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PRODUCTS OF $\mathcal{N}$-CONNECTED GROUPS

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ABSTRACT. Two subgroups H and K of a finite group G are said to be $\mathcal{N}$-connected if the subgroup generated by x and y is a nilpotent group, for every pair of elements x in H and y in K. This paper is devoted to the study of pairwise $\mathcal{N}$-connected and permutable products of finitely many groups, in the framework of formation and Fitting class theory.

1. Introduction

All groups considered in this paper are finite.

The contents of this paper relate to recent investigations on factorized groups whose factors are linked by some particular connections. The original starting point is the study of totally permutable supersoluble groups introduced by M. Asaad and A. Shaalan in [2] and extended to the framework of classes of groups by R. Maier in [16] for the first time. Here two subgroups H and K of a group G are said to be totally permutable if every subgroup of H permutes with every subgroup of K. Products of totally permutable groups have since been the object of thorough study, and much is known about their structure. We refer to [5], [9] for an account on this development in the framework of formation theory, to [11], [12], [13] in relation with Fitting classes and to [7] for more general information. In particular, R. Maier proved in [16] that saturated formations containing $\mathcal{U}$, the class of all supersoluble groups, are closed under the product of totally permutable groups. In the same paper he also made the following observation: If H and K are totally permutable subgroups of a group G, then $\langle x, y\rangle = \langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$ is a supersoluble group, for every pair of elements $x \in H$ and $y \in K$. Then he gave an example showing that his result does not hold if total permutability of the factors H and K involved is replaced by the weaker connection property '$\langle x, y\rangle$ is supersoluble for every $x \in H$ and $y \in K$'. This led A. Carocca [8] to introduce the concept of $\mathcal{L}$-connected subgroups, defined as follows: Given a non-empty class

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of groups $\mathcal{L}$, two subgroups H and K of a group G are said to be $\mathcal{L}$-connected if $\langle x, y\rangle \in \mathcal{L}$, for all $x \in H$ and $y \in K$. In [8] Carocca started an investigation of this property and, in particular, considered products of $\mathcal{N}$-connected groups, for the class $\mathcal{N}$ of all nilpotent groups. More precisely, he proved that saturated formations (containing $\mathcal{N}$) are closed under the product of pairwise permutable and $\mathcal{N}$-connected groups. This study was taken further in [3] in the soluble universe and for products of two $\mathcal{N}$-connected groups, mainly in the framework of formation theory. As pointed out in this paper, although total permutability and $\mathcal{N}$-connection are quite different properties, they are related in the sense that the first one is to supersolubility as the second one is to nilpotence. In fact, they have been the object of parallel and similar developments.

One of the aims of the present paper is to extend this study to the finite universe and to products of finitely many factors. First, a detailed account about the structure of $\mathcal{N}$-connected products of groups is provided. Then the behaviour of residuals and projectors associated to (saturated) formations in such products is studied. The above-mentioned comments about total permutability and $\mathcal{N}$-connection are made particularly clear when considering their relations with the 'duals' of formations, namely Fitting classes. In [13] a study of radicals and injectors associated to Fitting classes containing $\mathcal{U}$ in totally permutable products of groups was carried out. We show now that analogous results to those obtained can be stated if total permutability is replaced by $\mathcal{N}$-connection and for Fitting classes containing $\mathcal{N}$.

The notation is standard and mainly taken from [10]. We also refer to this book for the basic results on classes of groups.

2. Properties

We collect first some elementary properties of a product of pairwise $\mathcal{N}$-connected and permutable groups.

PROPOSITION 1. *Let the group $G = G_1G_2\cdots G_r$ be the product of the pairwise $\mathcal{N}$-connected and permutable subgroups $G_1, G_2, \ldots, G_r$. Then the following properties hold:*

(1) ([8, Theorem 2], [3, Lemma 2]) $[G_i^{\mathcal{N}}, G_j] = 1$, *for all $i, j \in \{1, 2, \ldots, r\}$, $i \neq j$. In particular, $G_i^{\mathcal{N}}$ is a normal subgroup of G, for all $i \in \{1, 2, \ldots, r\}$.*

(2) *G_i is a subnormal subgroup of G, for all $i \in \{1, 2, \ldots, r\}$.*

(3) *If $(G_i)_p \in \mathrm{Syl}_p(G_i)$, for each $i \in \{1, 2, \ldots, r\}$ and a prime p, then $(G_i)_p(G_j)_p = (G_j)_p(G_i)_p \in \mathrm{Syl}_p(G_iG_j)$, for all $i, j \in \{1, 2, \ldots, r\}$, and $(G_1)_p \cdots (G_r)_p \in \mathrm{Syl}_p(G)$.*

Moreover, if $P \in \mathrm{Syl}_p(G)$, then $P \cap G_i \in \mathrm{Syl}_p(G_i)$, for all $i \in \{1, 2, \ldots, r\}$, and $P = (P \cap G_1)\cdots(P \cap G_r)$.

(4) *If X_i is a p-subgroup of G_i, for each $i \in \{1,2,\dots,r\}$ and a prime p, then $\langle X_1,\dots,X_r\rangle$ is a p-subgroup of G.*
(5) *If X_i is a nilpotent subgroup of G_i, for each $i \in \{1,2,\dots,r\}$, then $\langle X_1,\dots,X_r\rangle$ is nilpotent.*
(6) $G^{\mathcal{N}} = G_1^{\mathcal{N}} \cdots G_r^{\mathcal{N}}$.
(7) *If $I, J \subseteq \{1,2,\dots,r\}$ and $I \cap J = \emptyset$, then the subgroups $\prod_{i\in I} G_i$ and $\prod_{j\in J} G_j$ are $\mathcal{N}$-connected.*
(8) *If $I, J \subseteq \{1,2,\dots,r\}$ and $I \cap J = \emptyset$, then $[\prod_{i\in I} G_i, \prod_{j\in J} G_j] \leq Z_\infty(G)$. In particular, $(\prod_{i\in I} G_i) \cap (\prod_{j\in J} G_j) \leq Z_\infty(G)$.*
(9) *If X_i is a π-subgroup of G_i, for a set of primes π and for each $i \in \{1,2,\dots,r\}$, then $\langle X_1,\dots,X_r\rangle$ is a π-group.*

Proof. (2) We argue by induction on $|G|$. If G_i is nilpotent, for all $i \in \{1,\dots,r\}$, then $G \in \mathcal{N}$, by [8, Theorem 2], and the result follows.

Assume that there exists $j \in \{1,2,\dots,r\}$ such that G_j is not nilpotent. Then $1 \neq G_j^{\mathcal{N}} \trianglelefteq G$. By the inductive hypothesis on the factor group $G/G_j^{\mathcal{N}}$, we obtain that G_j and $G_iG_j^{\mathcal{N}}$, for all $i \neq j$, are subnormal subgroups of G. But G_i is normal in $G_iG_j^{\mathcal{N}}$, for all $i \neq j$, and so we are done.

(3) Let $i, j \in \{1,2,\dots,r\}$, $i \neq j$. We note that $G_i^{\mathcal{N}}(G_i)_p \trianglelefteq G_i$. Then $\mathrm{Syl}_p(G_i) = \{(G_i)_p^t : t \in G_i^{\mathcal{N}}\}$, for every i.

Since $G_iG_j = G_jG_i$, we know by [1, Lemma 1.3.2] that there exist $X \in \mathrm{Syl}_p(G_i)$ and $Y \in \mathrm{Syl}_p(G_j)$ such that $XY = YX \in \mathrm{Syl}_p(G_iG_j)$. Then there exist $t \in G_i^{\mathcal{N}}$ and $s \in G_j^{\mathcal{N}}$ such that $(G_i)_p(G_j)_p = X^tY^s = X^{ts}Y^{ts} = (XY)^{ts} = (YX)^{ts} = Y^sX^t = (G_j)_p(G_i)_p$.

The remainder is now clear from (2).

(4) This follows easily from (3).

(5) This follows from (4), taking into account that $[(G_i)_p, (G_j)_q] = 1$, for every $(G_i)_p \in \mathrm{Syl}_p(G_i)$, $(G_j)_q \in \mathrm{Syl}_q(G_j)$, for all prime numbers $p \neq q$ and $i, j \in \{1,2,\dots,r\}$, $i \neq j$.

(6) This is clear because $G/(G_1^{\mathcal{N}} \cdots G_r^{\mathcal{N}}) \in \mathrm{N}_0(\mathcal{N}) = \mathcal{N}$ from (2).

(7) Let $I = \{i_1,\dots,i_m\}$ and $J = \{j_1,\dots,j_n\}$. Let $a \in \prod_{i\in I} G_i$ and $b \in \prod_{j\in J} G_j$. Then $a = a_{i_1}\cdots a_{i_m}$ and $b = b_{j_1}\cdots b_{j_n}$, for some $a_{i_l} \in G_{i_l}$ and $b_{j_t} \in G_{j_t}$, $l = 1,\dots,m$, $t = 1,\dots,n$. So $\langle a,b\rangle \leq \langle\langle a_{i_1}\rangle,\dots,\langle a_{i_m}\rangle,\langle b_{j_1}\rangle,\dots,\langle b_{j_n}\rangle\rangle$, which is nilpotent, by (5).

(8) By (7) we can assume that the group $G = AB$ is the product of two $\mathcal{N}$-connected subgroups A, B and it is enough to prove that $[A,B] \leq Z_\infty(G)$. Clearly, $[A^{\mathcal{N}}, \langle B^G\rangle] = [B^{\mathcal{N}}, \langle A^G\rangle] = 1$, from (1), and consequently $[G^{\mathcal{N}}, [A,B]] = [A^{\mathcal{N}}B^{\mathcal{N}}, [A,B]] \leq [A^{\mathcal{N}}B^{\mathcal{N}}, \langle A^G\rangle \cap \langle B^G\rangle] = 1$, by (6).

On the other hand, we consider $A = A^{\mathcal{N}}X$ and $B = B^{\mathcal{N}}Y$, where $X \in \mathrm{Proj}_{\mathcal{N}}(A)$ and $Y \in \mathrm{Proj}_{\mathcal{N}}(B)$. Then $G = A^{\mathcal{N}}B^{\mathcal{N}}\langle X,Y\rangle$ and $\langle X,Y\rangle \in \mathcal{N}$ by (5). Moreover, $[A,B] = [A^{\mathcal{N}}X, B^{\mathcal{N}}Y] = [X,Y] \leq \langle X,Y\rangle$.

Therefore, $[A, B] \leq C_{\langle X,Y \rangle}(G^{\mathcal{N}}) \leq Z_\infty(G)$ by [10, Theorem IV, 6.14].

Finally, $(A \cap B)Z_\infty(G)/Z_\infty(G)$ is contained in $Z(G/Z_\infty(G)) = 1$ and we are done.

(9) By step (7) and arguing by induction on the number of factors, we can assume that the group G is the product of two $\mathcal{N}$-connected subgroups A and B, and we will prove that $\langle X, Y \rangle$ is a π-group, whenever X is a π-subgroup of A and Y is a π-subgroup of B.

We argue by induction on $|G|$. If $[A, B] = 1$, then $\langle X, Y \rangle = XY$ and the result is clear. Otherwise, there exists a minimal normal subgroup N of G such that $1 \neq N \leq [A, B] \leq Z_\infty(G)$. In particular, $N \leq Z(G)$ and N is a q-group, for some prime q. By the inductive hypothesis, $\langle X, Y \rangle N/N \cong \langle X, Y \rangle/(\langle X, Y \rangle \cap N)$ is a π-group. If $q \in \pi$, then $\langle X, Y \rangle$ is a π-group and we are done. Otherwise, $\langle X, Y \rangle \cap N \leq [X, Y]$, because $\langle X, Y \rangle = XY[X, Y]$ and so $[X, Y]$ contains the Sylow q-subgroups of $\langle X, Y \rangle$. But this implies that $\langle X, Y \rangle \cap N \leq \langle X, Y \rangle' \cap Z(\langle X, Y \rangle) \leq \phi(\langle X, Y \rangle)$. Consequently, $\langle X, Y \rangle \cap N = 1$ and $\langle X, Y \rangle$ is a π-group. □

Remark. The concept of $\mathcal{N}$-connectedness is related to the concept of strong cosubnormality, introduced by Knapp [14] and defined as follows:

Definition ([14, Definition 3.1]). Let G be a group and let A, B be subgroups of G. A is called *strongly cosubnormal* with B if for any subgroups $A_1 \leq A$ and $B_1 \leq B$ we have that A_1 and B_1 are cosubnormal, that is, both are subnormal subgroups of their join $\langle A_1, B_1 \rangle$.

Strongly cosubnormal subgroups are characterized by the following result:

Theorem ([14, Theorem 3.3]). *Let A, B be subgroups of a group G. Then the following are equivalent:*

(a) *A is strongly cosubnormal with B.*
(b) $[A, B] \leq Z_\infty(\langle A, B \rangle)$.

It is clear that two strongly cosubnormal subgroups are $\mathcal{N}$-connected. It is not difficult to check that the arguments used in the proof of Proposition 1 (8) provide an alternative proof that (a) implies (b) in the preceding theorem. Note also that for permutable subgroups A and B strong cosubnormality and $\mathcal{N}$-connectedness are actually equivalent by Proposition 1 (8) and Knapp's theorem. This equivalence does not hold in general as the next example shows.

Example. Let $N = \langle n_1, n_2, n_3, n_4 \rangle$ be an elementary abelian group of order 3^4. Define automorphisms $x_1, \ldots, x_4, y_1, y_2$ and z of N in the following way: x_i inverts n_i and fixes n_j for $j \neq i$, $i = 1, \ldots, 4$, y_1 fixes n_3, n_4 and interchanges n_1 and n_2, y_2 fixes n_1, n_2 and interchanges n_3 and n_4, and finally z interchanges n_1 and n_3 as well as n_2 and n_4. These automorphisms generate a subgroup U of $\mathrm{Aut}(N)$, $U \cong (Z_2 \sim_{\mathrm{reg}} Z_2) \sim_{\mathrm{reg}} Z_2$. Let $H = [N]U$ be the semidirect product of N with U with respect to the given action of U on N. Consider $A = \langle z^{n_1 n_4} \rangle$, $B = \langle y_1 x_3, x_1 x_2 \rangle$ and $G = \langle A, B \rangle \leq H$. It is not difficult to check that

$$\langle z^{n_1 n_4}, y_1 x_3 \rangle = \langle z, y_1 x_3 \rangle^{n_1 n_2 n_4^{-1}},$$

$$\langle z^{n_1 n_4}, x_1 x_2 \rangle = \langle z, x_1 x_2 \rangle^{n_3^{-1} n_4},$$

$$\langle z^{n_1 n_4}, y_1 x_1 x_2 x_3 \rangle = \langle z, y_1 x_1 x_2 x_3 \rangle^{n_1 n_2^{-1}}$$

and $[z^{n_1 n_4}, y_1 x_3, x_1 x_2] = n_2^{-1}$, which does not centralize $\langle A, B \rangle$. This means that A and B are nilpotent $\mathcal{N}$-connected subgroups of G, but G is not nilpotent, so A and B are not strongly cosubnormal.

Remark. Wielandt ([17, p. 166]; see also [15, p. 238]) asked whether two subgroups A and B of a finite group are cosubnormal if there exists a positive integer n such that $[b, {}_n a] \in A$ and $[a, {}_n b] \in B$ for all $a \in A$ and $b \in B$. The groups A and B of the preceding example provide a negative answer (with $n = 3$).

3. Projectors

In [8] and [3] the behaviour of products of $\mathcal{N}$-connected permutable subgroups with regard to formation theory was studied. In the sequel we will take this study further.

In particular, the following result was proved in [8, Theorem 2]:

Theorem 1. *Let the group $G = G_1 G_2 \cdots G_r$ be the product of the pairwise $\mathcal{N}$-connected and permutable subgroups $G_1, G_2, \ldots, G_r$ and let $\mathcal{F}$ be a saturated formation. If $G_i \in \mathcal{F}$, for all $i \in \{1, 2, \ldots, r\}$, then $G \in \mathcal{F}$.*

In fact, in the original statement of this result, the saturated formation $\mathcal{F}$ is assumed to contain $\mathcal{N}$, but the same proof shows that this hypothesis is really not necessary.

The following lemma was proved in [3] for the soluble universe.

Lemma 1. *Let $\mathcal{F}$ be a formation containing $\mathcal{N}$. Let the group $G = AB$ be the product of the $\mathcal{N}$-connected subgroups A and B. If $A, B \in \mathcal{F}$, then $G \in \mathcal{F}$.*

Proof. Assume that $\langle H, K \rangle$ is a group generated by the $\mathcal{N}$-connected subgroups H and K. Note that, as in the case of a product of $\mathcal{N}$-connected groups, $H^{\mathcal{N}}$ centralizes K and, in particular, $H^{\mathcal{N}}$ is a normal subgroup of $\langle H, K \rangle$. Assume now that $H \in \mathcal{F}$ and $K \in \mathcal{N}$. Let X be an $\mathcal{N}$-projector of H and assume that $\langle X, K \rangle$ is nilpotent. Arguing as in the proof of [6, Theorem, Step 2], and replacing the supersoluble residual by the nilpotent residual, and the supersoluble projector by the nilpotent projector, we deduce that $\langle H, K\rangle = \langle K^X\rangle H \in \mathcal{F}$.

Now let $G = AB$ be as in the statement of the lemma. By the previous paragraph and Proposition 1 (5), if either A or B is nilpotent, then $G = AB \in \mathcal{F}$. Otherwise, if X and Y are nilpotent projectors of A and B, respectively, then $\langle X, Y\rangle$, $\langle A, Y\rangle$ and $\langle B, Y\rangle$ belong to $\mathcal{F}$. Now the result follows by arguing as in the proof of [6, Theorem, Step 1], with replacements analogous to those above involving $\mathcal{N}$ and $\mathcal{U}$ and the join of the nilpotent projectors instead of the product. □

Now, from Proposition 1 (7), the following result is easily obtained:

Lemma 2. *Let $\mathcal{F}$ be a formation containing $\mathcal{N}$. Let the group $G = G_1G_2\cdots G_r$ be the product of the pairwise $\mathcal{N}$-connected and permutable subgroups $G_1, G_2, \ldots, G_r$. If $G_i \in \mathcal{F}$, for all $i \in \{1, 2, \ldots, r\}$, then $G \in \mathcal{F}$.*

We notice that, for arbitrary formations, the hypothesis $\mathcal{N} \subseteq \mathcal{F}$ in the above lemma is necessary. To see this, we can consider, for instance, the formation $\mathcal{F}$ of all elementary abelian p-groups, for a prime p. Let $G = Z_p \sim_{\text{reg}} Z_p$ be the regular wreath product of Z_p with Z_p. Clearly, G is the product of the $\mathcal{N}$-connected subgroups $Z_p^\natural$, the base group of G, and a suitable subgroup Z_p, and both subgroups belong to $\mathcal{F}$, but G does not.

The behaviour of the $\mathcal{F}$-projectors when $\mathcal{F}$ is a saturated formation containing $\mathcal{N}$, as well as the behaviour of the $\mathcal{F}$-residuals in products of $\mathcal{N}$-connected groups, were studied in [3] in the soluble universe. In the following we provide some extensions of these results, in particular, to the universe of all finite groups.

We recall that if $\mathcal{F}$ is a Schunck class (in particular, if $\mathcal{F}$ is a saturated formation), each finite group G has $\mathcal{F}$-projectors [10, Theorem III, 3.10].

Lemma 3. *Let the group $G = AB$ be the product of the $\mathcal{N}$-connected subgroups A and B. If $X \in \mathrm{Proj}_{\mathcal{N}}(A)$ and $Y \in \mathrm{Proj}_{\mathcal{N}}(B)$, then $XY = YX \in \mathrm{Proj}_{\mathcal{N}}(G)$.*

Proof. We argue by induction on $|G|$. Let $C = Z_\infty(G)$. We notice that $G/C = (AC/C)(BC/C)$ is a central product, because $[A, B] \le Z_\infty(G)$. Then $(XYC/C) = (XC/C)(YC/C) \in \mathrm{Proj}_{\mathcal{N}}(G/C)$ by [10, Theorem III, 6.3]. If $C = 1$, we are done, and in any case we have $XYC \in \mathrm{Proj}_{\mathcal{N}}(G)$, because $XYC \in \mathcal{N}$.

Since $G^{\mathcal{N}} = A^{\mathcal{N}}B^{\mathcal{N}}$ by Proposition 1 (6), we have that

$$\begin{aligned} C_G(G^{\mathcal{N}}) &= C_{AB}(A^{\mathcal{N}}B^{\mathcal{N}}) = C_{AB}(A^{\mathcal{N}}) \cap C_{AB}(B^{\mathcal{N}}) \\ &= \big(C_A(A^{\mathcal{N}})B\big) \cap \big(C_B(B^{\mathcal{N}})A\big) = C_A(A^{\mathcal{N}})C_B(B^{\mathcal{N}})(A \cap B) \\ &= C_A(A^{\mathcal{N}})C_B(B^{\mathcal{N}}). \end{aligned}$$

Hence

$$C \le C_G(G^{\mathcal{N}}) = C_A(A^{\mathcal{N}})C_B(B^{\mathcal{N}}) \le Z_\infty(A)Z(A^{\mathcal{N}})Z_\infty(B)Z(B^{\mathcal{N}})$$

by [10, Theorem IV, 6.13] and so $XYC \le XZ(A^{\mathcal{N}})YZ(B^{\mathcal{N}})$, since $Z_\infty(A) \le X$ and $Z_\infty(B) \le Y$. Let $A_1 = XZ(A^{\mathcal{N}}) \le A$, $B_1 = YZ(B^{\mathcal{N}}) \le B$, and $R = A_1B_1$. We note that A_1 and B_1 are $\mathcal{N}$-connected and that R is a soluble group because $Z(A^{\mathcal{N}})Z(B^{\mathcal{N}})$ is abelian and $\langle X, Y\rangle$ is nilpotent.

Assume that $R < G$. Since $X \in \mathrm{Proj}_{\mathcal{N}}(A_1)$ and $Y \in \mathrm{Proj}_{\mathcal{N}}(B_1)$, it follows by the inductive hypothesis that $XY = YX \in \mathrm{Proj}_{\mathcal{N}}(R)$. But $XY \le XYC \le R$ and $XYC \in \mathcal{N}$, which implies that $XY = XYC \in \mathrm{Proj}_{\mathcal{N}}(G)$, and the result follows.

Consider now the case $G = R$. If $Z(A^{\mathcal{N}}) = Z(B^{\mathcal{N}}) = 1$, then $G = XY \in \mathcal{N}$ and we are done. So, we can suppose without loss of generality that $Z(A^{\mathcal{N}}) \neq 1$. Since $Z(A^{\mathcal{N}}) \trianglelefteq G$, we can consider a minimal normal subgroup N of G contained in $Z(A^{\mathcal{N}})$. By the inductive hypothesis we deduce that $(XN/N)(YN/N) = (YN/N)(XN/N) \in \mathrm{Proj}_{\mathcal{N}}(G/N)$.

Assume that $XYN < G$. Since $XN \le A$, we have that $X \in \mathrm{Proj}_{\mathcal{N}}(AN)$. Then $XY = YX \in \mathrm{Proj}_{\mathcal{N}}((XN)Y)$, by the inductive hypothesis. Consequently, since $XYN/N \in \mathrm{Proj}_{\mathcal{N}}(G/N)$, we have that $XY \in \mathrm{Proj}_{\mathcal{N}}(G)$.

Therefore we can assume that $G = XYN$. Then $G/N \in \mathcal{N}$ and so $G^{\mathcal{N}} \le N$. We can suppose that $G^{\mathcal{N}} = N$ and so $G^{\mathcal{N}}$ is abelian. Since $XY \subseteq \langle X, Y\rangle \in \mathcal{N}$ and $G = N\langle X, Y\rangle$, there exists $T \in \mathrm{Proj}_{\mathcal{N}}(G)$ such that $\langle X, Y\rangle \le T$, by [10, Lemma III, 3.14]. Moreover, $G = XYN = TN$ and $T \cap N = 1$ by [10, Theorem IV, 5.18]. Then $XY \subseteq T$ and $|XY| = |T|$, which implies finally that $XY = T \in \mathrm{Proj}_{\mathcal{N}}(G)$. □

Proposition 2. *Let the group $G = G_1G_2\cdots G_r$ be the product of pairwise $\mathcal{N}$-connected and permutable subgroups $G_1, G_2, \ldots, G_r$. Then*

$$Z_\infty(G) = Z_\infty(G_1)Z_\infty(G_2)\cdots Z_\infty(G_r).$$

Proof. By Proposition 1 (7) and induction it suffices to prove the assertion for the case of two factors. So let the group $G = AB$ be the product of the $\mathcal{N}$-connected subgroups A and B. Let $X \in \mathrm{Proj}_{\mathcal{N}}(A)$ and $Y \in \mathrm{Proj}_{\mathcal{N}}(B)$. By Lemma 3, $XY \in \mathrm{Proj}_{\mathcal{N}}(G)$. Then, by [10, Theorem IV, 6.14]

$$\begin{aligned} Z_\infty(G) &= C_{XY}(G^{\mathcal{N}}) = C_{XY}(A^{\mathcal{N}}B^{\mathcal{N}}) = C_{XY}(A^{\mathcal{N}}) \cap C_{XY}(B^{\mathcal{N}}) \\ &= \big(C_X(A^{\mathcal{N}})Y\big) \cap \big(C_Y(B^{\mathcal{N}})X\big) = C_X(A^{\mathcal{N}})C_Y(B^{\mathcal{N}})(X \cap Y) \\ &= C_X(A^{\mathcal{N}})C_Y(B^{\mathcal{N}}) = Z_\infty(A)Z_\infty(B). \end{aligned}$$

□

Theorem 2. *Let $\mathcal{F}$ be a saturated formation and let the group $G = G_1G_2\cdots G_r$ be the product of pairwise $\mathcal{N}$-connected and permutable subgroups $G_1, G_2, \ldots, G_r$. If $X_i \in \mathrm{Proj}_{\mathcal{F}}(G_i)$, for every $i \in \{1, 2, \ldots, r\}$, then $X_1\cdots X_r$ is a pairwise permutable product of the subgroups $X_1, \ldots, X_r$ and $X_1\cdots X_r \in$*

$\mathrm{Proj}_{\mathcal{F}}(G)$. *Moreover, if G has a unique conjugacy class of $\mathcal{F}$-projectors, then every $\mathcal{F}$-projector of G has this form.*

Proof. By Proposition 1 (7) and induction it suffices to prove the first assertion for the case of two factors. So let the group $G = AB$ be the product of the $\mathcal{N}$-connected subgroups A and B. Let $X \in \mathrm{Proj}_{\mathcal{F}}(A)$ and $Y \in \mathrm{Proj}_{\mathcal{F}}(B)$. Let $C = Z_\infty(G) = Z_\infty(A)Z_\infty(B)$, by Proposition 2. Since $[A,B] \leq C$, $G/C = (AC/C)(BC/C)$ is a central product and then $(XC/C)(YC/C) \in \mathrm{Proj}_{\mathcal{F}}(G/C)$ by [10, Theorem III, 6.3]. Note that this theorem is valid in our context since $\mathrm{D}_0\mathcal{F} = \mathcal{F}$. But

$$XYC = XYZ_\infty(A)Z_\infty(B) = XYO_{\pi'}(Z_\infty(G)),$$

for $\pi = \mathrm{char}(\mathcal{F})$, because $O_\pi(Z_\infty(A)) \leq X$ and $O_\pi(Z_\infty(B)) \leq Y$. Moreover,

$$\begin{aligned}\langle X,Y\rangle = XY[X,Y] &\leq XY(Z_\infty(G) \cap \langle X,Y\rangle) \leq XYO_\pi(Z_\infty(G)) \\ &= XYO_\pi(Z_\infty(A))O_\pi(Z_\infty(B)) = XY,\end{aligned}$$

by Proposition 1 (9), because X,Y are π-groups. Then $XY = YX \in \mathcal{F}$ by Theorem 1, and XY is an $\mathcal{F}$-maximal subgroup of XYC. Consequently, $XY \in \mathrm{Proj}_{\mathcal{F}}(XYC)$ by [10, III, Lemma 3.14], which implies that $XY \in \mathrm{Proj}_{\mathcal{F}}(G)$.

Now let $G = G_1G_2\cdots G_r$ be the product of pairwise $\mathcal{N}$-connected and permutable subgroups $G_1, G_2, \ldots, G_r$. Assume that G has a unique conjugacy class of $\mathcal{F}$-projectors and let $T \in \mathrm{Proj}_{\mathcal{F}}(G)$. Then $T = (X_1X_2\cdots X_r)^g$ for some $g = g_1g_2\cdots g_r$, with $g_i \in G_i$ and $X_i \in \mathrm{Proj}_{\mathcal{F}}(G_i)$, for every $i \in \{1,2,\ldots,r\}$. Since $[G_i, \prod_{j=1, j\neq i}^r G_j] \leq Z_\infty(G) = C$, by Proposition 1 (8), we have that $G/C = \prod_{i=1}^r (G_iC/C)$ is a central product. Therefore $TC/C = \big(\prod_{i=1}^r X_i^{g_i}\big)C/C$, whence $T \times O_{\pi'}(C) = (\prod_{i=1}^r X_i^{g_i}) \times O_{\pi'}(C)$. Consequently, $T = \prod_{i=1}^r X_i^{g_i}$, where $X_i^{g_i} \in \mathrm{Proj}_{\mathcal{F}}(G_i)$, for every $i \in \{1,2,\ldots,r\}$. □

PROPOSITION 3. *Let $\mathcal{F}$ be a formation and let the group $G = G_1G_2\cdots G_r$ be the product of the pairwise $\mathcal{N}$-connected and permutable subgroups $G_1, G_2, \ldots, G_r$. If either $\mathcal{F}$ is saturated or $\mathcal{N} \subseteq \mathcal{F} \subseteq \mathcal{S}$, then $G^{\mathcal{F}} = G_1^{\mathcal{F}}\cdots G_r^{\mathcal{F}}$. In particular, if $G \in \mathcal{F}$, then $G_i \in \mathcal{F}$, for all $i \in \{1,2,\ldots,r\}$.*

Proof. We prove the result for $r = 2$. The general case follows by a straightforward inductive argument on the number of factors and Proposition 1 (7). So we consider $G = G_1G_2$ as above.

Assume first that $\mathcal{N} \subseteq \mathcal{F} \subseteq \mathcal{S}$. If $G \in \mathcal{F}$, then, in particular, G is soluble, and so G_1 and G_2 belong to $\mathcal{F}$ by [3, Theorem 4]. The remainder is easily proved by an argument similar to that in [3, Theorem 2].

Assume now that $\mathcal{F}$ is saturated. We claim that if $G \in \mathcal{F}$, then G_1 and G_2 belong to $\mathcal{F}$. Let $X_i \in \mathrm{Proj}_{\mathcal{F}}(G_i)$, for every $i = 1, 2$. Then $X_1X_2 = X_2X_1 \in \mathrm{Proj}_{\mathcal{F}}(G)$, by Theorem 2, and so $G = X_1X_2$ because $G \in \mathcal{F}$. Consequently,

$G_i = X_i(X_{3-i} \cap G_i)$, for $i = 1, 2$. Now, we have that, for every $i = 1, 2$, $X_{3-i} \cap G_i \leq O_\pi(G_{3-i} \cap G_i) \in \mathcal{N}_\pi \subseteq \mathcal{F}$, for $\pi = \operatorname{char}(\mathcal{F})$, $X_i \in \mathcal{F}$, and X_i and $X_{3-i} \cap G_i$ are $\mathcal{N}$-connected. Then it follows from Theorem 1 that $G_i \in \mathcal{F}$, for every $i = 1, 2$.

This implies that $G_i G^{\mathcal{F}}/G^{\mathcal{F}} \in \mathcal{F}$, for every $i = 1, 2$, because $G/G^{\mathcal{F}}$ is an $\mathcal{F}$-group and the product of the $\mathcal{N}$-connected subgroups $G_1 G^{\mathcal{F}}/G^{\mathcal{F}}$ and $G_2 G^{\mathcal{F}}/G^{\mathcal{F}}$. In particular, $G_i^{\mathcal{F}} \leq G^{\mathcal{F}}$, for $i = 1, 2$, and so $\langle G_1^{\mathcal{F}}, G_2^{\mathcal{F}} \rangle \leq G^{\mathcal{F}}$.

If $X_i \in \operatorname{Proj}_{\mathcal{F}}(G_i)$, for $i = 1, 2$, then $X_1 X_2 \in \operatorname{Proj}_{\mathcal{F}}(G)$ and, in particular, $X_1 X_2$ is a π-group, for $\pi = \operatorname{char}(\mathcal{F})$. Since $\mathcal{N}_\pi \subseteq \mathcal{F}$, we have that

$$\begin{aligned} G_i^{\mathcal{F}} \leq G_i^{\mathcal{N}_\pi} &= \bigcap_{p \in \pi} O^p(G_i) \\ &= \bigcap_{p \in \pi} \langle (G_i)_q : q \text{ a prime}, \, q \neq p, \, (G_i)_q \in \operatorname{Syl}_q(G_i) \rangle \\ &\leq \bigcap_{p \in \pi} C_G(O^{p'}(G_{3-i})) \leq C_G(X_{3-i}), \end{aligned}$$

for every $i = 1, 2$. Consequently, $\langle G_1^{\mathcal{F}}, G_2^{\mathcal{F}} \rangle$ is normal in $\langle G_1^{\mathcal{F}}, G_2^{\mathcal{F}} \rangle X_1 X_2 = G$. Then it follows that $G^{\mathcal{F}} \leq \langle G_1^{\mathcal{F}}, G_2^{\mathcal{F}} \rangle$ and so $G^{\mathcal{F}} = \langle G_1^{\mathcal{F}}, G_2^{\mathcal{F}} \rangle$.

We argue now by induction on $|G|$ to prove that $G^{\mathcal{F}} = G_1^{\mathcal{F}} G_2^{\mathcal{F}}$.

If $G_1^{\mathcal{F}}$ and $G_2^{\mathcal{F}}$ are π'-groups, then $G^{\mathcal{F}} = \langle G_1^{\mathcal{F}}, G_2^{\mathcal{F}} \rangle$ is a π'-group by Proposition 1 (9). Then we have that $G = G^{\mathcal{F}} X_1 X_2 = G_1^{\mathcal{F}} G_2^{\mathcal{F}} X_1 X_2$, which implies that $|G^{\mathcal{F}}| = |G_1^{\mathcal{F}} G_2^{\mathcal{F}}|$ and so $G^{\mathcal{F}} = G_1^{\mathcal{F}} G_2^{\mathcal{F}}$.

We may assume that there exists a prime $p \in \pi$ such that p divides the order of $G_1^{\mathcal{F}}$. In particular, $O^{p'}(G_1^{\mathcal{F}}) \neq 1$. We notice that $O^{p'}(G_1^{\mathcal{F}})$ is normal in G because it is normal in G_1 and it is centralized by $G_2 = G_2^{\mathcal{F}} X_2$ by the above argument. Let $N = O^{p'}(G_1^{\mathcal{F}})$. By the inductive hypothesis we obtain that $G^{\mathcal{F}}/N = (G/N)^{\mathcal{F}} = (G_1 N/N)^{\mathcal{F}} (G_2 N/N)^{\mathcal{F}} = (G_1^{\mathcal{F}}/N)(G_2^{\mathcal{F}} N/N) = (G_1^{\mathcal{F}} G_2^{\mathcal{F}})/N$, that is, $G^{\mathcal{F}} = G_1^{\mathcal{F}} G_2^{\mathcal{F}}$ and we are done. □

REMARK. In general, for an arbitrary formation $\mathcal{F}$ of finite groups the $\mathcal{F}$-residual does not respect products of $\mathcal{N}$-connected groups, not even for direct products (see [10, X, 1, Exercise 12]).

Even if $\mathcal{F}$ is a formation of soluble groups, the condition $\mathcal{N} \subseteq \mathcal{F}$ in Proposition 3 is necessary as the example after Lemma 2 shows.

COROLLARY 1. *Let $\mathcal{F}$ be a saturated formation and let the group $G = G_1 G_2 \cdots G_r$ be the product of pairwise $\mathcal{N}$-connected and permutable subgroups $G_1, G_2, \ldots, G_r$. Let $X_i \in \operatorname{Proj}_{\mathcal{F}}(G_i)$, for every $i \in \{1, 2, \ldots, r\}$, and let $P = X_1 \cdots X_r \in \operatorname{Proj}_{\mathcal{F}}(G)$. Then $X_i = P \cap G_i$, for every $i \in \{1, 2, \ldots, r\}$.*

If we assume moreover that $\mathcal{N} \subseteq \mathcal{F}$, then $G_i \cap (\prod_{j=1, j \neq i}^{r} G_j) \leq P$. In particular, for $r = 2$, such an $\mathcal{F}$-projector P of G is factorized, that is,

$P = (P \cap G_1)(P \cap G_2)$ and $G_1 \cap G_2 \leq P$. If, in addition, G has a unique conjugacy class of $\mathcal{F}$-projectors, all $\mathcal{F}$-projectors of G are factorized.

Proof. Assume first that $r = 2$. Then $P = X_1X_2 \subseteq (P \cap G_1)(P \cap G_2)$. That is, $P = (P \cap G_1)(P \cap G_2) \in \mathcal{F}$ and P is a product of the $\mathcal{N}$-connected groups $P \cap G_1$ and $P \cap G_2$. By Proposition 3, we have that $P \cap G_1$ and $P \cap G_2$ belong to $\mathcal{F}$. Then $X_1 = P \cap G_1$ and $X_2 = P \cap G_2$ because of the $\mathcal{F}$-maximality of X_1 and X_2.

Now the general case follows easily taking into account Proposition 1 (7).

The remainder of the proof is clear from Proposition 1 (8), since $Z_\infty(G) \leq P$ if $\mathcal{N} \subseteq \mathcal{F}$, and Theorem 2. □

4. Fitting classes

Some information about the behaviour of radicals and injectors for a Fitting class containing $\mathcal{U}$, the class of all supersoluble groups, in products of totally permutable groups was obtained in [13]. We recall that two subgroups H and K of a group G are totally permutable if every subgroup of H permutes with every subgroup of K. Propositions 4, 5 and 6 below show that statements analogous to those of Theorem 1, Proposition 6 and Theorem 2 of [13] remain true if we consider products of $\mathcal{N}$-connected groups instead of products of totally permutable groups and if the Fitting class $\mathcal{F}$ under consideration contains $\mathcal{N}$. The properties obtained in Proposition 1 and Proposition 2 allow us to deduce these results arguing as in the proofs given in [13], with the obvious simplifications both in the statements of the results and in the arguments. It is worth mentioning the following facts:

(1) Let the group $G = G_1G_2 \cdots G_r$ be the product of the pairwise $\mathcal{N}$-connected and permutable subgroups $G_1, G_2, \ldots, G_r$. Since, by Proposition 1 (2), G_i is subnormal in G, for all $i \in \{1, 2, \ldots, r\}$, it is obvious that $G \in \mathcal{F}$ if and only if $G_i \in \mathcal{F}$, for all $i \in \{1, 2, \ldots, r\}$, for any Fitting class $\mathcal{F}$.

(2) Let $\mathcal{F}$ be a Fitting class. By Proposition 1 (7) the following are equivalent:
   (i) If a group $G = AB$ is the product of the $\mathcal{N}$-connected subgroups A and B, then $G_\mathcal{F} = A_\mathcal{F}B_\mathcal{F}$.
   (ii) If a group $G = G_1G_2 \cdots G_r$ is the product of the pairwise $\mathcal{N}$-connected and permutable subgroups G_1, $G_2, \ldots$, G_r, then $G_\mathcal{F} = (G_1)_\mathcal{F} \cdots (G_r)_\mathcal{F}$.

   An analogous statement is true if we consider soluble groups and, in each case, an $\mathcal{F}$-injector I of G instead of the $\mathcal{F}$-radical $G_\mathcal{F}$ and $I \cap X$ instead of $X_\mathcal{F}$, where X stands for any of the totally permutable subgroups of G under consideration.

(3) If the group $G = AB$ is the product of the $\mathcal{N}$-connected subgroups A and B, then from Proposition 2 and Proposition 1 (8) it follows that

$G/Z_\infty(G) = (AZ_\infty(G)/Z_\infty(G))(BZ_\infty(G)/Z_\infty(G))$ is a direct product. As a consequence, if X and Y are subgroups of A and B, respectively, such that $Z_\infty(A) \leq X$ and $Z_\infty(B) \leq Y$, then $\langle X, Y\rangle = XY$.

Moreover, if $\mathcal{F}$ is a Fitting class containing $\mathcal{N}$, then $Z_\infty(G) \leq G_\mathcal{F}$.

Using these facts, the arguments used in the proofs of Theorem 1, Proposition 6 and Theorem 2 of [13] yield the following results.

Proposition 4. *Let $\mathcal{F}$ be a Fischer class containing $\mathcal{N}$. If the group $G = G_1G_2\cdots G_r$ is the product of the pairwise $\mathcal{N}$-connected and permutable subgroups $G_1, G_2, \ldots, G_r$, then $G_\mathcal{F} = (G_1)_\mathcal{F}\cdots(G_r)_\mathcal{F}$.*

Proposition 5. *Let $\mathcal{F}$ be a Fitting class containing $\mathcal{N}$. Let the soluble group $G = G_1G_2\cdots G_r$ be the product of the pairwise $\mathcal{N}$-connected and permutable subgroups $G_1, G_2, \ldots, G_r$. Assume that there exists an $\mathcal{F}$-injector I of G such that $I = (I\cap G_1)\cdots(I\cap G_r)$. Then the following hold:*

(i) *$G_\mathcal{F} = (G_1)_\mathcal{F}\cdots(G_r)_\mathcal{F}$ and $(G_i)_\mathcal{F} = G_i \cap G_\mathcal{F}$, for all $i \in \{1, 2, \ldots, r\}$.*
(ii) *If $J \in \mathrm{Inj}_\mathcal{F}(G)$, then $J = (J\cap G_1)\cdots(J\cap G_r)$ and $J\cap G_i \in \mathrm{Inj}_\mathcal{F}(G_i)$, for every $i = 1, \ldots, r$.*
(iii) *If $I_i \in \mathrm{Inj}_\mathcal{F}(G_i)$, for every $i = 1, \ldots, r$, then $J = I_1\cdots I_r \in \mathrm{Inj}_\mathcal{F}(G)$ and $I_i = J\cap G_i$, for every $i = 1, \ldots, r$.*

Proposition 6. *For a Fitting class $\mathcal{F}$ containing $\mathcal{N}$, the following statements are equivalent:*

(i) *If a soluble group $G = AB$ is the product of the $\mathcal{N}$-connected subgroups A and B, then $G_\mathcal{F} = A_\mathcal{F}B_\mathcal{F}$.*
(ii) *If a soluble group $G = AB$ is the product of the $\mathcal{N}$-connected subgroups A and B,and $I \in \mathrm{Inj}_\mathcal{F}(G)$, then $I = (I\cap A)(I\cap B)$.*

Moreover, in this case and for such a soluble group $G = AB$, if $I \in \mathrm{Inj}_\mathcal{F}(A)$ and $J \in \mathrm{Inj}_\mathcal{F}(B)$, then $IJ \in \mathrm{Inj}_\mathcal{F}(G)$. Furthermore, the $\mathcal{F}$-radical and the $\mathcal{F}$-injectors of G are factorized.

Obviously, if $\mathcal{F}$-radicals associated to a Fitting class $\mathcal{F}$ are factorized in $\mathcal{N}$-connected products of groups, as stated in Proposition 4, then $\mathcal{F}$ is a Lockett class, since direct products are $\mathcal{N}$-connected. It is not known if the converse is also true in general. The following result shows that this holds apart from Fischer classes also for Lockett classes with other additional closure properties. Its proof is straightforward, taking into account that if G is a product of $\mathcal{N}$-connected groups, then $G/Z_\infty(G)$ is a direct product.

Proposition 7. *Let $\mathcal{F}$ be a Lockett class satisfying the following properties:*

(1) *Whenever* $G \in \mathcal{F}$ *and* $N \leq Z(G)$*, then* $G/N \in \mathcal{F}$.
(2) *Whenever* $G/N \in \mathcal{F}$ *and* $N \leq Z(G)$*, then* $G \in \mathcal{F}$.

(For instance, $\mathcal{F} = \mathcal{N}\Diamond\mathcal{X} = (\, G : G/F(G) \in \mathcal{X}\,)$ *for a Lockett class* $\mathcal{X}$*.) If the group* $G = G_1G_2\cdots G_r$ *is the product of the pairwise* $\mathcal{N}$*-connected and permutable subgroups* $G_1, G_2, \ldots, G_r$*, then* $G_{\mathcal{F}} = (G_1)_{\mathcal{F}} \cdots (G_r)_{\mathcal{F}}$.

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