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BESOV FUNCTIONS AND VANISHING EXPONENTIAL INTEGRABILITY

DAVID R. ADAMS AND RITVA HURRI-SYRJÄNEN

Abstract. We prove a general vanishing exponential integrability result for Besov functions. In a basic case, this allows us to improve the known $\mathcal{O}(1)$ estimate to a $o(1)$ estimate. It also leads to improvements of differentiability results for Besov functions.

1. Introduction

A non-negative function $v(x)\,, x \in \mathbb{R}^n$, is said to satisfy the vanishing exponential integrability condition if there is a constant $\beta > 0$, independent of v and the radius r of the Euclidean n-ball $B^n(x_0, r)$, such that

$$\lim_{r\to 0} ⨍_{B^n(x_0,r)} \left(e^{\beta v(x)} - 1\right)\, dx = 0 \tag{1.1}$$

for all $x_0 \in \mathbb{R}^n \backslash E$, where E is an exceptional set for some universal set function σ strictly stronger than Lebesgue measure on $\mathbb{R}^n$. The set function σ might be a Hausdorff capacity (content) or an L^p-capacity. In (1.1), the bar on the integral sign denotes the integral average over $B^n(x_0, r)$. In our basic case, Theorem 1.3, $v(x)$ will be $|u(x) - u(x_0)|^{q/(q-1)}$, where $u \in \Lambda_\alpha^{p,q}(\mathbb{R}^n)$, the standard class of Besov functions on $\mathbb{R}^n$. The set function σ is given by $\sigma = [H_{p,q,h}, A_{\alpha,p,q}]$, the Neugebauer bracket of the two capacities (see Definition 2.7), where $A_{\alpha,p,q}$ is the Besov capacity associated with the space $\Lambda_\alpha^{p,q}(\mathbb{R}^n)$, $\alpha p = n$, and $H_{p,q,h}$ is a certain Hausdorff capacity with the measure function $h(t) = (\log 1/t)^{1-q}$,

$$H_{p,q,h} = \begin{cases} H^{h^{p/q}}\,, & \text{if } q < p, \\ (H^h)^{p/q}\,, & \text{if } p \le q, \end{cases} \tag{1.2}$$

$1 < p, q < \infty$. For the definitions of these capacities see Definition 2.3 and Remark 2.8. We now state our main theorem.

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1.3. THEOREM. *Let $u \in \Lambda_\alpha^{p,q}(\mathbb{R}^n)$ and $\alpha p = n$. Then there exists a constant $\beta > 0$ independent of u and $r > 0$ such that*

$$\fint_{B^n(x_0,r)} \Big(\exp(\beta|u(x) - u(x_0)|^{q/(q-1)}) - 1\Big)\, dx = o(1)$$

as $r \to 0$ for $[H_{p,q,h}, A_{\alpha,p,q}]$-a.e. $x_0 \in \mathbb{R}^n$, where the Hausdorff capacity $H_{p,q,h}$ with the measure function $h(t) = (\log 1/t)^{1-q}$ is defined in (1.2).

In our general case, Theorem 6.1, $v(x)$ is $(r^{-m}|u(x) - P^m_{x_0}(x)|)^{q/(q-1)}$, where $P^m_{x_0}$ is the mth order Taylor polynomial for u centered at x_0. Here $m \in [1,\alpha)$ and $\sigma = [H_{p,q,h}, A_{\alpha-m,p,q}]$, $(\alpha - m)p = n$. It is remarkable that the integral average in (1.1) of the exponential function is $o(1)$ as $r \to 0$ when $v(x) = |u(x) - u(x_0)|^{q/(q-1)}$. Previously, C. J. Neugebauer [8, Proof of Theorem 2] showed that the integral in (1.1) over the exponential function with this function $v(x)$ was merely $\mathcal{O}(1)$ as $r \to 0$. Even in the paper [2, Theorem 2] by the first author the result is $\mathcal{O}(1)$. As an application of Theorem 6.1 some earlier differentiability results by J. R. Dorronsoro [7] for functions in the Besov space $\Lambda_\alpha^{p,q}$, $1 \le p < \infty$, $1 \le q < \infty$, $(\alpha - m)p = n$, are improved; see Section 7.

The definitions and previously known results which we need are recalled in Section 2. A capacitary average result is shown in Section 3 and lower bounds for the Besov capacity are proved in Section 4. The proof for Theorem 1.3 is presented in Section 5. The result for the general case, when $v(x) = (r^{-m}|u(x) - P^m_{x_0}(x)|)^{q/(q-1)}$, is proved in Section 6. Differentiability results for Besov functions are briefly considered in Section 7.

2. Preliminaries

Let $\alpha > 0$, $1 < p < \infty$, and $1 < q < \infty$ throughout the paper. Recall that a function sequence $f = \{f_k\}_0^\infty$ is in $l^q(L^p)$ if

$$\|f\|_{l^q(L^p)} = \left(\sum_{k=0}^\infty \|f_k\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q} < \infty\,.$$

Let $\eta \in C_0^\infty(\mathbb{R}^n)$. Set $\eta_k(x) = 2^{nk}\eta(2^k x)$ for $k = 0, 1, 2, \ldots$. A representation theorem for Besov spaces, [4, Theorem 4.1.7], states that a function u belongs to a Besov space $\Lambda_\alpha^{p,q}$ if and only if there is a function sequence $f = \{f_k\}_0^\infty \in l^q(L^p)$ such that

$$u = \mathcal{H}_\alpha f = \sum_{k=0}^\infty 2^{-\alpha k}\eta_k * f_k\,. \tag{2.1}$$

Further,

$$\|f\|_{l^q(L^p)} \sim \|\mathcal{H}_\alpha f\|_{\Lambda_\alpha^{p,q}}\,,$$

where the norm $\| * \|_{\Lambda_\alpha^{p,q}}$ is the Besov norm (see [4, Chapter 4]).

2.2. Remark. The notation $\sim$ means 'is comparable to'.

The potential representation can be used to define the Besov capacity $A_{\alpha,p,q}(*)$.

2.3. Definition ([4, Definition 4.4.2 and Remark]). Let $1 < p < \infty$, $1 < q < \infty$, and $0 < \alpha < \infty$. Let $E \subset \mathbb{R}^n$ be arbitrary. Then

$$A_{\alpha,p,q}(E) = \inf \left\{ \|f\|^p_{l^q(L^p)} : f \geq 0, \mathcal{H}_\alpha f(x) \geq 1 \text{ on } E \right\} .$$

Let $B^n(x_0, r)$ be a ball in $\mathbb{R}^n$ with a center x_0 and radius $r > 0$. The Besov capacity for a ball $B^n(x_0, r)$ is given in the following lemma whenever the radius is sufficiently small.

2.4. Lemma ([3, Theorem 3.5]). *Let $1 < p < \infty$, $1 < q < \infty$, and $0 < \alpha < \infty$. For sufficiently small r, and any $x_0 \in \mathbb{R}^n$,*

$$A_{\alpha,p,q}(B^n(x_0, r)) \sim \left(\log \frac{1}{r} \right)^{p(1-q)/q} ,$$

whenever $\alpha p = n$.

We use the following notation:

$$s = \begin{cases} q/p\,, & \text{if } p \leq q, \\ 1\,, & \text{if } p > q. \end{cases} \tag{2.5}$$

The strong type estimates for the Besov capacity are also needed.

2.6. Theorem ([5, Theorem1]). *Let $u \in \Lambda^{p,q}_\alpha(\mathbb{R}^n)$, $1 < p, q < \infty$, and $0 < \alpha < \infty$. There is a constant $c = c(\alpha, p, q, n)$ such that*

$$\int_0^\infty \left(A_{\alpha,p,q}(\{x \in \mathbb{R}^n : |u(x)| \geq t\}) \right)^s \, dt^{sp} \leq c \|u\|^{sp}_{\Lambda^{p,q}_\alpha} ,$$

where s is defined in (2.5).

Next, we introduce Neugebauer's bracket of two capacities.

2.7. Definition ([9, p. 304], [1, V.4]). Let $E \subset \mathbb{R}^n$. Given two capacities cap_1 and cap_2, set

$$[\mathrm{cap}_1, \mathrm{cap}_2](E) = \inf\{\mathrm{cap}_1(E_1) + \mathrm{cap}_2(E_2)\},$$

where the infimum is over all disjoint partitions E_1, E_2 of $E = E_1 \cup E_2$.

2.8. Remark. The Hausdorff capacity is denoted by H^h. Here $h = h(t)$ is a monotone increasing function of $t \geq 0$, and

$$H^h(K) = \inf \sum_{j=0}^\infty h(r_j),$$

where the infimum is over all countable coverings of K by balls and r_j denotes the radius of the jth ball of such a cover.

2.9. Remark ([6, Chapter 4, Definition 4.1, Proposition 4.2]). We recall the equivalent norm for the Lorentz spaces $L(p,q)$:

$$\left(\int_0^\infty (t^p|\{x : |f(x)| > t\}|)^{q/p} \frac{dt}{t}\right)^{1/q} \sim \|f\|_{L(p,q)} < \infty\,.$$

Note that $\|f\|_{L(p,p)} = \|f\|_{L^p}$ and $L(p,p) = L^p$ is the classical Lebesgue space. One always has $L(p,q_1) \subset L(p,q_2)$ if $q_1 < q_2$.

Throughout the paper, the letter c will denote various constants which may differ from one formula to the next even within a single string of estimates.

3. Capacitary averages

Let $\alpha p = n$ throughout this section. We define s as in (2.5). For a Besov function $v \in \Lambda_\alpha^{p,q}(\mathbb{R}^n)$ we consider the maximal function

$$\begin{aligned}(3.1)\quad &\mathcal{M}_{\alpha,s}(v)(x_0)\\ &= \sup_{r>0} A_{\alpha,p,q}(B^n(x_0,r))^{-s} \int_0^\infty \left(A_{\alpha,p,q}(B^n(x_0,r) \cap [v > t])\right)^s \, dt^{sp},\end{aligned}$$

where $v = \mathcal{H}_\alpha f$ with $f \geq 0$. The set $\{x : v(x) > t\}$ is abbreviated by $[v > t]$. For a function $u \in \Lambda_\alpha^{p,q}(\mathbb{R}^n)$ we write

$$E_t(r) = B^n(x_0,r) \cap \{x : |u(x) - u(x_0)| > t\}. \tag{3.2}$$

To show (in Theorem 3.8 below) that the capacity average satisfies

$$A_{\alpha,p,q}(B^n(x_0,r))^{-s} \int_0^\infty \left(A_{\alpha,p,q}(E_t(r))\right)^s \, dt^{sp} \to 0$$

when r goes to zero, we need the following lemma.

3.3. Lemma. *Let $\alpha p = n$. Then*

$$[(H^h)^{p/q}, A_{\alpha,p,q}]\left(\{x : \mathcal{M}_{\alpha,q/p}(\mathcal{H}_\alpha f)(x) > t^q\}\right) \leq \frac{c}{t^p}\|f\|^p_{l^q(L^p)}\,,\ 1 < p \leq q,$$

and

$$[H^{h^{p/q}}, A_{\alpha,p,q}]\left(\{x : \mathcal{M}_{\alpha,1}(\mathcal{H}_\alpha f)(x) > t^p\}\right) \leq \frac{c}{t^p}\|f\|^p_{l^q(L^p)}\,,\ q < p,$$

where $h(t) = (\log 1/t)^{1-q}$.

Proof. We write $f^r = \{f_k^r\}$, where $f_k^r = f_k \cdot \chi_{B(x_0,2r)}$ and $g = \{g_k\}$, $g_k = f_k - f_k^r$; here $\chi_{B(x_0,2r)}$ is the characteristic function of a ball $B(x_0,2r)$. Then

$$\mathcal{H}_\alpha f(x) = \mathcal{H}_\alpha f^r(x) + \mathcal{H}_\alpha g(x)\,.$$

So

$$\mathcal{M}_{\alpha,s}(\mathcal{H}_\alpha f)(x) \le c\mathcal{M}_{\alpha,s}(\mathcal{H}_\alpha f^r)(x) + c\mathcal{M}_{\alpha,s}(\mathcal{H}_\alpha g)(x)\,, \tag{3.4}$$

where s is defined in (2.5). By Lemma 2.4 and Theorem 2.6,

$$\begin{aligned}
&A_{\alpha,p,q}(B^n(x_0,r))^{-s}\int_0^\infty \left(A_{\alpha,p,q}(B^n(x_0,r)\cap[\mathcal{H}_\alpha f^r > t])\right)^s \, dt^{sp} \\
&\quad \le c\left(\log\frac{1}{r}\right)^{ps(q-1)/q} \|\mathcal{H}_\alpha f^r\|^{sp}_{\Lambda^{pq}_\alpha} \\
&\quad \sim c\left(\log\frac{1}{r}\right)^{ps(q-1)/q} \left(\sum_{k=0}^\infty \|f^r_k\|^q_{L^p(\mathbb{R}^n)}\right)^{ps/q} \\
&\quad = c\left(\log\frac{1}{r}\right)^{ps(q-1)/q} \left[\sum_{k=0}^\infty \left(\int_{B^n(x_0,2r)} f_k(y)^p\, dy\right)^{q/p}\right]^{ps/q} .
\end{aligned}$$

Now set, for $t > 0$,

$$K_t = \left\{ x\, : \sup_{r>0} \left(\log\frac{1}{r}\right)^{ps(q-1)/q} \left[\sum_{k=0}^\infty \left(\int_{B^n(x,r)} f_k(y)^p\, dy\right)^{q/p}\right]^{ps/q} > t^{sp} \right\} .$$

For each $x \in K_t$ there exists a ball B_{r_x} centered at x and of radius r_x such that

$$\left(\log\frac{1}{r_x}\right)^{ps(1-q)/q} < \frac{1}{t^{sp}} \left[\sum_{k=0}^\infty \left(\int_{B_{r_x}} f_k(y)^p\, dy\right)^{q/p}\right]^{ps/q} .$$

By a standard covering argument there exists a sequence of disjoint balls $\{B_j\}$ with radius r_j such that $\{5B_j\}$ covers K_t:

$$\sum_{j=0}^\infty \left(\log\frac{1}{r_j}\right)^{ps(1-q)/q} \le \frac{1}{t^{sp}} \sum_{j=0}^\infty \left[\sum_{k=0}^\infty \left(\int_{B_j} f_k(y)^p\, dy\right)^{q/p}\right]^{ps/q} . \tag{3.5}$$

When $s = q/p$ in (3.5), then $q/p \ge 1$ and hence

$$\sum_{j=0}^\infty \left(\log\frac{1}{r_j}\right)^{1-q} \le \frac{1}{t^q} \sum_{k=0}^\infty \left(\sum_{j=0}^\infty \int_{B_j} f_k(y)^p\, dy\right)^{q/p} .$$

This then gives the estimate

$$H^{h_1}(K_t) \le \frac{c}{t^q}\|f\|^q_{l^q(L^p)}\,,$$

which means

$$(H^{h_1}(K_t))^{p/q} \le \frac{c}{t^p}\|f\|^p_{l^q(L^p)}\,, \tag{3.6}$$

for $p \le q$, $h_1(t) = (\log 1/t)^{1-q}$.

When $s = 1$ in (3.5), we have

$$\sum_{j=0}^{\infty} \left(\log \frac{1}{r_j}\right)^{p(1-q)/q} \le \frac{1}{t^p} \sum_{j=0}^{\infty} \left[\sum_{k=0}^{\infty} \left(\int_{B_j} f_k(y)^p \, dy\right)^{q/p}\right]^{p/q}$$
$$\le \frac{1}{t^p} \left[\sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} \int_{B_j} f_k(y)^p \, dy\right)^{q/p}\right]^{p/q}$$

by the generalized Minkowski inequality, [10, p. 271]. This gives

$$H^{h_2}(K_t) \le \frac{c}{t^p} \|f\|^p_{l^q(L^p)}, \tag{3.7}$$

where $h_2(t) = (\log 1/t)^{p(1-q)/q}$.

On the other hand,

$$\mathcal{M}_{\alpha,s}(\mathcal{H}_\alpha g)(x_0) \le c\mathcal{H}_\alpha f(x_0)^{sp}\,.$$

Hence, by (3.4) we have

$$\{x : \mathcal{M}_{\alpha,s}(\mathcal{H}_\alpha f)(x) > t^{sp}\} \subset K_{ct} \cup \{x : (\mathcal{H}_\alpha f(x))^{sp} > ct^{sp}\}\,,$$

where K_{ct} is estimated in terms of H^{h_i}, $i = 1, 2$, as in (3.6) and (3.7). The set $\{x : (\mathcal{H}_\alpha f(x))^{sp} > t^{sp}\}$ can be estimated in terms of $A_{\alpha,p,q}$ via a weak-type capacity estimate by the definition, Definition 2.3. We have from Definition 2.7 the estimates

$$[(H^h)^{p/q}, A_{\alpha,p,q}]\left(\{x : \mathcal{M}_{\alpha,q/p}(\mathcal{H}_\alpha f)(x) > t^q\}\right) \le \frac{c}{t^p}\|f\|^p_{l^q(L^p)}\,,\ 1 < p \le q,$$

and

$$[H^{h^{p/q}}, A_{\alpha,p,q}]\left(\{x : \mathcal{M}_{\alpha,1}(\mathcal{H}_\alpha f)(x) > t^p\}\right) \le \frac{c}{t^p}\|f\|^p_{l^q(L^p)}\,,\ q < p,$$

where $h(t) = (\log 1/t)^{1-q}$. □

We are now in a position to prove a key result.

3.8. Theorem. *Let $u \in \Lambda^{p,q}_\alpha(\mathbb{R}^n)$ with $\alpha p = n$. Let $h(t) = (\log 1/t)^{1-q}$. If $E_t(r) = B^n(x_0, r) \cap \{x : |u(x) - u(x_0)| > t\}$, then*

$$\lim_{r\to 0} A_{\alpha,p,q}(B^n(x_0,r))^{-s} \int_0^\infty \left(A_{\alpha,p,q}(E_t(r))\right)^s \, dt^{sp} = 0 \tag{3.9}$$

for $[(H^h)^{p/q}, A_{\alpha,p,q}]$-a.e. x_0 whenever $s = q/p$ and $p \le q$. If $s = 1$ and $p > q$, (3.9) holds for $[H^{h^{p/q}}, A_{\alpha,p,q}]$-a.e. x_0.

Proof. We consider only the case $s = q/p \geq 1$; the case $s = 1$ is similar. Denote $\mathcal{C} = [(H^h)^{p/q}, A_{\alpha,p,q}]$ for convenience. By the triangle inequality,

$$\begin{aligned}
\int_0^\infty & \mathcal{C}\left(B^n(x_0,r) \cap \{x : |u(x) - u(x_0)| > t\}\right)^s \, dt^{ps} \\
&\leq \int_0^\infty \mathcal{C}\left(B^n(x_0,r) \cap \{x : |u(x)| + |u(x_0)| > t\}\right)^s \, dt^{ps} \\
&\leq \int_0^\infty \mathcal{C}\left(B^n(x_0,r) \cap (\{x : |u(x)| > t/2\} \cup \{x : |u(x_0)| > t/2\})\right)^s \, dt^{ps} \\
&\leq c\Bigg(\int_0^\infty \mathcal{C}\left(B^n(x_0,r) \cap \{x : |u(x)| > t/2\}\right)^s \, dt^{ps} \\
&\qquad + \int_0^\infty \mathcal{C}\left(B^n(x_0,r) \cap \{x : |u(x_0)| > t/2\}\right)^s \, dt^{ps}\Bigg).
\end{aligned}$$

We introduce the notation

$$av\mathcal{C}(u,r)(x_0) = \mathcal{C}(B^n(x_0,r))^{-s} \int_0^\infty \mathcal{C}(E_t(r))^s \, dt^{ps}\,, \tag{3.10}$$

where $E_t(r)$ is defined as in (3.2). Hence, for $u = \mathcal{H}_\alpha f$, assuming, without loss of generality, $f \geq 0$, $u \geq 0$, we have

$$av\mathcal{C}(u,r)(x_0) \leq c(\mathcal{M}_{\alpha,s}(\mathcal{H}_\alpha f)(x_0) + (\mathcal{H}_\alpha f(x_0))^{ps})\,.$$

Thus,

$$\lim_{r\to 0} av\mathcal{C}(u,r)(x_0) \leq c(\mathcal{M}_{\alpha,s}(\mathcal{H}_\alpha f)(x_0) + (\mathcal{H}_\alpha f(x_0))^{ps})\,.$$

We have to show that

$$\mathcal{C}(\{x_0 : \lim_{r\to 0} av\mathcal{C}(u,r)(x_0) > \lambda^{ps}\}) = 0$$

for any $\lambda > 0$. By the above estimate and the previous lemma

$$\begin{aligned}
\mathcal{C}&\Big(\{x_0 : \lim_{r\to 0} av\mathcal{C}(u,r)(x_0) > \lambda^{ps}\}\Big) \\
&\qquad \leq \frac{c}{\lambda^p}\|f\|^p_{l^q(L^p)} + \mathcal{C}\left(\{x_0 : (\mathcal{H}_\alpha f(x_0))^{ps} > (\lambda/2)^{ps}\}\right)\,.
\end{aligned}$$

By Definition 2.3 the weak type estimate

$$\mathcal{C}(\{x_0 : (\mathcal{H}_\alpha f(x_0))^{ps} > (\lambda/2)^{ps}\}) \leq \frac{c}{\lambda^p}\|f\|^p_{l^q(L^p)}$$

holds. We use the standard argument. Sequences of C_0^∞ functions are dense in $l^q(L^p)$. Let $f = f - \psi + \psi$, where $f = \{f_k\}_0^\infty \in l^q(L^p)$ and $\psi = \{\psi_k\}_0^\infty$ with $\psi_k \in C_0^\infty$, and f_k a sequence $\{\psi_k^j\}$ with $\|f_k - \psi_k^j\|_{L^p} \to 0$ as $j \to \infty$.

Then, for any $\lambda > 0$,

$$\begin{aligned}\mathcal{C}(\{x_0 : \lim_{r\to 0} av\mathcal{C}(\mathcal{H}_\alpha f, r)(x_0) > \lambda^{ps}\}) \\ &\leq \mathcal{C}(\{x_0 : \lim_{r\to 0} av\mathcal{C}(\mathcal{H}_\alpha f - \mathcal{H}_\alpha \psi, r)(x_0) > \lambda^{ps}\}) \\ &\leq \frac{c}{\lambda^p}\|f-\psi\|^p_{l^q(L^p)} < c\frac{\epsilon}{\lambda^p},\end{aligned}$$

where $\epsilon > 0$ can be taken arbitrary small. Hence,

$$\mathcal{C}(\{x_0 : \lim_{r\to 0} av\mathcal{C}(\mathcal{H}_\alpha f, r)(x_0) > \lambda^{ps}\}) = 0$$

for all $\lambda > 0$, and thus

$$\mathcal{C}(\{x_0 : \lim_{r\to 0} av\mathcal{C}(\mathcal{H}_\alpha f, r)(x_0) > 0\}) = 0\,,$$

and the claim follows. □

4. Lower bounds for $A_{\alpha,p,q}$-capacity

When $\alpha p = n$, by [2, Theorem 4] there exist constants $a > 0$ and $c < \infty$ independent of a ball B_1 and a function u such that

$$⨍_{B_1} \exp(a|u(x)|^{q/(q-1)})\,dx \leq c$$

with $\|u\|_{\Lambda^{p,q}_\alpha} \leq 1$. See [8, Theorem 2].

4.1. Lemma. *Let $\alpha p = n$. There are constants a and $c > 0$ independent of the set E such that*

$$A_{\alpha,p,q}(E) \geq a\left(\log\frac{c}{|E|}\right)^{p(1-q)/q}\,,$$

for all $E \Subset B_1$, where B_1 is some fixed ball in $\mathbb{R}^n$ and $|E|$ is small enough.

Proof. We use the estimate (27) in [2, Theorem 4] for a function $u = g/\|g\|_{\Lambda^{p,q}_\alpha}$. Let $g(x) \geq 1$ on E, so $E \subset \{x \in B_1 : |g(x)| \geq 1\}$, and hence

$$\begin{aligned}c &\geq \int_{B_1} \exp\left(a\left(\frac{|g(x)|}{\|g\|_{\Lambda^{p,q}_\alpha}}\right)^{q/(q-1)}\right)dx \\ &\geq \int_{\{x\in B_1:|g(x)|\geq 1\}} \exp\left(a\left(\frac{1}{\|g\|_{\Lambda^{p,q}_\alpha}}\right)^{q/(q-1)}\right)dx \\ &\geq |E|\exp\left(\frac{a}{\|g\|^{q/(q-1)}_{\Lambda^{p,q}_\alpha}}\right).\end{aligned}$$

Thus

$$\|g\|_{\Lambda^{p,q}_\alpha} \geq \left(a\log\frac{c}{|E|}\right)^{p(1-q)/q}$$

and

$$A_{\alpha,p,q}(E) \geq \left(a \log \frac{c}{|E|}\right)^{p(1-q)/q} .$$

4.2. Remark. According to (3.10) we can write briefly

$$avA_{\alpha,p,q}(u,r)(x_0) = A_{\alpha,p,q}(B^n(x_0,r))^{-s} \int_0^\infty A_{\alpha,p,q}(E_t(r))^s \, dt^{sp} . \tag{4.3}$$

(Recall the definition of s from (2.5).)

4.4. Lemma. *Let $u \in \Lambda_\alpha^{p,q}(\mathbb{R}^n)$ with $\alpha p = n$. Then there exists a number $r_0 > 0$ and a constant $c = c(n, r_0, x_0) > 0$, such that for $[H_{p,q,h}, A_{\alpha,p,q}]$-a.e. x_0 and all $r < r_0$, when r and t are related by*

$$avA_{\alpha,p,q}(u,r)(x_0)^{1/2sp} < t, \tag{4.5}$$

we have

$$A_{\alpha,p,q}(E_t(r)) \geq c\left(\log \frac{|B^n(x_0,r)|}{|E_t(r)|}\right)^{p(1-q)/q} , \tag{4.6}$$

where $H_{p,q,h}$ is defined in (1.2).

Proof. By Lemma 2.4 and Lemma 4.1 we have for sufficiently small r

$$\frac{A_{\alpha,p,q}(E_t(r))}{A_{\alpha,p,q}(B^n(x_0,r))} \geq c\left(\frac{\log \frac{1}{|E_t(r)|}}{\log \frac{1}{|B^n(x_0,r)|}}\right)^{p(1-q)/q} \tag{4.7}$$

$$= c\left(\frac{\log \frac{|B^n(x_0,r)|}{|E_t(r)|} + \log \frac{1}{|B^n(x_0,r)|}}{\log \frac{1}{|B^n(x_0,r)|}}\right)^{p(1-q)/q}$$

$$= c\left(1 + \frac{\log \frac{|B^n(x_0,r)|}{|E_t(r)|}}{\log \frac{1}{|B^n(x_0,r)|}}\right)^{p(1-q)/q} . \tag{4.8}$$

By (4.3),

$$avA_{\alpha,p,q}(u,r)(x_0) \geq \frac{A_{\alpha,p,q}(E_t(r))^s}{A_{\alpha,p,q}(B^n(x_0,r))^s} t^{sp} \tag{4.9}$$

for all $r, t > 0$. For all r and t satisfying $avA_{\alpha,p,q}(u,r)(x_0)^{1/2sp} < t$,

$$avA_{\alpha,p,q}(u,r)(x_0)^{1/2} \geq \frac{A_{\alpha,p,q}(E_t(r))^s}{A_{\alpha,p,q}(B^n(x_0,r))^s} .$$

Since by Theorem 3.8 for $[H_{p,q,h}, A_{\alpha,p,q}]$-a.e. x_0 we have $avA_{\alpha,p,q}(u,r)(x_0) \to 0$ as r tends to zero,

$$\frac{A_{\alpha,p,q}(E_t(r))^s}{A_{\alpha,p,q}(B^n(x_0,r))^s} \to 0$$

uniformly for $t > avA_{\alpha,p,q}(u,r)(x_0)^{1/2sp}$. By (4.7) also

$$\left(1+\frac{\log\frac{|B^n(x_0,r)|}{|E_t(r)|}}{\log\frac{1}{|B^n(x_0,r)|}}\right)^{sp(1-q)/q} \to 0 \text{ as } r\to 0\,.$$

Hence, for $[H_{p,q,h}, A_{\alpha,p,q}]$-a.e. x_0 there is $r_0>0$ such that for all $r<r_0$ and all $t > avA_{\alpha,p,q}(u,r)(x_0)^{1/2sp}$,

$$A_{\alpha,p,q}(E_t(r)) \ge cA_{\alpha,p,q}(B^n(x_0,r))\left(\frac{\log\frac{|B^n(x_0,r)|}{|E_t(r)|}}{\log\frac{1}{|B^n(x_0,r)|}}\right)^{p(1-q)/q},$$

and by Lemma 2.4 there is $r_1>0$ such that for all $r<r_1\le r_0$ and all $t > avA_{\alpha,p,q}(u,r)(x_0)^{1/2sp}$,

$$A_{\alpha,p,q}(E_t(r)) \ge c\left(\log\frac{|B^n(x_0,r)|}{|E_t(r)|}\right)^{p(1-q)/q}. \qquad \square$$

5. Vanishing exponential integrability

We are ready to prove our main result.

Proof of Theorem 1.3. We take $\sigma := avA_{\alpha,p,q}(u,r)(x_0)^{1/sp}$ as in (4.3). Using the elementary inequality $\log t \le kt^{1/k}$, for all t and $k>0$, we can estimate (4.6) below by

$$\left(\log\frac{|B^n(x_0,r)|}{|E_t(r)|}\right)^{p(1-q)/q} \ge \left[k\left(\frac{|B^n(x_0,r)|}{|E_t(r)|}\right)^{1/k}\right]^{p(1-q)/q}. \tag{5.1}$$

Let $p>q$. Thus using (4.3), Lemma 2.4, (4.6) and (5.1) we have

$$\sigma^{sp} \ge ck^{p(1-q)/q}\left(\log\frac{1}{r}\right)^{p(q-1)/kq}|B^n(x_0,r)|^{p(1-q)/kq}\int_\sigma^\infty |E_t(r)|^{p(q-1)/kq}\,dt^{sp}.$$

Here,

$$\int_{\sigma^{1/2}}^\infty |E_t(r)|^{p(q-1)/kq}\,dt^{sp} = \int_{\sigma^{1/2}}^\infty \left(t^{kq/(q-1)}|E_t(r)|\right)^{p(q-1)/kq}\frac{dt}{t}\,.$$

If the integration were extended to $[0,\infty)$, then the above integral would be the classical $L(qk/(q-1),p)$-Lorentz norm of $|u-u(x_0)|$ over the ball $B^n(x_0,r)$ to the power p. Recall that $L(qk/(q-1),p)\subset L(qk/(q-1),qk/(q-1))$ as soon as $p<qk/(q-1)$, which is equivalent to $k>p(q-1)/q$; see [6, Chapter

4, Proposition 4.2]. Hence for $E_\sigma(r)$ (see (3.2)),

$$\begin{aligned}\int_\sigma^\infty |E_t(r)|^{p(q-1)/kq}\,dt^{sp} &\geq \|u-u(x_0)\|^p_{L(kq/(q-1),p)(E_\sigma(r))}\\ &\geq \|u-u(x_0)\|^p_{L(kq/(q-1),kq/(q-1))(E_\sigma(r))}\\ &= \|u-u(x_0)\|^p_{L^{kq/(q-1)}(E_\sigma(r))}\\ &= \left(\int_{E_\sigma(r)} |u(x)-u(x_0)|^{kq/(q-1)}\,dx\right)^{p(q-1)/kq}.\end{aligned}$$

Hence, for $p > q$ and $s = 1$,

$$|B^n(x_0,r)|^{-1}\int_{E_\sigma(r)} |u(x)-u(x_0)|^{kq/(q-1)}\,dx \leq c\left(\log\frac{1}{r}\right)^{-1} k^k\sigma^{spkq/p(q-1)}.$$

We use this to estimate the terms of the series expansion of the exponentials. Thus we write

$$\begin{aligned}&\fint_{B^n(x_0,r)} \left(\exp(\beta|u(x)-u(x_0)|^{q/(q-1)})-1\right)dx \\ &\qquad = \sum_{j=1}^\infty \frac{\beta^j}{j!}\fint_{B^n(x_0,r)} |u(x)-u(x_0)|^{jq/(q-1)}\,dx.\end{aligned}\tag{5.2}$$

We now break up this integral into two parts, corresponding to the sets $E_\sigma(r)$ and $B^n(x_0,r)\backslash E_\sigma(r)$. In the latter case, (5.2) does not exceed

$$\sum_{j=1}^\infty \frac{\beta^j}{j!}\sigma^{jq/(q-1)}.\tag{5.3}$$

In the former case with $k = j$, we obtain for the series the bound

$$\sum_{j=[p(q-1)/q]}^\infty \frac{\beta^j}{j!}j^j(c\sigma)^{jq/(q-1)}\tag{5.4}$$

for all $r < r_0$. The case $j < p(q-1)/q$ is handled by the Hölder inequality. Thus, since (5.3) and (5.4) tend to zero with σ, the vanishing exponential integrability results are valid when $p > q$.

The case $p \leq q$ is handled in a similar manner. □

6. Vanishing exponential integrability: the general case

The mth order Taylor polynomial of a function $u \in \Lambda_\alpha^{p,q}$ at x_0 is denoted by $P^m_{x_0}$.

6.1. THEOREM. *Let $u \in \Lambda_\alpha^{p,q}(\mathbb{R}^n)$, $0 < \alpha < n$. Let $m \in \mathbb{Z}^+$ be such that $1 \le m < \alpha$, and suppose further that $(\alpha - m)p = n$. Then there exists a constant $\beta > 0$ independent of u and $r > 0$ such that*

$$\fint_{B^n(x_0,r)} \left(e^{\beta(r^{-m}|u(y)-P^m_{x_0}(y)|)^{q/(q-1)}} - 1 \right) dy = o(1)$$

as $r \to 0$ for $[H_{p,q,h}, A_{\alpha-m,p,q}]$-a.e. $x_0 \in \mathbb{R}^n$, where $H_{p,q,h}$ is the Hausdorff capacity with the measure function $h(t) = (\log 1/t)^{1-q}$ as in (1.2).

Proof. From Taylor's formula for C^m-functions u,

$$u(y) = P^{m-1}_{x_0}(y) + m \sum_{|\gamma|=m} \frac{1}{\gamma!} \left[\int_0^1 (1-t)^{m-1} D^\gamma u((1-t)x_0 + ty)\, dt \right] (y - x_0)^\gamma.$$

If $u = \mathcal{H}_\alpha f$ with $f \in C_0^\infty(\mathbb{R}^n)$, then

$$|\mathcal{H}_\alpha f(y) - P^{m-1}_{x_0}(y)| \le m \sum_{|\gamma|=m} \frac{1}{\gamma!} \left| \int_0^1 D^\gamma \mathcal{H}_\alpha f((1-t)x_0 + ty)\, dt \right| |y - x_0|^m .$$

By (2.1),

$$|D^\gamma \mathcal{H}_\alpha f(y)| \le c \mathcal{H}_{\alpha-m} f(y),$$

where the function η occurring in the representation of $\mathcal{H}_{\alpha-m} f$ is different from the one used in the $\mathcal{H}_\alpha f$-case. This is an abuse of notation, but it is acceptable since the estimates we give depend only on f, and not on η. Let $y \in B^n(x_0, r)$. By the mean value theorem there is a point $t_0 = t_0(y) \in (0,1)$ such that

$$|\mathcal{H}_\alpha f(y) - P^{m-1}_{x_0}(y)| < c \mathcal{H}_{\alpha-m} f(x_0 + t_0(y - x_0)) r^m .$$

We may assume $s = 1$. Thus

$$\begin{aligned}
&\int_0^\infty A_{\alpha-m,p,q} \left(B^n(x_0,r) \cap \left[\frac{|\mathcal{H}_\alpha f(y) - P^{m-1}_{x_0}(y)|}{r^m} > \lambda \right] \right) d\lambda^p \\
&\quad \le c \int_0^\infty A_{\alpha-m,p,q} \left(B^n(x_0,r) \cap [\mathcal{H}_{\alpha-m} f(x_0 + t_0(y)(y - x_0)) > \lambda] \right) d\lambda^p \\
&\quad \le c \int_0^\infty A_{\alpha-m,p,q} \left(B^n(x_0,r) \cap [\mathcal{H}_{\alpha-m} f > \lambda] \right) d\lambda^p ,
\end{aligned}$$

where $|x_0 + t_0(y)(y - x_0)| \le t_0(y)|y - x_0| < r$. Write briefly

$$\mathcal{K}_\lambda(r; \mathcal{H}_\alpha f - P^{m-1}_{x_0}) = B^n(x_0,r) \cap \{y : |\mathcal{H}_\alpha f(y) - P^{m-1}_{x_0}(y)| > \lambda\}.$$

Taking the supremum over $r > 0$ we obtain

$$\begin{aligned}
&\sup_{r>0} r^{-mp} \frac{1}{A_{\alpha-m,p,q}(B^n(x_0,r))} \int_0^\infty A_{\alpha-m,p,q}(\mathcal{K}_\lambda(r; \mathcal{H}_\alpha f - P^{m-1}_{x_0}))\, d\lambda^p \\
&\quad \le \mathcal{M}_{\alpha-m,1}(\mathcal{H}_{\alpha-m} f)(x_0).
\end{aligned}$$

This reduces the general case to the basic one: We are able to apply Lemma 3.3 as in the proof of Theorem 3.8 to obtain

$$\lim_{r\to 0} r^{-mp} A_{\alpha-m,p,q}(B^n(x_0,r)) \int_0^\infty A_{\alpha-m,p,q}(\mathcal{K}_\lambda(r;\mathcal{H}_\alpha f - P^m_{x_0}))\, d\lambda^p = 0\,.$$

This is used in the same way as Theorem 3.8 was used in the proof of Theorem 1.3 to obtain the claim. The case $s = 1$ is complete. The case $s = q/p$ is proved in a similar manner. □

7. Differentiability for Besov functions

The (s,m)-differentiability means

$$\left(\fint_{B^n(x_0,r)} |u(y) - P^m_{x_0}(y)|^s\, dy\right)^{1/s} = o(r^m) \text{ as } r \to 0\,.$$

Here, $P^m_{x_0}$ denotes the $[m]$th order Taylor polynomial of u at x_0. We refer also to [11, Section 3.5]. J. R. Dorronsoro [7] proved differentiability results for functions in the Besov spaces; one of his theorems is as follows.

7.1. Theorem ([7, Theorem 2]). *If $u \in \Lambda^{p,q}_\alpha$, $1 \le p < \infty$, $\alpha p \le n$, $1 \le q \le p$ and β with $0 \le \beta < \alpha$ is given, u has an $(np/(n-\alpha p), \beta)$-differential $[H^{n-(\alpha-\beta)p}, A_{\alpha-\beta,p,q}]$-a.e.*

Theorems 1.3 and 6.1 imply the following result.

7.2. Theorem. *Let $u \in \Lambda^{p,q}_\alpha(\mathbb{R}^n)$, $1 < p < \infty$, $1 < q < \infty$, $(\alpha - m)p = n$ with $m \in [0,\alpha)$ given. Then for any $s < \infty$, u has an (s,m)-differential $[H_{p,q,h}, A_{\alpha-m,p,q}]$-a.e. $x \in \mathbb{R}^n$, where $H_{\alpha,p,q}$ is defined as in (1.2).*

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David R. Adams, University of Kentucky, Department of Mathematics, 715 Patterson Office Tower, Lexington, KY 40506–0027, USA

E-mail address: dave@ms.uky.edu

Ritva Hurri-Syrjänen, University of Helsinki, Department of Mathematics, P.O. Box 4, FIN-00014 University of Helsinki, Finland

E-mail address: ritva.hurri-syrjanen@helsinki.fi