = \frac{\sinh B}{\sin\beta};$$

(3) Cosine law: $\cosh A = \cosh B\cosh C - \sinh B\sinh C\cos\alpha$.

Since $0, \tilde{z}, w$ also satisfy (5), the cosine law gives that the angles $\angle 0\tilde{z}w$ and $\angle \tilde{z}w0$ are at most $\pi/2$. Similarly, the same holds for the angles $\angle 0\tilde{z}\tilde{w}$, $\angle \tilde{z}\tilde{w}0$.

Therefore, to prove the lemma is enough to show that when $A < B < C$, $\alpha, \beta \in [0, \pi/2]$, and the lengths A, B are kept fixed, the angle α will continuously decrease as γ increases. Indeed, when $\gamma \geq \pi/2$, this follows from the first equality in relation (2) above, and if $\gamma < \pi/2$, it follows from the second equality in (2) together with (1). □

Next, we state a comparison result between triangles in M and triangles in the disk $\mathbb{D}$.

LEMMA 3. *Take $0, \tilde{z}, w \in \mathbb{D}$ satisfying (5). Consider triangles in M and $\mathbb{D}$ given by $T^M = (m_0, m_1, m_2)$ and $(w, 0, \tilde{z})$, respectively. If*

$$d_M(m_0, m_1) = d_{\mathbb{D}}(0, w), \quad d_M(m_1, m_2) \geq d_{\mathbb{D}}(0, \tilde{z}), \quad \angle m_0m_1m_2 \geq \angle w0\tilde{z},$$

then

(8) $$\angle m_1m_2m_0 \leq \angle 0\tilde{z}w.$$

Proof. Let $z \in \mathbb{D}$ be so that the triangle $(0, z, w)$ is equal to $(0, \tilde{z}, w) \cup (\tilde{z}, z, w)$, and let $T^{\text{alek}} \in \mathbb{D}$ be an Aleksandrov triangle of T^M (see Section 2.2). Without loss of generality, we can take m_1^{alek} to be 0, m_0^{alek} to be $\tilde{w}$ in Lemma 2, and m_2^{alek} to be z above. The result then follows from Theorem AT and Lemma 2. □

4. Proof of Theorem 1

We now take Theorem 3 for granted.

Let H be the curve in the hyperbolic disk $\mathbb{D}$ of constant curvature $+1$ parameterized by (hyperbolic) arclength with $H(0) = 0$. Theorem 3 gives

$$d_M\left(\sigma(0), \sigma(s)\right) \geq d_{\mathbb{D}}\left(0, H(s)\right) \to \infty \text{ as } s \to \infty$$

and

$$\int_{-\infty}^{\infty} \frac{1}{2\cosh^2\left(\frac{d_M(\sigma(0),\sigma(s))}{2}\right)} ds \leq \int_{-\infty}^{\infty} \frac{1}{2\cosh^2\left(\frac{d_{\mathbb{D}}(0,H(s))}{2}\right)} ds = \pi, \tag{9}$$

where the last equality follows from the following observation:

$$\text{Euclidean-length}(H) = \int_H |ds| = \int_H \frac{1-|s|^2}{2} \, \frac{2}{1-|s|^2} |ds|.$$

In order to express the integrand in terms of hyperbolic distance, note that if $z \in \mathbb{D}$, then

$$\frac{1}{1-|z|^2} = \frac{\frac{1+|z|}{1-|z|} + \frac{1-|z|}{1+|z|}}{4} + \frac{1}{2} = \frac{e^{d_{\mathbb{D}}(0,z)} + e^{-d_{\mathbb{D}}(0,z)} + 2}{4} = \cosh^2\left(\frac{1}{2} d_{\mathbb{D}}(0,z)\right).$$

This, together with (4), implies the equality in (9).

5. Proof of Theorem 3

In applications, we shall only need the case where $b = 1$. The proof will be given in this case only; the proof in the general case is similar.

We first prove a polygonal line version of Theorem 3 (Section 5.1). Then (Section 5.2) we describe a procedure that allows us to derive the general comparison theorem for curves from this discrete version.

5.1. Comparison theorem for polygonal lines. We fix a point $m_0 \in M$ which will serve as a reference point; in the case of the disk $\mathbb{D}$, we take 0.

We denote by $P^M_{(\alpha_1,\ldots,\alpha_{n-1};L)}$ a piecewise geodesic polygonal line in M parameterized by arclength, consisting of a number of geodesic arcs, whose sides, $\gamma_1, \ldots, \gamma_n$, are of the same length L, connecting vertices $m_0, \ldots, m_n$, so that

$$m_0 = \gamma_1(0), \quad m_n = \gamma_n(L), \quad m_i = \gamma_i(L) = \gamma_{i+1}(0), \quad i = 1, \ldots, n-1,$$

and such that the α_i's are angles between consecutive segments,

$$\alpha_i = \angle\left((\gamma_i)'(L), (\gamma_{i+1})'(0)\right) \quad (\in [-\pi, \pi]).$$

We want to compare polygonal lines in M with polygonal lines in the disk $\mathbb{D}$. We denote these by P^M and P, respectively.

To each polygonal line P^M in M we associate a region, the *fanshaped frame*, which consists of the union of the boundaries of the geodesic triangles T_k^M in M with vertices at the points (m_0, m_k, m_{k+1}), $k = 1, \ldots, n-1$, and *vertex* at m_0.

Roughly speaking, with this notation, Theorem AT states that to any 2-polygonal line in M a comparison polygonal line in $\mathbb{D}$ can be associated such that its curvature and distance between endpoints are at most those of the polygonal line in M. Our aim in this section is to extend this to n-polygonal lines.

In what follows, let $\beta = 2\arctan(\sinh(L/2)) > 0$ (see Section 2.3); we take $P_{(\beta,\ldots,\beta;L)}$ as the *comparison n-polygonal line* in $\mathbb{D}$ with

$$z_0 = P_{(\beta,\ldots,\beta;L)}(0) = 0, \quad z_k = P_{(\beta,\ldots,\beta;L)}(kL),$$

where the angles are kept fixed. (The appropriate value for the angles comes from the fact that the curve $\sigma \in M$ of Theorem 1 has curvature bounded above by 1, and thus the polygonal line in M, $P^M_{(\alpha_1,\ldots,\alpha_{n-1};L)}$, has angles bounded above by this value of β.)

THEOREM 4. *Let $P^M_{(\alpha_1,\ldots,\alpha_{n-1};L)}$ and $P_{(\beta,\ldots,\beta;L)}$ as above. If $|\alpha_k| \leq \beta$, then*

$$d_M(m_0, m_n) \geq d(z_0, z_n). \tag{10}$$

Fix the polygonal lines and consider the (fixed) triangles with vertices m_k, m_l, m_j, and corresponding triangles with vertices z_k, z_l, z_j. In the course of the proof we will construct some triangles in $\mathbb{D}$ which will serve as intermediate objects for comparison. A convenient notation for angles of triangles of the fanshaped frame will be the following:

- $C^M(k; l, j)$, for $0 \leq j < k < l$: the interior (central) angle at vertex m_k of the triangle in M of vertices m_k, m_l and m_j.
- $I^M(k; j-1, j)$, for any k, $j-1$, j distinct: the interior (lateral) angle at vertex m_k of the triangle in M of vertices m_k, m_{j-1} and m_j.

The same quantities without the superscript M will be the (corresponding) magnitudes in $\mathbb{D}$.

Proof. We shall prove (10) by induction on the number of triangles $T^M_k = (m_0, m_k, m_{k+1})$, $k = 1, \ldots, n-1$, of the fanshaped frame.

The case $n = 2$ follows easily from Theorem AT and the cosine rule in the hyperbolic disk. Indeed, consider the triangle with vertices m_0, m_1, m_2, and proceed as follows:

(i) Since $|\alpha_1| \leq \beta$, we get directly $C^M(1; 2, 0) \geq C(1; 2, 0)$.

(ii) Consider now an Aleksandrov triangle. From Theorem AT, (i) and the hyperbolic cosine rule we get $d_M(m_0, m_2) \geq d_{\mathbb{D}}(z_0, z_2)$.

(iii) Lemma 3 together with (i) and $d(m_1, m_2) = d(z_1, z_2) = L$ gives $I^M(2; 0, 1) \leq I(2; 0, 1)$.

For the general case, consider the $(n-1)-$polygonal line $P^M_{(L;\alpha_1,\ldots,\alpha_{n-2})}$ and the triangles of its fanshaped frame T^M_k and suppose that for $k \leq n-1$

(a) $C^M(k-1; k, 0) \geq C(k-1; k, 0)$,

(b) $d_M(m_0, m_k) \geq d_{\mathbb{D}}(z_0, z_k)$,
(c) $I^M(n-1; j-1, j) \leq I(n-1; j-1, j)$ for $j = 1, \ldots, n-2$.

We shall verify that we can continue the process.

(a') $C^M(n-1; n, 0) \geq C(n-1; n, 0)$. Consider the fanshaped frame with vortex at m_{n-1}, and the corresponding frame with vortex at z_{n-1}.

Since the vertices $z_0, \ldots, z_n$ lie on a curve of constant curvature 1, the angles at vertex z_{n-1} of the triangles of the fanshape frame in $\mathbb{D}$ trivially verify

$$\pi = C(n-1; n, 0) + \sum_{j=1}^{n-2} I(n-1; j-1, j) + \beta. \tag{11}$$

Take the exponential map, so that $\exp_{m_{n-1}}(0) = m_{n-1}$. Consider the points $\hat{m}_0, \ldots, \hat{m}_{n-2}, \hat{m}_n$ lying on the surface of the unit ball, S, that satisfy

$$\begin{aligned} d_S(\hat{m}_k, \hat{m}_{k-1}) &= I^M(n-1; k-1, k) \quad \text{for } k = 0, \ldots, n-2, \\ d_S(\hat{m}_0, \hat{m}_n) &= C^M(n-1; n, 0), \\ d_S(\hat{m}_{n-2}, \hat{m}_n) &= \pi - |\alpha_{n-1}|. \end{aligned}$$

By the triangle inequality for $d_S(\hat{m}_{n-2}, \hat{m}_n)$,

$$\pi \leq C^M(n-1; n, 0) + \sum_{j=1}^{n-2} I^M(n-1; j-1, j) + |\alpha_{n-1}|. \tag{12}$$

Subtracting (11) from (12), using $|\alpha_i| \leq \beta$ and (c) above, (a') follows.

Observe that (a') also implies that $C^M(1; n, 0) \geq C(1; n, 0)$, since one can consider, by reordering the vertices, the triangle $T = (m_0, m_1, m_n)$.

(b') $d_M(m_0, m_n) \geq d_{\mathbb{D}}(z_0, z_n)$. Consider the triangle with vertices m_0, m_{n-1}, m_n, and its comparison triangle with vertices z_0, z_{n-1}, z_n.

Construct an Aleksandrov triangle in $\mathbb{D}$. Theorem AT together with (a') gives $C^{\text{alek}}(n-1; n, 0) \geq C^M(n-1; n, 0) \geq C(n-1; n, 0)$. By induction, $d_M(m_0, m_{n-1}) \geq d(z_0, z_{n-1})$. We claim that (b') follows from the cosine rule in $\mathbb{D}$. Indeed, z_0, z_{n-1}, z_n satisfy (5), for the vertices z_k lie on a curve of constant curvature 1, $d_{\mathbb{D}}(z_{n-1}, z_n) = d_{\mathbb{D}}(z_0, z_1)$, and the distance to z_0 increases as we move along the horocycle and, therefore, $d_{\mathbb{D}}(z_0, z_n) \geq d_{\mathbb{D}}(z_0, z_{n-1}) \geq L$. Thus, $\angle z_{n-1} z_0 z_n < \pi/2$, and this angle at z_0 remains below $\pi/2$ while the edge of length $d(z_0, z_{n-1})$ is lengthened to an edge of length $d_M(m_0, m_{n-1})$. The cosine rule gives the desired inequality.

(c') $I^M(n; k-1, k) \leq I(n; k-1, k)$. It is enough to prove this for $k = 1$, since all other cases follow from (c) by reordering the vertices. Consider the triangles $T^M = (m_0, m_1, m_n)$ and $T = (z_0, z_1, z_n)$. We claim that Lemma 3 applies.

First observe that, by the remark in (b'), z_0, z_1, z_n also satisfy (5). Therefore $d_{\mathbb{D}}(z_0, z_n) \geq d_{\mathbb{D}}(z_1, z_n) \geq L$. Note that $d_M(m_0, m_1) = d_{\mathbb{D}}(z_0, z_1) = L$ and, by the observation in (a'), $C^M(1; n, 0) \geq C(1; n, 0)$. Finally, $d_M(m_1, m_n) \geq d_{\mathbb{D}}(z_1, z_n)$; to see this, simply take the new $(n-1)$-polygonal line $\tilde{P}^M_{(\alpha_2,\ldots,\alpha_{n-1};L)}$ starting at the point $\tilde{m}_0 = m_1$ and its comparison polygonal line $\tilde{P}_{(\beta,\ldots,\beta;L)}$ starting at $\tilde{z}_0 = z_1$. The claim now follows from (b). Thus, by Lemma 3, $I(n; 0, 1) \geq I^M(n; 0, 1)$. □

5.2. Comparison theorem for curves. We consider a curve σ in M parametrized by arclength. On σ, we choose a point m_0; we may assume that σ is parametrized so that $m_0 = \sigma(0)$.

Given $\epsilon > 0$, we consider as the model space for comparison the surface of constant sectional curvature $-(1+\epsilon)^2$ and take the curve of constant curvature $1+\epsilon$, say $H_\epsilon(s)$, in $\mathbb{D}$ to be the unique curve parametrized by arclength that satisfies $H(0) = 0$ and is tangent to $\partial\mathbb{D}$ at $e^{\imath\pi}$. We want to show that

$$d_M(m_0, \sigma(s)) \geq d(0, H_\epsilon(s)) \quad \text{for all } s. \tag{13}$$

The general case will then follow by letting $\epsilon \to 0$.

Fix a small stepsize L of the form $L = s/n$, where n is a positive integer. We approximate $H(s)$ and the given curve $\sigma(s)$ by polygonal lines $P_{(\beta,\ldots,\beta;L)}$ and $P^M_{(\alpha_1,\ldots,\alpha_{n-1};L)}$, respectively, in the following way:

To approximate H_ϵ, set $z_0 = 0 \in H_\epsilon$ and determine successive points $\{z_k\}_{k=1}^n$, $z_k \in H_\epsilon$, so that $d(z_k, z_{k+1}) = L$ for $k = 0, \ldots, n-1$. The angles of the polygonal line constructed in this way (see Section 2.3) are all equal to $\beta = 2\arctan(\sinh(1+\epsilon)L/2)$.

To approximate σ, we set $m_0 = m \in \sigma$ and determine successive points $\{m_k\}_{k=1}^n$, $m_i \in \sigma$, so that $d_M(m_k, m_{k+1}) = L$ for $k = 0, \ldots, n-1$. The angles of the polygonal line constructed in this way satisfy $|\alpha_i| \leq \beta$ if L is sufficiently small. This, of course, is due to the fact that the curvature of σ is strictly less than $1+\epsilon$ in absolute value (see Section 2.3).

By Theorem 4, $d_M(m_0, P^M_{(\alpha_1,\ldots,\alpha_{n-1};L)}(nL)) \geq d(0, P_{(\beta,\ldots,\beta;L)}(nL))$. Letting $L \to 0$, we get (13).

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