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ON SURFACES WITH CONSTANT MEAN CURVATURE IN HYPERBOLIC SPACE

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Abstract. It is shown that for a complete surface with constant mean curvature $H > 1$ in $\mathbb{H}^3$ with boundary and finite index the distance function to the boundary is bounded. We apply this result to establish a sharp height estimate for certain geodesic graphs with noncompact boundary. We also show that a geodesically complete, embedded surface in $\mathbb{H}^3$ with constant mean curvature $H > 1$ and bounded Gaussian curvature is proper and has an $\epsilon-$tubular neighborhood on its mean convex side that is embedded. Finally, we use this last result to obtain a monotonicity formula for such a surface.

1. Introduction

Surfaces of constant mean curvature in Euclidean, spherical and hyperbolic space have been a natural object of investigation in the study of differential geometry of submanifolds. They appear as critical points to the variational problem of minimizing the area function for compactly supported variations that leave constant the volume "enclosed" by the surface, and have a natural physical interpretation as "soap bubbles".

In the present paper, we establish some results on surfaces with constant mean curvature in hyperbolic 3-space. They are extensions of recent results, not yet published, obtained by Meeks-Rosenberg and Ros-Rosenberg for surfaces with nonzero constant mean curvature in Euclidean 3-space.

It turns out that while surfaces with constant mean curvature 1 in hyperbolic space share many properties with minimal surfaces in Euclidean space (cf. [6], [9], [20]), surfaces of constant mean curvature greater than 1 in hyperbolic space are naturally associated with nonzero constant mean curvature surfaces in Euclidean space (cf. [14]). So, except for Proposition 3 in Section 3, all results in this paper are for surfaces with constant mean curvature greater than 1.

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We prove that for a complete surface with constant mean curvature $H > 1$ in $\mathbb{H}^3$ with boundary and finite index the distance function to the boundary is bounded (Theorem 1). In particular, if the boundary is compact, the surface is compact. If, in addition, the surface is strongly stable, the bound for this function is a constant that depends only on H (see Section 2 for definitions).

In 1969, Serrin [19] proved the following sharp height estimate for compact graphs with constant mean curvature in Euclidean space. If $\Omega \subset \mathbb{R}^2$ is a bounded domain in $\mathbb{R}^2$ and $f : \Omega \to \mathbb{R}$ is a smooth, positive function that equals zero on $\partial\Omega$ and whose graph M is a surface with nonzero constant mean curvature H in $\mathbb{R}^3$, then the highest point of M is within a distance at most $1/H$ from Ω.

In the work mentioned above, Rosenberg and Ros proved a Euclidean version of our Theorem 1 for nonzero constant mean curvature surfaces. By using this result, they were able to obtain Serrin's height estimate without assuming Ω to be bounded.

On the other hand, in [14], Korevaar et al. generalized Serrin's theorem to compact geodesic graphs (see Definition 1 in Section 3) with constant mean curvature $H > 1$ in hyperbolic space. They obtained the sharp height estimate $\operatorname{arctanh}(1/H)$. By applying Theorem 1, we show that the height estimate $\operatorname{arctanh}(1/H)$ is also valid for a general geodesic graph with constant mean curvature $H > 1$ in $\mathbb{H}^3$ defined on an unbounded domain Ω, provided it has finite index and boundary $\partial\Omega$.

It should be remarked that, unlike graphs with constant mean curvature in Euclidean space, geodesic graphs with constant mean curvature in hyperbolic space are not always stable. In Section 3, we give an example of an unbounded geodesic graph of constant mean curvature, defined on a non-compact domain Ω, with infinite index and boundary $\partial\Omega$. We also give a sufficient condition for a geodesic graph with nonzero constant mean curvature in $\mathbb{H}^3$ to be (strongly) stable (Proposition 3).

In [15] W. Meeks and H. Rosenberg obtained results on properness and volume growth of surfaces with nonzero constant mean curvature in Euclidean 3-space. Here, we extend some of these results to surfaces with constant mean curvature $H > 1$ in hyperbolic space.

We apply a corollary of Theorem 1 (previously proved by Silveira [20]) to show that a (geodesically) complete embedded surface in $\mathbb{H}^3$ with constant mean curvature $H > 1$ and bounded Gaussian curvature is proper. In turn, this result is used to prove that such a surface has an ϵ-tubular neighborhood on its mean convex side that is embedded. Then we apply this last property to obtain a monotonicity formula.

The paper is organized as follows. In Section 2 we recall the definition of stability of surfaces with constant mean curvature in hyperbolic space and some results related to this subject. We also state the interior maximum principle for the mean curvature equation, which is a tool frequently used throughout the paper. In Section 3 we prove the results on stability and height estimates for geodesic graphs with constant mean curvature $H > 1$. Finally, in Section 4, after reviewing some basic facts on laminations that will be used in the proof of the main theorem of this section, we prove the theorems on properness and volume growth mentioned above.

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2. Preliminaries

2.1. Stability of constant mean curvature surfaces in $\mathbb{H}^3$. In what follows, we fix some notation and recall some results on the stability of constant mean curvature surfaces in hyperbolic space. For further details on this subject we refer to [3] and [20].

We use the upper half-space model for $\mathbb{H}^3$, i.e.,

$$\mathbb{R}^3_+ = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3; x_3 > 0\right\},$$

endowed with the metric

$$\langle , \rangle = \frac{dx_1^2 + dx_2^2 + dx_3^2}{x_3^2}.$$

Let $\phi : M^2 \to \mathbb{H}^3$ be an isometric immersion of a smooth complete orientable surface M into the hyperbolic space $\mathbb{H}^3$. By *complete* we mean that all divergent paths in M have infinite length. Thus, with this definition, M can have nonempty boundary. As usual, sometimes we will identify M with its image under ϕ, $\phi(M)$.

Let $\mathbf{N}$ be a unit normal vector field on M. Recall that the second fundamental form B of M is given by

$$BX = \nabla_X \mathbf{N},\ X \in TM,$$

where ∇ stands for the Riemannian connection of $\mathbb{H}^3$.

If $\{e_1, e_2\}$ is an orthonormal frame on M, the *norm* of B is, by definition,

$$|B|^2 = \sum_{i,j=1}^{2} \langle \nabla_{e_i} \mathbf{N}, e_j \rangle^2.$$

The *mean curvature function* and the *mean curvature vector* on M are given, respectively, by

$$H = \frac{1}{2}\text{trace}\,(B) = \frac{1}{2}\sum_{i=1}^{2}\langle \nabla_{e_i}\mathbf{N}, e_i\rangle \quad \text{and} \quad \mathbf{H} = H\mathbf{N}.$$

Notice that, unlike the mean curvature function H, the mean curvature vector $\mathbf{H}$ does not depend on the chosen orientation $\mathbf{N}$. When H is not identically zero, we say that M is *oriented by its mean curvature vector* if $\langle \mathbf{H}, \mathbf{N} \rangle \geq 0$.

Henceforth, we call a surface with constant mean curvature $H \neq 0$ an *H-surface*.

Consider a bounded domain $D \subset M$ and a smooth variation of D, $\Phi : (-\epsilon, \epsilon) \times D \to \mathbb{H}^3$, that fixes the boundary; i.e., for each $t \in (-\epsilon, \epsilon)$, $\phi_t = \Phi(t, .)$ is an immersion, $\phi_0 = \phi$ and, for all t, $\phi_t|_{\partial D} = \phi|_{\partial D}$. The *area* and *volume* functions of Φ are

$$A(t) = \int_D dD_t, \quad V(t) = \int_{[o,t] \times D} \Phi^* d\mathbb{H}^3,$$

where dD_t is the volume element of D in the metric induced by ϕ_t and $\Phi^* d\mathbb{H}^3$ is the pull-back of the volume element of $\mathbb{H}^3$.

We call $F = \frac{\partial \Phi}{\partial t}|_{t=0}$ the *variation vector field* of Φ. The variation Φ is called *normal* if F is parallel to N, and *volume-preserving* if $V(t) = V(0)$ for all t. Let $f = \langle F, N \rangle$. In [3] it was shown that

$$\frac{dA}{dt}(0) = -\int_D 2HfdM \text{ and } \frac{dV}{dt}(0) = \int_D fdM.$$

Therefore, H-surfaces are characterized as critical points to the variational problem of minimizing the area function for volume-preserving variations that fix the boundary. In this context, questions related to stability naturally arise. So, if M has constant mean curvature $H \neq 0$, we say that D is *stable* if $A''(0) \geq 0$ for all volume-preserving variations of D that fix the boundary; otherwise we say that D is *unstable*. If $A''(0) \geq 0$ for all variations of D that fix the boundary (but are not necessarily volume-preserving), we say that D is *strongly stable*. M is called *stable* (resp. *strongly stable*), if all bounded domains $D \subset M$ are stable (resp. strongly stable).

Let

$$\mathcal{F}_D = \left\{ f \in C^\infty(D); f|_{\partial D} \equiv 0, \int_D fdM = 0 \right\},$$
$$\mathcal{G}_D = \{ f \in C^\infty(D); f|_{\partial D} \equiv 0 \}.$$

As was proved in [3], each $f \in \mathcal{F}_D$ determines a volume-preserving normal variation of D. Suppose M has constant mean curvature $H \neq 0$. Consider $f \in \mathcal{F}_D$ and let Φ be the variation of D determined by f. Define $J(t) = A(t) + 2HV(t), t \in (-\epsilon, \epsilon)$. A calculation gives

$$J''(0) = -\int_D \frac{\partial H_t}{\partial t}(0) f dM. \tag{1}$$

Propositions (2.5) and (2.7) in [3] give the following result.

Proposition 1 (Barbosa-do Carmo-Eschenburg). *$D \subset M$ is stable if and only if $J''(0) \geq 0$ for all $f \in \mathcal{F}_D$. Moreover, $\frac{\partial H_t}{\partial t}(0, p) = Lf(p)$. Here $L = \Delta + (|B|^2 - 2)$ is the stability operator of M and Δ is the Laplacian on M.*

Remark 1. In [2] the identity $\frac{\partial H_t}{\partial t}(0, p) = Lf(p)$ was actually proved for H-surfaces in Euclidean space. However, as the authors of [3] remarked, the proof applies to the more general case when the ambient space is a simply-connected complete Riemannian manifold with constant sectional curvature.

We call *index* of L in M (or *index* of M) the supremum, over compact domains $D \subset M$, of the number of negative eigenvalues of L acting on functions $f \in \mathcal{G}_D$. It should be noted that the eigenvalue problem we are considering here is $Lf + \lambda f = 0$, so M is strongly stable if and only if the first eigenvalue of L is non-negative. Also, by a result in [20], if M is stable, L has index at most one.

The following proposition was proved in [11].

Proposition 2 (Fischer-Colbrie). *If M has finite index, then there is a compact set Ω in M so that $M - \Omega$ is strongly stable and there exists a positive function u on M so that $Lu = 0$ on $M - \Omega$.*

Remark 2. As was shown in the proof of Proposition 2 in [11], if M is strongly stable, then the set Ω is empty. Thus, in this case, the positive function u in the statement satisfies $Lu = 0$ on all of M.

2.2. The maximum principle for the mean curvature equation. In many of the results obtained here, the interior maximum principle for the mean curvature equation, as stated below, will play a crucial role. This principle is based on the fact that a function f whose graph has mean curvature H (not necessarily constant) satisfies a quasi-linear elliptic equation. Details and proofs can be found, for example, in [10].

Suppose M_1 and M_2 are two smooth oriented surfaces of $\mathbb{H}^3$ which are tangent at an interior point p and have at p the same oriented normal. In this case p is called a *point of common tangency*, and we say that M_1 *lies above* M_2 *near* p, if, when we express the surfaces M_1 and M_2 (near p) as graphs of functions f_1 and f_2 over the common tangent plane through p, we have $f_1 \geq f_2$ in a neighborhood of p.

Maximum Principle. *Let M_1 and M_2 be oriented surfaces in $\mathbb{H}^3$ and H_1 and H_2 their respective mean curvature functions. Suppose M_1 and M_2 have a point p of common tangency. Then, if $H_1 \leq H_2$ near p, it is not true that M_1 lies above M_2, unless M_1 coincides with M_2 near p.*

3. Stability and height estimates for geodesic H-graphs

THEOREM 1. *Let (M, ds^2) be a complete H-surface in $\mathbb{H}^3$, $H > 1$, with possibly nonempty boundary ∂M and finite index. Then there is a constant $C > 0$ such that, for all p in M,*

$$\operatorname{dist}(p, \partial M) < C.$$

Proof. Consider the stability operator of M in $\mathbb{H}^3$, $L = \Delta + (|B|^2 - 2)$. By Proposition 2 in Section 2, there is a compact set Ω in M and a differentiable function $u : M \to \mathbb{R}$, $u > 0$, such that $Lu = 0$ on $M - \Omega$. Denoting by K the intrinsic Gaussian curvature of M, we have $|B|^2 = 4H^2 - 2(K+1)$. So, on $M - \Omega$,

$$Lu = \Delta u - 2Ku + (4H^2 - 4)u = 0. \tag{2}$$

Given $p \in M - \Omega$, consider a geodesic ball $B_R(p)$ in $M - \Omega$ with center at p and radius R such that $\partial B_R(p) \cap \partial(M - \Omega)$ is empty. As was proved in [11], there exists a curve $\gamma(s) : [0, \bar{R}] \to M - \Omega$, from p to $\partial B_R(p)$, parametrized by arclength in the metric ds^2, that is a minimizing geodesic in the metric $u^2 ds^2$. Since Ω is compact and clearly $\bar{R} \geq R$, it is sufficient to prove that there exists a constant $C_0 > 0$ satisfying $\bar{R} < C_0$.

Let $\tilde{s}$ be the arclength parameter of γ in the metric $u^2 ds^2$ and $\bar{R}_0$ the length of γ in this metric. Consider a variation of γ given by $f\eta$, where η is the unit normal of $\gamma(\tilde{s})$ and f is a differentiable function satisfying $f(0) = f(\bar{R}_0) = 0$.

Since γ is a minimizing geodesic in the metric $u^2 ds^2$, by the second variation formula we have

$$\int_0^{\bar{R}_0} \left\{ \left(\frac{df}{d\tilde{s}} \right)^2 - \widetilde{K} f^2 \right\} d\tilde{s} \geq 0, \tag{3}$$

where $\widetilde{K}$ denotes the intrinsic Gaussian curvature of M in the $u^2 ds^2$ metric. $\widetilde{K}$ is related to K by

$$\widetilde{K} = \frac{1}{u^2}(K - \Delta \ln u), \qquad \Delta = \Delta_{ds}. \tag{4}$$

Denoting by ∇ the gradient on (M, ds^2), we have

$$\Delta \ln u = \frac{1}{u^2}(u\Delta u - |\nabla u|^2). \tag{5}$$

Since $H^2 - K \geq 1$, (2) and (5) give

$$K - \Delta \ln u \geq c + \frac{|\nabla u|^2}{u^2}, \qquad c = 3(H^2 - 1) > 0.$$

Using this last inequality, (3), (4) and the identity $df/d\tilde{s} = u^{-1}df/ds$, we obtain

$$\int_0^{\bar{R}} \left(\frac{c}{u} + \frac{|\nabla u|^2}{u^3}\right) f^2 ds \le \int_0^{\bar{R}} \left(\frac{K - \Delta \ln u}{u}\right) f^2 ds = \int_0^{\bar{R}} \widetilde{K} f^2 u ds \tag{6}$$

$$\le \int_0^{\bar{R}} \left(\frac{df}{d\tilde{s}}\right)^2 u ds = \int_0^{\bar{R}} \frac{1}{u} \left(\frac{df}{ds}\right)^2 ds.$$

Now, let ψ be such that $f = \sqrt{u}\psi$, $\psi(0) = \psi(\bar{R}) = 0$. Then,

$$\frac{1}{u}(f')^2 = (\psi')^2 + u^{-1}u'\psi\psi' + \frac{1}{4}u^{-2}(u')^2\psi^2, \tag{7}$$

where $'$ denotes derivation with respect to the variable s. From (6),

$$\int_0^{\bar{R}} \frac{1}{u}(f')^2 ds \ge \int_0^{\bar{R}} \left(c + \frac{|\nabla u|^2}{u^2}\right) \psi^2 ds.$$

Since $u'(s) = \langle \nabla u, \gamma'(s) \rangle$, we have $(u')^2 \le |\nabla u|^2$. From this, (7), and the last inequality above we get

$$\int_0^{\bar{R}} \left((\psi')^2 + u^{-1}u'\psi\psi' - \frac{3}{4}u^{-2}(u')^2\psi^2 - c\psi^2\right) ds \ge 0.$$

Applying the inequality $a^2 + b^2 \ge 2ab$ with $a = (\sqrt{6}/2)u^{-1}u'\psi$ and $b = (\sqrt{6}/3)\psi'$ we obtain

$$\frac{3}{4}u^{-2}(u')^2\psi^2 + \frac{1}{3}(\psi')^2 \ge u^{-1}u'\psi\psi'.$$

Therefore,

$$\int_0^{\bar{R}} \left(\frac{4}{3}(\psi')^2 - c\psi^2\right) ds \ge 0.$$

Finally, integration by parts gives

$$\int_0^{\bar{R}} \left(\frac{4}{3}\psi'' + c\psi\right) \psi ds \le 0. \tag{8}$$

We now take $\psi(s) = \sin\left(\pi s/\bar{R}\right)$, $s \in [0, \bar{R}]$. Then (8) yields

$$\int_0^{\bar{R}} \left(c - \frac{4\pi^2}{3\bar{R}^2}\right) \sin^2\left(\frac{\pi s}{\bar{R}}\right) ds \le 0.$$

Thus we must have

$$c - \frac{4\pi^2}{3\bar{R}^2} \le 0,$$

that is,

$$\bar{R} \le \frac{2}{3}\frac{\pi}{\sqrt{H^2 - 1}},$$

as desired. □

Corollary 1. *Let M be a complete H-surface in $\mathbb{H}^3$, $H > 1$, with finite index and compact boundary. Then M is compact.*

Remark 3. In the case when ∂M is empty the result of Corollary 1 was obtained by Silveira [20].

In the proof of Theorem 1 above, we have shown that if R is the radius of a geodesic ball whose closure is in $M - \Omega$, then R is bounded by a constant that depends only on H. From this and (the proof of) Proposition 2 in Section 3 (see Remark 2), we have the following corollary.

Corollary 2. *If M is a complete, strongly stable H-surface with non-empty boundary in $\mathbb{H}^3, H > 1$, then for all $p \in M$*

$$\operatorname{dist}(p, \partial M) \leq \frac{2}{3} \frac{\pi}{\sqrt{H^2 - 1}}.$$

Remark 4. Since Proposition 2 is still valid when M is an H-surface in an arbitrary complete 3-manifold (see [11]), Theorem 1, as well as Corollaries 1 and 2, still hold if $\mathbb{H}^3$ is replaced by a complete 3-manifold with constant sectional curvature -1.

Remark 5. We do not know if the constant $C(H) = (2/3)\pi/\sqrt{H^2 - 1}$ in Corollary 2 is sharp. For example, if S is a geodesic sphere in $\mathbb{H}^3$ with constant mean curvature H, it is well known that a hyperbolic hemisphere S' of S is strongly stable. Let $p \in S'$ be the farthest point from $\partial S'$. An easy calculation gives

$$\operatorname{dist}(p, \partial S') = \frac{1}{2} \frac{\pi}{\sqrt{H^2 - 1}} = \frac{3}{4} C(H).$$

Definition 1. A *geodesic graph* in $\mathbb{H}^3$ is a graph in the following system of coordinates: Let Ω be a domain in a totally geodesic plane P and let ρ be a real function that associates to each $q \in \Omega$ a point on the geodesic γ_q, through q, orthogonal to P, at hyperbolic distance $\rho(q)$ from P. A *geodesic H-graph* is a geodesic graph with constant mean curvature $H \neq 0$.

Although H-graphs in $\mathbb{R}^3$ are strongly stable, the behavior of geodesic H-graphs in $\mathbb{H}^3$ is quite different, as is shown by the following example.

Example 1. Consider a Euclidean cone in $\mathbb{R}^3$ with vertex at the origin and axis parallel to $(0, 0, 1)$. The part of this cone in $\mathbb{R}^3_+$ is a cylinder $\mathcal{C}$ with constant mean curvature greater than 1 in $\mathbb{H}^3$. Let P be a totally geodesic plane of $\mathbb{H}^3$ that intersects the axis of $\mathcal{C}$ orthogonally at a point $O \in \mathbb{H}^3$. $\mathcal{C}$ divides $\mathbb{H}^3$ into two components, one of which is mean convex, namely the component to which the mean curvature vector of $\mathcal{C}$ points. Let Ω be the intersection of the mean convex component with P. Then the part of $\mathcal{C}$ above

P is an unbounded geodesic graph M over $\Omega - \{O\}$ with boundary in P (see Figure 1). By computing the eigenvalues of the stability operator of $\mathcal{C}$, it can be shown that M has infinite index (cf. [4]). Since on M the distance function to the boundary is unbounded, Theorem 1 also implies that M has infinite index.

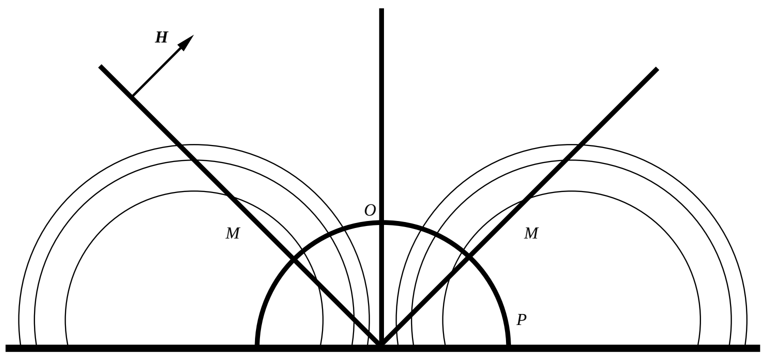

FIGURE 1. A geodesic H-graph in $\mathbb{H}^3$ with infinite index.

Let $\mathcal{C}$ be the cylinder of the example above and $\mathcal{C}^-, \mathcal{C}^+$ the two connected components of $\mathcal{C}$ determined by the vertical totally geodesic plane $Q = \{(x_1, x_2, x_3) \in \mathbb{H}^3; x_2 = 0\} \subset \mathbb{H}^3$. As was proved in [4, Prop. 1, p. 108], each of these components is a strongly stable geodesic graph over a domain $\Omega \subset Q$. More generally, it was proved in [4] that half-Delaunay surfaces in $\mathbb{H}^{n+1}$ are strongly stable. Notice that $\mathcal{C}^-$ and $\mathcal{C}^+$ are also Euclidean graphs over their Euclidean orthogonal projections to Q. This motivates the following definition and proposition.

DEFINITION 2. Let $Q \subset \mathbb{H}^3$ be a vertical totally geodesic plane in $\mathbb{H}^3$. We say that a geodesic H-graph M over a domain in Q is *horizontal* if M is also a Euclidean graph over its Euclidean orthogonal projection to Q, i.e., if there is a vector $\mathbf{e}$, normal to Q, such that $\langle \mathbf{H}, \mathbf{e} \rangle > 0$, where $\mathbf{H}$ is the mean curvature vector of M.

PROPOSITION 3. *Let M be a horizontal geodesic H-graph in $\mathbb{H}^3$. Then M is strongly stable.*

Proof. Consider in M the orientation given by its mean curvature vector $\mathbf{H}$. Let $D \subset M$ be a compact domain in M and λ the first eigenvalue of the stability operator L acting on functions $f \in \mathcal{G}_D$. To prove that D is strongly stable, it is sufficient to prove that $\lambda \geq 0$.

Suppose $\lambda < 0$ and let $f \in \mathcal{G}_D$ be an eigenfunction of L associated to λ, i.e., $Lf + \lambda f = 0$. Let ϕ_t be the normal variation of D determined by f. Since $-f$ is also an eigenfunction of L associated to λ and f does not change sign in D (see [8, p. 20]), we can assume that f is positive on D. So, for all $p \in D$ we have

$$\frac{\partial H_t}{\partial t}(0, p) = Lf(p) = -\lambda f(p) > 0. \tag{9}$$

Given $\delta > 0$, denote by D_δ the image of D under a horizontal translation of magnitude δ in the $\mathbf{e}$ direction. Since M is horizontal, D and D_δ are disjoint. Now, $f > 0$ and $f|_{\partial D} \equiv 0$. Thus there exist t and δ such that $\phi_t(D)$ and D_δ have an interior point of common tangency q, with D_δ above $\phi_t(D)$ near q. Therefore, since horizontal translations are isometries of $\mathbb{H}^3$, the maximum principle implies $H \geq H_t(q)$ (see Figure 2).

On the other hand, by (9), the mean curvature of the variation ϕ_t increases with respect to t, i.e., $H_t(q) > H$. This contradiction proves that $\lambda \geq 0$ and therefore D is strongly stable. □

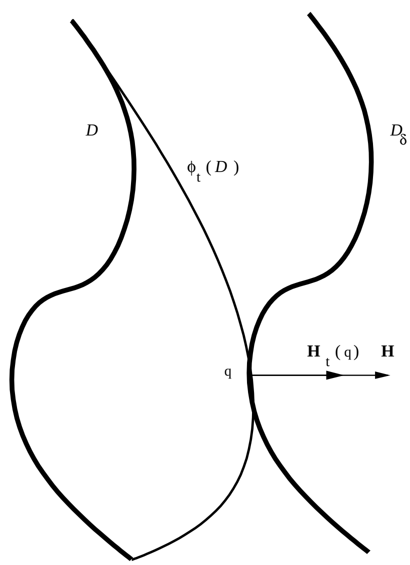

FIGURE 2

REMARK 6. After ad hoc modifications in Definitions 1 and 2, Proposition 3 can be easily extended to the hyperbolic spaces $\mathbb{H}^{n+1}$, $n > 2$. Therefore, it generalizes the hyperbolic part of Proposition 1 in [4] mentioned above.

THEOREM 2. *Let P be a totally geodesic plane in $\mathbb{H}^3$, $\Omega \subset P$ an unbounded domain in P with boundary $\partial\Omega$ and $\rho \in C^\infty(\Omega)$ a positive function such that $\rho|_{\partial\Omega} \equiv 0$. Let M be the geodesic graph determined by ρ. If M is an H-surface of $\mathbb{H}^3$, $H > 1$, with finite index, then*

$$\sup_\Omega \rho \leq \operatorname{arctanh}\left(\frac{1}{H}\right).$$

Proof. Given $p \in M$, denote by $\rho(p)$ the hyperbolic distance to p from P. M has finite index and, for all $p \in M$, $\rho(p) \leq \operatorname{dist}(p, \partial M)$. Hence, by Theorem 1, ρ is bounded. Suppose there is a point $p \in M$ and a constant $\delta > 0$ satisfying

$$\rho(p) = d + \delta, \quad d = \operatorname{arctanh}\left(\frac{1}{H}\right).$$

After an ambient isometry, we can assume P is the plane $\{x_1^2 + x_2^2 + x_3^2 = 1\}$ and the orthogonal projection of p on P is $O = (0, 0, 1)$. Let γ be the vertical geodesic through O and P^* the totally geodesic plane of $\mathbb{H}^3$, orthogonal to γ, at a distance $\delta/2$ from O. Clearly, the part of M above P^* is a geodesic H-graph M^* over a domain in P^*, with $p \in M^*$ and $\partial M^* \subset P^*$. Moreover, since ρ is bounded and the distance function from P^* to P is unbounded, M^* is compact (see Figure 3). Then, from (the proof of) Lemma 3.3 in [14], we obtain that all points $q \in M^*$ satisfy $\rho^*(q) \leq d$, where $\rho^*(q)$ denotes the distance to q from P^*. But, by construction, $\rho^*(p) = d + \delta/2$. This contradiction shows any point $p \in M$ satisfies $\rho(p) \leq d$ and proves the theorem. □

REMARK 7. The above height estimate is best possible for, given $H > 1$, there exists a cylinder $\mathcal{C}$ in $\mathbb{H}^3$ with constant mean curvature H. The points of the (strongly stable) geodesic graphs $\mathcal{C}^+$ and $\mathcal{C}^-$ defined above that project on the axis of $\mathcal{C}$ are within a distance $d = \operatorname{arctanh}(1/H)$ from Q. Hence the assertion follows.

4. Properness and volume growth of H-surfaces in $\mathbb{H}^3$

4.1. Holonomy of laminations. In the proof of the main theorem of this section, we shall deal with holonomy of laminations. We therefore summarize some concepts and results on this subject. For a detailed presentation of this theory, we refer the reader to Chapter 11 of [7].

Intuitively, a lamination of a smooth manifold M is a foliation of M with some leaves removed. More precisely, let M be a smooth manifold which is covered by a collection of open sets U_i such that, for each of these sets, there exists a diffeomorphism $\varphi_i : U_i \to V_i \times W_i \subset \mathbb{R}^k \times W$, where W is a closed subset of $\mathbb{R}^l$ and V_i and W_i are open in $\mathbb{R}^k$ and W, respectively. M is called a smooth *lamination* if the overlap maps $\varphi_j \varphi_i^{-1}$ are of the form

$$(x, \lambda) \to (\varphi_{ij}(x, \lambda), \Lambda_{ij}(\lambda)),$$

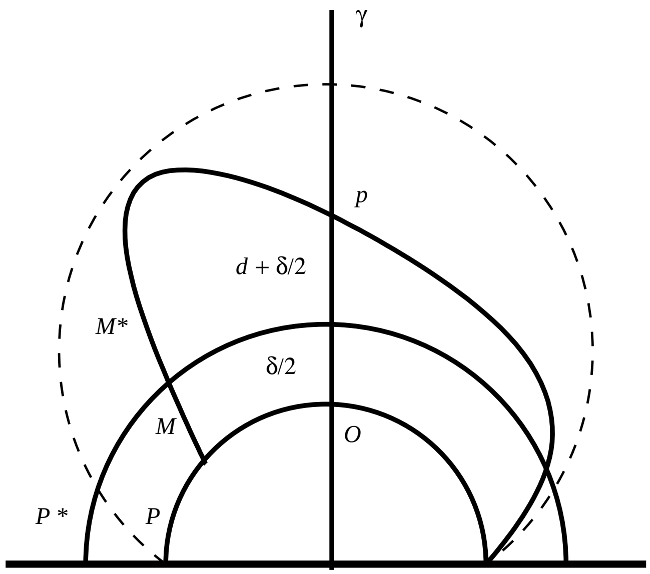

FIGURE 3

where $x \to \varphi_{ij}(x,\lambda)$ is smooth for each λ. The charts (U_i,φ_i) are called *lamination charts*. We define a relation in M by saying that two points p,q in M are related if, for any $\lambda \in W$, there are subsets $\alpha_i = \varphi_i^{-1}(V_i \times \lambda), i = 1,\dots,n$, such that $p \in \alpha_1, q \in \alpha_n$, and $\alpha_i \cap \alpha_{i+1}$ is nonempty. The equivalence classes of this relation are called the *leaves* of the lamination. Thus, a lamination is defined in the same way as a foliation except that, in local charts, where the leaves of a foliation are the submanifolds $\mathbb{R}^k \times \{\lambda\}, \lambda \in \mathbb{R}^l$, the leaves of a lamination are the submanifolds $\mathbb{R}^k \times \{\lambda\}$, for λ in some closed subset of $\mathbb{R}^l$.

Let L be a leaf of a lamination M and $p \in M$. A *transversal* to L through p, Σ_p, is a set $\varphi_i^{-1}(\{x\} \times W_i)$, where (U_i,φ_i) is a lamination chart such that $p \in U_i$ and $\varphi_i(p) = x$. If γ is a loop in L based at p, then γ is covered by a chain of lamination charts $U_i,\dots,U_n$. Thus, if L' is a leaf close enough to L and $q \in L' \cap \Sigma_p$, we can use this chain to "lift" γ to a path γ_q, with initial point q. Define

$$\begin{array}{rccc} h_\gamma : & \Sigma_p & \to & \Sigma_p \\ & q & \to & \gamma_q(1)\,, \end{array}$$

where $\gamma_q(1)$ is the endpoint of γ_q. For suitably chosen Σ_p, h_γ is a diffeomorphism with $h_\gamma^{-1} = h_{-\gamma}$.

Recall that given topological spaces X, Y and $g : X \to Y$, the *germ* of g at $x_0 \in X$ is the equivalence class of functions that coincide with g in a neighborhood of x_0 in X.

Denote by $\widehat{h}_\gamma$ the germ of h_γ at p and let

$$\mathcal{H}_p(L) = \left\{ \widehat{h}_\gamma\,;\, \gamma \text{ is a loop in } L \text{ based at } p \right\}.$$

The set $\mathcal{H}_p(L)$ is easily given a group structure by composition, i.e., $\widehat{h}_\gamma \widehat{h}_{\gamma'} = \widehat{h_\gamma \circ h_{\gamma'}}$, and is called the *holonomy group* of L at p.

Now, it can be shown that if γ and γ' are homotopic loops in L based at p, then $h_\gamma \equiv h_{\gamma'}$. Therefore, the map

$$\begin{array}{rccc} \widehat{h}: & \pi_1(L,p) & \to & \mathcal{H}_p(L) \\ & \langle\gamma\rangle & \to & \widehat{h}_\gamma \end{array}$$

is a group homomorphism, called the *holonomy homomorphism* of the leaf L at p.

Holonomy is a natural tool to study the behavior of the leaves in a neighborhood of a leaf L of a lamination. In this context, a very important result is the Reeb stability theorem (see [7, Prop. 11.4.8, p. 301]). Applied to the particular case when the leaf under consideration is compact and simply-connected, the Reeb stability theorem yields the following result.

THEOREM 3 (Reeb). *Let L be a compact simply-connected leaf of a lamination M and $p \in L$. Then there is a transversal Σ_p through p such that all leaves of M meeting Σ_p are homeomorphic to L.*

4.2. Properness and volume growth. In the sequel, all *complete* surfaces in $\mathbb{H}^3$ are assumed to be *geodesically complete*, i.e., with empty boundary.

THEOREM 4. *Let M be a complete embedded H-surface in $\mathbb{H}^3$, $H > 1$, with bounded Gaussian curvature. Then M is properly embedded.*

Proof. Suppose that M is not proper. In this case, since M is embedded, there is an accumulation point p of M such that $p \notin M$ (see [13, p. 38]). We shall show that there exists a complete embedded H-surface in $\mathbb{H}^3$ containing p and disjoint from M.

Let $\{p_k\} \subset M$ be a sequence of points on M converging to p. Since M is an H-surface with bounded Gaussian curvature, the second fundamental form of M is bounded. Therefore there is a $\delta > 0$ such that for all $q \in M$, M is locally a graph over a disk of radius δ in T_qM, $D_\delta(q)$, centered at the origin. Notice that the uniform boundedness of the second fundamental form of M gives also a C^1-uniform bound for the function defining this local graph since, in this case, the unit normal vector field of M has bounded variation. So, this function, as well as its gradient, are bounded by a constant independent of q.

Suppose the tangent planes $T_{p_k}M$ converge to a (Euclidean) plane P at p. Fix one of the two connected components of $\mathbb{H}^3 - P$ and consider η, the unit normal vector to P at p that points to this component. Take a subsequence of $\{p_k\}$, which we also denote by $\{p_k\}$, such that $\langle \mathbf{H}(p_k), \eta\rangle$ has a fixed sign; i.e., we consider only those points p_k for which $\mathbf{H}(p_k)$ points to the same component. Since M is locally a graph over each $D_\delta(p_k) \subset T_{p_k}M$, for k sufficiently large, each of these local graphs is a graph of a function u_k on $D_{\delta/2}(p) \subset P$. Each u_k is a solution of a quasi-linear elliptic partial differential equation of second order (the mean curvature equation) with a C^1 uniform bound. Therefore there is a subsequence u_{k_n} of u_k and a differentiable function u_∞ on $D_{\delta/2}(p)$ such that u_{k_n} converges to u_∞ in the C^∞-topology and the graph of u_∞ is an H-graph (see [12]). Let L be the H-graph of u_∞ and notice that L is tangent to P at p. Observe that L does not depend on the choice of the sequence p_k nor on the choice of sequence of convergent tangent planes. Indeed, if $T_{q_k}M$ converges to a plane Q at p, $Q \neq P$, then Q is transversal to P. So, for large k , the local graphs at p_k and q_k intersect, which contradicts the assumption that M is embedded. Moreover, L and M are disjoint; otherwise, by the maximum principle, L and M would intersect transversally, which contradicts again the embeddedness of M.

Now the sequence $(p_k, \mathbf{N}(p_k))$ is bounded in the unit normal bundle of M. Thus there is a subsequence of $T_{p_k}M$ that converges to a plane P at p. Since the limit tangent plane is unique, it follows that $T_{p_k}M$ converges.

Notice that all boundary points of L are accumulation points of M. Therefore, by reasoning as above for these points, we obtain an H-surface in $\mathbb{H}^3$ that we will also denote by L, containing p and disjoint from M. Moreover, L is geodesically complete and embedded. The embeddedness of L follows from the embeddedness of M, for if L were not embedded in a neighborhood of a point $\bar{p} \in L$, there would be local graphs of M near $\bar{p}$ that would intersect. That L is geodesically complete can be seen as follows. As we have seen, L is covered by local graphs of functions defined on disks of radius $\delta/2$ of their tangent planes. Therefore, for each $\bar{p} \in L$, there is a lower bound l_0 for the length of the geodesic rays of L issuing from $\bar{p}$ and contained in the local graph of L at $\bar{p}$. Now let $\gamma : [0, s_0) \to L$ be a geodesic in L, not defined at s_0, and choose $s' < s_0$ such that $s_0 - s' < l_0$. Since the geodesic ray issuing from $\bar{p} = \gamma(s')$ in the direction $\gamma'(s')$ has length greater than l_0, it extends γ to s_0. This shows that L is geodesically complete. Thus the closure of M in $\mathbb{H}^3$, $\overline{M}$, is a complete (not connected) embedded H-surface of $\mathbb{H}^3$.

We give $\overline{M}$ a lamination structure by choosing local charts (U_i, φ_i) such that the open sets U_i are contained in the local graphs described above. It is easily seen that these charts are lamination charts and that the leaves of this lamination are M and the accumulation surfaces L. Let L be a leaf in $\overline{M}$,

$L \neq M$, and consider the ϵ-normal bundle of L in $\mathbb{H}^3$,

$$N_\epsilon(L) = \left\{(p, v); p \in L, v \in T_pL^\perp, |v| \leq \epsilon\right\}.$$

Let $\widetilde{N_\epsilon(L)}$ be the universal covering space of $N_\epsilon(L)$ and notice that $\widetilde{N_\epsilon(L)}$ is diffeomorphic to $\widetilde{L} \times D(\epsilon)$, where $\widetilde{L}$ is the universal covering space of L and $D(\epsilon)$ is a geodesic segment of $\mathbb{H}^3$ of length 2ϵ.

Since L has bounded second fundamental form, for some $\epsilon > 0$, there are no focal points of L within a distance ϵ from L. Thus the projection of $\widetilde{N_\epsilon(L)}$ in $\mathbb{H}^3$ via the exponential map of $N_\epsilon(L)$ into $\mathbb{H}^3$ is a local immersion. Considering in $\widetilde{N_\epsilon(L)}$ the metric induced by this projection, we see that the lamination $\overline{M}$ determines a lamination of $\widetilde{N_\epsilon(L)}$ by H-surfaces.

Suppose that $\widetilde{L}$ is stable. Then, since $H > 1$, by Corollary 1 (see also Remark 4) $\widetilde{L}$ is compact. Hence L is compact, and since L is an embedded H-surface, by a theorem of Alexandrov [1], L is a geodesic sphere. In particular, L is simply connected. Notice that L is not an isolated leaf since M accumulates on L. Thus, by Theorem 3, the liftings of L via the holonomy of the lamination to the leaves near L are homeomorphisms. Therefore these leaves are geodesic spheres, pairwise disjoint, and of constant mean curvature H, which clearly is a contradiction. Thus it suffices to show $\widetilde{L}$ is stable.

Since $\widetilde{L}$ is simply-connected, we can exhaust it by compact, simply connected domains D. Consider in $\widetilde{L}$ the orientation given by its mean curvature vector $\mathbf{H}$. As in the proof of Proposition 3, suppose that λ, the first eigenvalue of the stability operator L of M, is negative. If $f \in \mathcal{G}_D$ satisfies $Lf + \lambda f = 0$, we can assume f to be positive on D, which implies $\frac{\partial H}{\partial t}(0, p) > 0$. The liftings of D, via the holonomy of the lamination, to the leaves near D, give distinct H-disks D_i that are pairwise disjoint and which project diffeomorphically onto D by the normal projection of $\widetilde{N_\epsilon(L)}$ to $\widetilde{L}$. Therefore, each D_i is a graph, in the normal bundle, of functions f_i on D such that $f_i \to 0$. Suppose that, for all i, $f_i > 0$. Since the variation ϕ_t determined by f fixes the boundary of D, there exist t_0 and i such that $\phi_{t_0}(D)$ and D_i have an interior point of common tangency q, with D_i above $\phi_{t_0}(D)$ near q. Thus the maximum principle implies $H \geq H_{t_0}$ at q, which contradicts the fact that the mean curvature increases. If the functions f_i are negative, we obtain a contradiction by replacing, in the argument above, the function f by the function $-f$. Therefore $\widetilde{L}$ is strongly stable and this concludes the proof. □

Let M be an H-surface in $\mathbb{H}^3$ and $N_\epsilon(M)$ the ϵ-normal bundle of M. Denote by $N^*_\epsilon(M)$ the set of points (p, v) in $N_\epsilon(M)$ at which $\langle \mathbf{H}(p), v\rangle \geq 0$.

Theorem 5. *Let M be a complete embedded H-surface in $\mathbb{H}^3$, $H > 1$, with bounded Gaussian curvature. Then there exists an $\epsilon > 0$ such that the exponential map*

$$\begin{array}{rccc} \exp : & N_\epsilon^*(M) & \to & \mathbb{H}^3 \\ & (p, v) & \to & \exp_p(v)\,, \end{array}$$

is an embedding.

Proof. By Theorem 4, M is proper. Suppose that for all $\epsilon > 0$, the exponential map from $N_\epsilon^*(M)$ to $\mathbb{H}^3$ is not an embedding. Since M has bounded Gaussian curvature, its second fundamental form is bounded. Thus there is an $\epsilon > 0$ such that $\exp : N_\epsilon^*(M) \to \mathbb{H}^3$, is a local immersion.

Since we are assuming exp is not an embedding, we can suppose that there exist points $p_i, q_i \in M$ satisfying $q_i = \exp_{p_i}(t_i \mathbf{N}(p_i))$ with $t_i \to 0$. Here, $\mathbf{N} = \frac{1}{H}\mathbf{H}$ and t_i is the smallest positive number such that $\exp_{p_i}(t_i \mathbf{N}(p_i))$ is in M (see Figure 4).

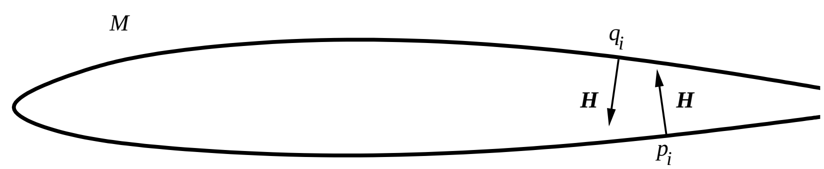

Figure 4

Notice that M separates $\mathbb{H}^3$ and that the mean curvature vectors of M at p_i and q_i point to the same component. Thus, for i sufficiently large we have

$$\langle \mathbf{H}(p_i), \mathbf{H}(q_i) \rangle \leq 0. \tag{10}$$

As before, the boundedness of the second fundamental form of M gives a $\delta > 0$ such that, for all $p \in M$, there is a neighborhood of p in M which is a graph over the disk of radius δ in T_pM centered at the origin. Fix a point $O \in \mathbb{H}^3$ and a (Euclidean) plane P of the tangent space of $\mathbb{H}^3$ at O. For each i, take an isometry of $\mathbb{H}^3$ that maps p_i to O and $T_{p_i}M$ to P, and let M_i and M_i' be, respectively, the images of the local graphs at p_i and q_i under this isometry. As in the proof of Theorem 4, there are H-graphs in $\mathbb{H}^3$, M_∞ and M_∞', and subsequences M_{i_k} and M_{i_k}' such that $M_{i_k} \overset{C^\infty}{\to} M_\infty$ and $M_{i_k}' \overset{C^\infty}{\to} M_\infty'$. Obviously, the tangent plane of M_∞ at O is P. Since M is embedded, the same is true for M_∞'. Therefore, M_∞ and M_∞' are H-graphs of functions $f, f' : D \subset P \to \mathbb{R}$, where D is a disk in P containing O, with O as an interior point of common tangency. Now, take the mean curvature vector of M_∞ at O, $\mathbf{H}$, as the positive direction for the construction of the graphs of f and f', and let H' be the mean curvature function of M_∞' in this orientation. Then, by the embeddedness of M and inequality (10), we have on D

$$f \leq f' \quad \text{and} \quad H' < 0 < H,$$

which clearly contradicts the maximum principle (see Figure 5). This shows that for some $\epsilon > 0$, $N^*_\epsilon(M)$ embeds in $\mathbb{H}^3$ under the exponential map, as claimed. □

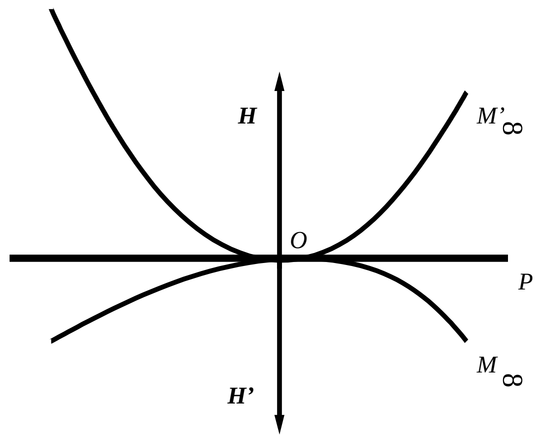

FIGURE 5

For the proof of the next theorem, we shall use the fact that the group of isometries of $\mathbb{H}^3$ has a discrete subgroup whose fundamental region has compact closure. If Γ is a discrete subgroup of the group of isometries of $\mathbb{H}^3$, a *fundamental region* for Γ is a connected open set $\mathcal{R}$ in $\mathbb{H}^3$ with the following properties:

- The members of $\{g\mathcal{R}; g \in \Gamma\}$ are mutually disjoint.
- $\mathbb{H}^3 = \bigcup \{g\overline{\mathcal{R}}; g \in \Gamma\}$, where $\overline{\mathcal{R}}$ is the closure of $\mathcal{R}$.

Thus, if $\mathcal{R}$ is a fundamental region of Γ, the images of $\overline{\mathcal{R}}$ under the elements of Γ cover $\mathbb{H}^3$ without interior overlappings. Such a cover is called a *tessellation* of $\mathbb{H}^3$, and the images of $\mathcal{R}$ under the elements of Γ are called the *regions* of the tessellation. In [5], Borel established the existence of compact hyperbolic manifolds in all dimensions by proving, for all n, the existence of fundamental regions with compact closures in $\mathbb{H}^{n+1}$.

Given a surface $M \subset \mathbb{H}^3$, let $M_R(p)$ be the intersection of M with $B_R(p)$, the geodesic ball in $\mathbb{H}^3$ centered at $p \in M$ and of radius R. As a consequence of Theorem 5 we have the following result.

THEOREM 6. *Suppose M is a complete embedded H-surface in $\mathbb{H}^3$, $H > 1$, with bounded Gaussian curvature. Then, for all $p \in M$ and sufficiently large $R > 0$, there exists a constant $C > 0$, independent of p and R, satisfying*

$$\frac{\mathrm{Vol}(M_R(p))}{\mathrm{Vol}(B_R(p))} < C.$$

Proof. By the hypothesis, M has bounded second fundamental form; i.e., if k_1, k_2 are the principal curvatures of M, there is a constant $c_0 > 0$ satisfying $k_i < c_0$, $i = 1, 2$. By Theorem 5, we can choose $\epsilon > 0$ such that $N^*_\epsilon(M)$ embeds in $\mathbb{H}^3$ under the exponential map. Take $R_0 > 0$ such that $R_0 < \min\{\epsilon, 2\mathrm{arcsinh}(1/2c_0)\}$ and consider $\Omega = N^*_{R_0/2}(M_{R_0}(p))$. The volume of Ω in $\mathbb{H}^3$ is given by (see [16])

$$\mathrm{Vol}(\Omega) = \int_0^{R_0/2} \int_{M_{R_0}} (\cosh t - k_1 \sinh t)(\cosh t - k_2 \sinh t) dM dt.$$

Since, for $i = 1, 2$ and $0 \leq t \leq R_0/2$,

$$\cosh t - k_i \sinh t \geq 1 - c_0 \sinh(R_0/2) > 1/2,$$

we have

$$\mathrm{Vol}(\Omega) \geq \frac{1}{8} R_0 \, \mathrm{Vol}(M_{R_0}(p)).$$

Let $q^* \in \Omega$ be the point of maximal distance from p in Ω and let $q \in M_{R_0}(p)$ be the hyperbolic orthogonal projection of q^* in $M_{R_0}(p)$. Then,

$$d(p, q^*) \leq d(p, q) + d(q, q^*) \leq R_0 + R_0/2 = 3R_0/2,$$

where d stands for the distance function on $\mathbb{H}^3$. Hence $\Omega \subset B_{3R_0/2}(p)$, which implies

$$\mathrm{Vol}(B_{3R_0/2}(p)) \geq \frac{1}{8} R_0 \, \mathrm{Vol}(M_{R_0}(p)). \tag{11}$$

Since, for all $R > 0$, $\mathrm{Vol}(B_R(p))$ is independent of p, (11) gives

$$\mathrm{Vol}(M_{R_0}(p)) < c_1, \tag{12}$$

where c_1 is a positive constant independent of p.

Consider a tessellation of $\mathbb{H}^3$ whose fundamental region $\mathcal{R}$ has compact closure. Given $R > R_0$, denote by Λ the set of regions of the tessellation that intersect $B_R(p)$, and let $\Lambda_1 \subset \Lambda$ be the subset of Λ whose elements are the regions contained in $B_R(p)$. Then,

$$\#(\Lambda_1) \leq \frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}(\mathcal{R})},$$

where # denotes cardinality. Now notice that if $\Lambda_2 = \Lambda - \Lambda_1$ and d_0 is the diameter of $\mathcal{R}$, all regions of Λ_2 are contained in $B_{R+d_0}(p)$. Moreover, it is easily seen that $\operatorname{Vol}(B_{R+d_0}(p))/\operatorname{Vol}(B_R(p))$ is bounded above by a constant $c_2 > 0$, independent of p and R. Therefore,

$$\#(\Lambda_2) \leq \frac{\operatorname{Vol}(B_{R+d_0}(p))}{\operatorname{Vol}(\mathcal{R})} < \frac{c_2 \operatorname{Vol}(B_R(p))}{\operatorname{Vol}(\mathcal{R})}.$$

Thus we have

$$\#(\Lambda) = \#(\Lambda_1) + \#(\Lambda_2) < (1 + c_2)\frac{\operatorname{Vol}(B_R(p))}{\operatorname{Vol}(\mathcal{R})} = c_3 \operatorname{Vol}(B_R(p)). \tag{13}$$

Since $\mathcal{R}$ has compact closure and the constant c_1 in inequality (12) is independent of p, the volume of M in each region is uniformly bounded; that is, there is a constant $c_4 > 0$ such that

$$\operatorname{Vol}(M \cap \mathcal{R}') < c_4$$

for any region $\mathcal{R}'$ of the tessellation. From this and (13) we obtain

$$\operatorname{Vol}(M_R(p)) < c_4 \#(\Lambda) < c_3 c_4 \operatorname{Vol}(B_R(p)) = C \operatorname{Vol}(B_R(p)), \tag{14}$$

as desired. □

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