

SHARP SUBELLIPTIC ESTIMATES FOR $n - 1$ FORMS ON FINITE TYPE DOMAINS

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ABSTRACT. Let $x_0 \in b\Omega$ in a smooth domain $\Omega \subset \mathbf{C}^n$, which is not assumed to be pseudoconvex. We define a finite type condition $R(L, x_0)$ for a vector field $L \in T^{1,0}(b\Omega)$, which equals the well-known type $c(L, x_0)$ in certain important cases. We prove that if $R(L, x_0) = m$, then a subelliptic estimate of order $1/m$ holds at x_0 for $(p, n - 1)$ forms.

1. Introduction

Let Ω be a domain in \mathbf{C}^n with smooth boundary. In conjunction with proving local regularity results for $\bar{\partial}$, Kohn and Nirenberg [20] introduced subelliptic estimates (see Definition 2). There are many results on subelliptic estimates in case Ω is pseudoconvex (see, for example, [18],[19],[3],[4],[5]). In particular, Catlin ([3],[4],[5]) established necessary and sufficient conditions for subelliptic estimates for (p, q) forms on smoothly bounded pseudoconvex domains.

In this paper we consider domains that are not pseudoconvex. We suppose that there is a $(1, 0)$ vector field L such that the Levi form λ_L is nonnegative near x_0 and also that L satisfies a certain type condition (see Definition 1) at a boundary point x_0 . We then establish a subelliptic estimate for $(p, n - 1)$ forms at x_0 .

Before we introduce this new type condition, we recall some of the known results on subelliptic estimates, both for the case when Ω is pseudoconvex and for the general case.

On pseudoconvex domains Kohn [19] used the vector field estimates of Rothschild and Stein [21] to prove that if the maximum order of contact of $n - 1$ dimensional manifolds with the boundary at x_0 (which is the type at x_0) is m , then there is a subelliptic estimate of order $\epsilon = 1/m$ at x_0 . By the work of Greiner [13] this is best possible in \mathbf{C}^2 . Later, Catlin [3] gave general necessary condition for subellipticity for (p, q) forms in \mathbf{C}^n . He showed that $\epsilon \leq 1/m$

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when the order of contact of a q -dimensional complex analytic variety with the boundary is m . In a fundamental work [5] he proved that if a domain is of finite type at x_0 (in the sense of D'Angelo), then there is a subelliptic estimate at x_0 . There are other proofs that establish Kohn's subelliptic estimate in \mathbf{C}^2 and in convex domains. Catlin [6] proved this estimate as a corollary of the construction of certain plurisubharmonic functions in pseudoconvex domains of finite type in \mathbf{C}^2 . Also, the construction of plurisubharmonic peak functions of Fornaess and Sibony [12] lead to the same result in \mathbf{C}^2 and in convex domains in \mathbf{C}^n .

On non-pseudoconvex domains Hörmander [17] proved that if the Levi form has $n - q$ positive eigenvalues or $q + 1$ negative eigenvalues at x_0 , then there is a subelliptic estimate of order $\epsilon = 1/2$ at x_0 for (p, q) forms. In the case when $q = n - 1$, this implies that if the Levi-form has one positive eigenvalue, then a subelliptic estimate of order $1/2$ holds for $(p, n - 1)$ forms. Subelliptic estimates on non-pseudoconvex domains were studied by Derridj [10] and Ho [14]. In [14] it was proved that if there is a vector field L of type $(1, 0)$ with non-negative Levi form and the type $c(L, x_0)$ equals m , then there is a subelliptic estimate for $(p, n - 1)$ forms. However, the value of ϵ obtained there is very weak, namely $\epsilon = 1/2^m$. There are attempts to improve this value ϵ to the expected value $1/m$ (see [15], [16]).

In this paper we define a new type $R(L, x_0)$ and prove that the expected order of the subelliptic estimate $\epsilon = 1/m$ is achieved if $R(L, x_0) = m$.

DEFINITION 1. Let $\Omega = \{z : r(z) < 0\}$ be a smooth domain in \mathbf{C}^n , $x_0 \in b\Omega$, and $L \in T^{1,0}(b\Omega)$. We define

$$R(L, x_0) = 2 + \min \{m \mid (\operatorname{Re}(aL))^m \lambda_L(x_0) \neq 0 \\ \text{for some } C^\infty \text{ function } a \text{ near } x_0\}.$$

We will discuss the relation between $R(L, x_0)$ and the well-known vector field type $c(L, x_0)$ in Section 3. Our main result is as follows.

MAIN THEOREM. Let $\Omega = \{z : r(z) < 0\}$ be a smooth domain in \mathbf{C}^n , and let $x_0 \in b\Omega$. Assume that there exists a vector field $L \in T^{1,0}(b\Omega)$ with

- (1) $\lambda_L = \partial\bar{\partial}r(L, \bar{L}) \geq 0$,
- (2) $R(L, x_0) = m$.

Then a subelliptic estimate of order $\epsilon = 1/m$ holds at x_0 for $(p, n - 1)$ forms.

REMARKS.

- (1) We do not assume that Ω is pseudoconvex. When it is, our result for $(p, n - 1)$ forms is well-known. It is the same as a theorem in Kohn [19] and is also contained in a result of Catlin for \mathbf{C}^2 [6].

- (2) The order of the subelliptic estimate $\epsilon = 1/m$ is sharp in all known cases. Though a general necessary condition theorem relating the type m of L (in the sense of either $c(L, x_0)$ or $R(L, x_0)$) and the value of ϵ in the subelliptic estimate has not been established, we know that the subelliptic estimate is sharp in a great variety of non-pseudoconvex domains (Ho [15]), and also in all pseudoconvex domains (Catlin [3]).

The proof of the Main Theorem follows the argument of Ho [16]. The key ingredient is the construction of real functions μ_k so that for some constant C and every k ,

$$2^k \lambda + X^2 \mu_k \geq C 2^{\frac{2k}{m+2}},$$

where $m+2$ is the $R(L, x_0)$ type of L , $X = \text{Re}(aL)$ for some smooth function a near x_0 , and λ is the Levi form of L . The functions μ_k may be compared with the plurisubharmonic function used by Catlin [6]. However, it is of interest to note here that the construction of μ_k only makes use of the Levi form λ and there is no explicit reference to the defining function r . This is in sharp contrast with the proofs of Catlin [6] and Fornaess and Sibony [12].

2. Preliminaries

Let Ω be a smoothly bounded domain in \mathbf{C}^n that is endowed with a smoothly varying inner product $\langle \cdot, \cdot \rangle_x$ on $T_x^{1,0}$ for $x \in \bar{\Omega}$, where $T_x^{1,0}$ denotes the holomorphic vectors at x . This inner product extends to the space of (p, q) forms at x . We define an inner product on (p, q) forms u and v by

$$(u, v) = \int_{\Omega} \langle u, v \rangle dV$$

and set

$$\|u\|^2 = (u, u).$$

We will use $\| |u| \|_{\epsilon}^2$ to denote the tangential Sobolev norm of u of order ϵ , and write $\bar{\partial}^*$ for the L^2 adjoint of $\bar{\partial}$. The domain of $\bar{\partial}^*$ is defined as usual by

$$\text{Dom}(\bar{\partial}^*) = \left\{ u \in L^{p,q}(\Omega) \mid \text{there exists } C > 0 \text{ with } |(\bar{\partial}f, u)| \leq C \|f\| \right. \\ \left. \text{for all } f \in \text{Dom}(\bar{\partial}) \right\}$$

DEFINITION 2. *Let Ω be a domain in \mathbf{C}^n . We say that a subelliptic estimate holds for (p, q) forms at $x_0 \in b\Omega$ if there is a neighborhood U of x_0 and there are constants $\epsilon > 0$ and $C > 0$, such that*

$$\| |u| \|_{\epsilon}^2 \leq C \left(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 + \|u\|^2 \right)$$

for all smooth (p, q) forms $u \in \text{Dom}(\bar{\partial}^*)$ with compact support in U .

Set $T^{1,0}(b\Omega) = T^{1,0} \cap T(b\Omega)$, and let L_1, L_2, \dots, L_n be C^∞ vector fields that form a basis of $T^{1,0}$, with $L_1, \dots, L_{n-1} \in T^{1,0}(b\Omega)$. We denote the “bad direction” by T , where $T = L_n - \bar{L}_n$. Let $\omega_1, \omega_2, \dots, \omega_n$ be $(1,0)$ forms dual to L_1, L_2, \dots, L_n . For the sake of simplicity, we will prove the Main Theorem for $(0, n - 1)$ forms only, but the proof is clearly equally valid for $(p, n - 1)$ forms. We will further assume that a $(0, n - 1)$ form u is of the form

$$u = u\bar{\omega}_1 \wedge \bar{\omega}_2 \wedge \dots \wedge \bar{\omega}_{n-1}.$$

(The use of u to denote both the form and its first coefficient will not lead to any confusion.) We discard the other coefficients u_I because these coefficients all involve $\bar{\omega}_n$ and hence are zero on $b\Omega$. Thus they satisfy

$$\|u_I\|_1^2 \leq C \left(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 + \|u\|^2 \right).$$

It is easy to see that

$$\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 \approx \|L_1u\|^2 + \|L_2u\|^2 + \dots + \|L_{n-1}u\|^2 + \|\bar{L}_nu\|^2.$$

(The reader may refer to Folland and Kohn [11] for these facts.)

If ϕ is a C^2 function on $\bar{\Omega}$, we define, for functions $u, v \in C(\bar{\Omega})$,

$$(u, v)_\phi = \int_\Omega u\bar{v}e^{-\phi} dV$$

and

$$\|u\|_\phi^2 = \int_\Omega |u|^2 e^{-\phi} dV.$$

3. Notions of vector field types

The following notion of type was introduced by Kohn ([18],[19]) to measure the vanishing order of the Levi-form in the holomorphic and anti-holomorphic directions that are tangential to the boundary.

DEFINITION 3 (Kohn [18]). *Let Ω be a domain in \mathbf{C}^n , let $p \in b\Omega$, and let $L \in T^{1,0}(b\Omega)$ be nonvanishing at p . We define $\text{type}_p L$ to be the smallest integer k such that there is an iterated commutator*

$$\langle \partial r, [\dots [[L_1, L_2], L_3], \dots, L_k] \rangle (p) \neq 0,$$

where each L_i is either L or \bar{L} , $[,]$ is the Lie bracket and \langle , \rangle is the contraction between co-tangent vectors and tangent vectors. If no such k exists, then we set $\text{type}_p L = \infty$.

Another notion of type was introduced by Bloom [1].

DEFINITION 4. *With the same notations as in the preceding definition, we define $c(L, p)$ to be 2 plus the smallest order of a polynomial f in L and \bar{L} such that $f(L, \bar{L})\lambda_L(p) \neq 0$. If no such f exists, then we set $c(L, p) = \infty$.*

In dimension $n = 2$ it is always true that $\text{type}_p L = c(L, p)$. For $n \geq 3$ it was proved by D'Angelo [8] that the two types are equal if the domain is pseudoconvex and one of the two values equals 4. D'Angelo also proved that for larger values of $N = \text{type}_p L$,

$$N \leq c(L, p) \leq \max(N, 2N - 6).$$

These partial results suggest that perhaps the two values are equal whenever Ω is pseudoconvex. These two vector types were used and discussed by other authors; see, for example, Bloom and Graham [2], Catlin [4], Sibony [22], and Talhoui [23].

Kohn [19] proved that the type can also be characterized by the order of contact of manifolds with the boundary:

THEOREM 5. *If $\Omega \subset \mathbf{C}^n$ is pseudoconvex and $p \in b\Omega$, then*

$$\min_{L \in T^{1,0}(b\Omega)} \text{type}_p L = \min_{L \in T^{1,0}(b\Omega)} c(L, p) = \Delta_{\text{reg}}^{n-1}(\Omega, p),$$

where $\Delta_{\text{reg}}^{n-1}(\Omega, p)$ is the maximum order of contact of $n - 1$ dimensional complex manifolds through p with the boundary.

D'Angelo[7] defined a type $\Delta(M, p)$ which is the maximum order of contact of complex analytic varieties with the boundary. It is this type value that Catlin used to prove the necessary and sufficient condition for subelliptic estimates on pseudoconvex domains.

We now discuss the relation between the types $c(L, p)$ and $R(L, p)$, defined in Section 1. It is clear that if $(\text{Re}(aL))^m \lambda_L(p) \neq 0$, then there is a polynomial f of degree m such that $f(L, \bar{L}) \lambda_L(p) \neq 0$. Hence

$$c(L, p) \leq R(L, p).$$

There are many cases in which we can prove the reverse inequality. In particular, this is true in the following two important cases.

PROPOSITION 6. *Let Ω be a smooth domain in \mathbf{C}^n and $p \in b\Omega$. Suppose that*

- (a) Ω is pseudoconvex, and $L \in T^{1,0}(b\Omega)$ minimizes $c(L, p)$, or
- (b) for some $L \in T^{1,0}(b\Omega)$, $\{L, \bar{L}, T\}$ is closed under the Lie bracket.

Then

$$c(L, p) = R(L, p).$$

Proof. (a) First we consider the case $n = 2$. If $c(L, p) = m$, then

$$(1) \quad L_1 L_2 \dots [L_i, L_{i+1}] L_{i+2} \dots L_{m-2} \lambda_L(p) = 0,$$

where L_i equals L or \bar{L} (see Kohn [19], Lemma 5.32 and Lemma 5.34). Now there is a choice of L_i such that $L_1 L_2 \dots L_{m-2} \lambda(p) \neq 0$ and $L_1 L_2 \dots L_j \lambda(p) = 0$ for all $j \leq m - 3$. Let $X = \text{Re } L$ and $Y = \text{Im } L$. We need to show that

$(aX + bY)^{m-2}\lambda(p) \neq 0$ for some real numbers a and b . This will imply that $(\operatorname{Re}(\alpha L))^{m-2}\lambda(p) \neq 0$ for some complex number α . Otherwise we have

$$(a^{m-2}X^{m-2} + a^{m-3}b(X^{m-3}Y + X^{m-4}YX + \dots + YX^{m-3}) + \dots + b^{m-2}Y^{m-2})\lambda(p) = 0$$

for all a and b , and hence

$$\begin{aligned} X^{m-2}\lambda(p) &= (X^{m-3}Y + X^{m-4}YX + \dots + YX^{m-3})\lambda(p) \\ &= \dots = Y^{m-2}\lambda(p) = 0. \end{aligned}$$

We use (1) to conclude that the terms $A_1A_2 \dots A_{m-2}\lambda(p)$, where each A_i equals X or Y , all vanish. But this contradicts the fact that, for some choice of the L_i , $L_1L_2 \dots L_{m-2}\lambda(p) \neq 0$.

Now suppose $n > 2$ and that L minimizes $c(L, p)$. We need to show that

$$L_1L_2 \dots [L_i, L_{i+1}]L_{i+2} \dots L_{m-2}\lambda_L(p) = 0$$

for all choices of $L_i \in \{L, \bar{L}\}$. This will then imply the desired conclusion as in the above case. If this is not true, then by Kohn’s definition of type we have

$$\min_{L \in T^{1,0}(b\Omega)} \operatorname{type}_p L < \min_{L \in (1)T^{1,0}(b\Omega)} c(L, p),$$

which contradicts Theorem 5.

(b) The proof is the same as in the case $n > 2$ of (a) since (1) is true by the assumption of the closedness of the Lie bracket. \square

EXAMPLE. The domain defined by $r = 2 \operatorname{Re} z_3 - |z_1 - \bar{z}_1 z_2|^2 + |z_2|^{10}$ is an example that does not satisfy assumption (b) in Proposition 6 for the following holomorphic vector field L at the origin:

$$L = \bar{z}_1 \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \left(-|z_1|^2 \bar{z}_2 + \bar{z}_1^2 |z_2|^2 - 5z_2^4 \bar{z}_2^5 \right) \frac{\partial}{\partial z_3}$$

The Levi-form of L is

$$\lambda_L = 2|z_1|^2 - |z_1|^2 |z_2|^2 - \bar{z}_1^2 z_2 - z_1^2 \bar{z}_2 + 25|z_2|^8 \geq 0$$

near the origin. But

$$[L, \bar{L}] = -z_1 \frac{\partial}{\partial z_1} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \alpha \frac{\partial}{\partial z_3} + \beta \frac{\partial}{\partial \bar{z}_3}$$

is not a linear combination of L, \bar{L} and $T = L_3 - \bar{L}_3$. Though the hypotheses of Proposition 6 are not satisfied here, the conclusion $c(L, p) = R(L, p)$ still holds: in fact, we have $c(L, p) = R(L, p) = 10$ in this case. We have not yet found an example in which both $\lambda_L \geq 0$ and $c(L, p) < R(L, p)$ hold.

For small values of $R(L, p)$, we can give the following estimates for $c(L, p)$:

PROPOSITION 7. *Let Ω be a domain in $\mathbf{C}^n, p \in b\Omega$, and $L \in T^{1,0}(b\Omega)$.*

- (1) If $R(L, p) = 4$, then $c(L, p) = 4$.
- (2) If $R(L, p) = 6$, then $5 \leq c(L, p) \leq 6$.
- (3) If $R(L, p) = 8$, then $6 \leq c(L, p) \leq 8$.
- (4) If $R(L, p) = 10$, then $7 \leq c(L, p) \leq 10$.

Proof. We first note that $R(L, p)$ is always even, since we are considering the derivatives of a real vector field on a non-negative function (see Kohn [19], Lemma 5.28). It is not clear whether the type $c(L, p)$ is always even. Also, note that

$$R(L, p) = 2 + \min \{ m \mid (aX + bY)^m \lambda_L(x_0) \neq 0 \text{ for some } C^\infty \text{ functions } a, b \text{ near } x_0 \}.$$

We now move on to the proof of the four statements. The first statement is trivial since $R(L, p) = 4$ implies $X\lambda = Y\lambda = 0$, which in turn implies $L\lambda = \bar{L}\lambda = 0$, from which $c(L, p) = 4$ follows. (All derivatives are evaluated at p .) We outline the proof of the fourth statement here; the proofs of the second and third statements are similar.

If $R(L, p) = 10$, then setting $X = \text{Re } L$ and $Y = \text{Im } L$ we have

$$(aX + bY)^4 \lambda = (aX + bY)^5 \lambda = (aX + bY)^6 \lambda = (aX + bY)^7 \lambda = 0$$

for all smooth functions a and b defined near the point p . We will show that we get the following linear system, which implies that all four terms are 0:

$$\begin{aligned}
 & XXXXY\lambda + XXYX\lambda + XYXX\lambda + YXXX\lambda = 0 \\
 & 4XXXXY\lambda + 3XXYX\lambda + 2XYXX\lambda + YXXX\lambda = 0 \\
 & 10XXXXY\lambda + 6XXYX\lambda + 3XYXX\lambda + YXXX\lambda = 0 \\
 & 20XXXXY\lambda + 10XXYX\lambda + 4XYXX\lambda + YXXX\lambda = 0
 \end{aligned}$$

Consider the equation $(aX + bY)^4 \lambda = 0$ for constants a and b . The left hand side of the first equation in the above system is just the coefficient of a^3b in $(aX + bY)^4 \lambda$, and we conclude that it equals to 0 as in the proof of Proposition 6.

Next we consider the equation $(aX + bY)^5 \lambda = 0$. This time we need to consider a and b as smooth functions. The expansion of $(aX + bY)^5 \lambda$ will involve X and Y derivatives of the functions a and b . We get

$$\begin{aligned}
 (aX + bY)^5 \lambda &= a^5 X^5 \lambda + \dots + b^5 Y^5 \lambda \\
 &+ a^4 (Xb)(4XXXXY\lambda + 3XXYX\lambda + 2XYXX\lambda + YXXX\lambda) \\
 &+ \dots = 0
 \end{aligned}$$

where \dots stands for terms involving the second, third, fourth, and fifth X and Y derivatives of the functions a and b . Since a and b are allowed to be any smooth functions, $(Xb)(p)$ is just another variable. Hence we conclude that the coefficient of $a^4(Xb)$ must be equal to 0. This gives the second equation

in the above system. In a similar manner, the third equation is obtained by considering the coefficient of $a^5(X^2b)$ in $(aX + bY)^6 \lambda = 0$, and the fourth equation comes from the coefficient of $a^6(X^3b)$ in $(aX + bY)^7 \lambda = 0$.

Next we need to set up six equations for the six terms $XXYY\lambda$, $XYXY\lambda$, $XYXX\lambda$, $YXXY\lambda$, $YXYX\lambda$, and $YYXX\lambda$. Here the computations and the equations involved are more complicated. By considering the coefficients of a^2b^2 in $(aX + bY)^4 \lambda = 0$, of $a^3b(Xb)$ in $(aX + bY)^5 \lambda = 0$, of $a^2b^2(Xa)$ in $(aX + bY)^5 \lambda = 0$, of $a^4b(X^2b)$ in $(aX + bY)^6 \lambda = 0$, of $a^2b^2(Xa)(Xb)$ in $(aX + bY)^6 \lambda = 0$, and of $a^5b(X^3b)$ in $(aX + bY)^7 \lambda = 0$, we get the following six equations:

$$\begin{aligned}
 XXYY\lambda + XYXY\lambda + XYXX\lambda + YXXY\lambda &+ YXYX\lambda + YYXX\lambda = 0 \\
 7XXYY\lambda + 6XYXY\lambda + 5XYXX\lambda + 5YXXY\lambda &+ 4YXYX\lambda + 3YYXX\lambda = 0 \\
 5XXYY\lambda + 2XYXY\lambda + 5XYXX\lambda + 5YXXY\lambda &+ 6YXYX\lambda + 7YYXX\lambda = 0 \\
 16XXYY\lambda + 13XYXY\lambda + 9XYXX\lambda + 11YXXY\lambda &+ 7YXYX\lambda + 4YYXX\lambda = 0 \\
 43XXYY\lambda + 44XYXY\lambda + 41XYXX\lambda + 43YXXY\lambda &+ 38YXYX\lambda + 31YYXX\lambda = 0 \\
 30XXYY\lambda + 24XYXY\lambda + 14XYXX\lambda + 21YXXY\lambda &+ 11YXYX\lambda + 5YYXX\lambda = 0
 \end{aligned}$$

This clearly implies that

$$XXYY\lambda = XYXY\lambda = XYXX\lambda = YXXY\lambda = YXYX\lambda = YYXX\lambda = 0.$$

Finally, in a similar way as in the proof of the first four equations above, we see that

$$XYYY\lambda = YXYX\lambda = YYXY\lambda = YYYX\lambda = 0.$$

Hence $c(L, p) \geq 7$. \square

4. Lemmas

We need the following lemmas for the proof of the Main Theorem. For the sake of convenience we will assume that $x_0 = 0$. The constants C in the following proofs are not necessarily the same at each occurrence.

LEMMA 8. *Let Ω be a domain in \mathbf{C}^n with smooth boundary, and let $L \in T^{1,0}(b\Omega)$ with $\lambda = \partial\bar{\partial}r(L, \bar{L}) \geq 0$. Then, given any $\epsilon > 0$, there exists a*

neighborhood U of the origin such that for any function $u \in C_0^\infty(U \cap \bar{\Omega})$ we have

$$(2) \quad \|Lu\|^2 \geq \|\bar{L}u\|^2 + \int_{b\Omega} \lambda |u|^2 dS - \epsilon \left(\sum_{i=1}^{n-1} \|L_i u\|^2 + \|\bar{L}_n u\|^2 \right) - O(\|u\|^2)$$

The proof is standard and can be found in Folland and Kohn [11] or Ho [14].

LEMMA 9. Let Ω be a domain in \mathbf{C}^n with smooth boundary, let X be a real tangential vector field defined in $\bar{\Omega}$, and let $\phi \in C^2(\bar{\Omega})$. Then, given any $\epsilon > 0$, there exists a neighborhood U of the origin such that for all functions $u \in C_0^\infty(U \cap \bar{\Omega})$ we have

$$(3) \quad \int_{\Omega} (X^2 \phi) |u|^2 e^{-\phi} dV \leq 4 \|(X\phi)u\|_{\phi}^2 + 2 \|Xu\|_{\phi}^2 + C \|u\|_{\phi}^2.$$

Furthermore, assume $\psi \in C^2(\bar{\Omega})$ is a bounded function and M is a constant so that $M \geq 4e^{\psi}$. Define $\phi = \frac{1}{M}e^{\psi}$ and $X = \text{Re}(aL)$ for some smooth function a . Then we have the following estimate:

$$(4) \quad \int_{\Omega} (X^2 \psi) |u|^2 e^{-(\phi-\psi)} dV + \int_{b\Omega} \lambda |u|^2 dS \\ \leq C (\|Lu\|^2 + \|u\|^2) + \epsilon \left(\sum_{i=1}^{n-1} \|L_i u\|^2 + \|\bar{L}_n u\|^2 \right).$$

Proof. Letting g denote a smooth function, we have

$$\begin{aligned} ((X^2 \phi)u, u)_{\phi} &= (X((X\phi)u), e^{-\phi}u) - ((X\phi)(Xu), e^{-\phi}u) \\ &= -((X\phi)u, X(e^{-\phi}u)) + (g(X\phi)u, e^{-\phi}u) \\ &\quad - ((X\phi)(Xu), e^{-\phi}u) \\ &= ((X\phi)u, e^{-\phi}(X\phi)u) - ((X\phi)u, e^{-\phi}Xu) + (g(X\phi)u, e^{-\phi}u) \\ &\quad - ((X\phi)(Xu), e^{-\phi}u) \\ &\leq 4 \|(X\phi)u\|_{\phi}^2 + 2 \|Xu\|_{\phi}^2 + C \|u\|_{\phi}^2 \quad . \end{aligned}$$

To prove the second statement, we note that if $\phi = \frac{1}{M}e^{\psi}$ and $M \geq 4e^{\psi}$, then

$$(5) \quad X^2 \phi - 4|X\phi|^2 \geq \frac{1}{M}e^{\psi} X^2 \psi.$$

Also, using Lemma 8 to estimate $\|\bar{L}u\|^2$ we get

$$(6) \quad \begin{aligned} \|Xu\|_\phi^2 &\leq C \left(\|Lu\|^2 + \|\bar{L}u\|^2 \right) \\ &\leq C \left(\|Lu\|^2 + \|u\|^2 \right) + \epsilon \left(\sum_{i=1}^{n-1} \|L_i u\|^2 + \|\bar{L}_n u\|^2 \right). \end{aligned}$$

If we put (5) and (6) into (3) and add the resulting inequality to inequality (2), we get the desired inequality (4). \square

LEMMA 10. *If there is a neighborhood U of the origin such that for all $k \geq 1$ we can find a function $\mu_k \in C^2(U \cap \bar{\Omega})$ that is uniformly bounded in k , and a constant $C > 0$ independent of k such that*

$$2^k \lambda + X^2 \mu_k \geq C 2^{\frac{2k}{N}}$$

in $U \cap \bar{\Omega}$, then there is a subelliptic estimate of order $\frac{1}{N}$ for $(p, n - 1)$ forms at the origin.

The proof of this lemma is very similar to the proof of Theorem 1 in Ho [16]; the only difference is that we need to use inequality (4) of Lemma 9 instead of inequality (8) in [16]. Hence we omit the proof here.

For the construction of μ_k we also need the following technical lemmas.

LEMMA 11. *Given any $\epsilon > 0$ and K large enough we can find a non-negative function $\mu \in C_0^\infty[0, \infty)$ that satisfies the following properties, where μ depends on K , but the constants C and M are independent of K .*

- (1) $\mu(x) = x$ for $x \leq 40$.
- (2) μ is supported in $[0, MK]$.
- (3) There exists x_0 independent of K such that $\mu'(x) = 0$ on $[x_0, \frac{1}{100}K]$
- (4) $\mu'(x) \geq 0$ if $x \leq \frac{1}{100}K$.
- (5) $x\mu''(x) \geq -\epsilon$ if $x \leq \frac{1}{100}K$.
- (6) $|\mu'(x)| \leq \frac{C}{K}$ if $x \geq \frac{1}{100}K$.
- (7) $|\mu''(x)| \leq \frac{C}{K^2}$ if $x \geq \frac{1}{100}K$.

Proof. By Lemma 3 of Ho [16] we can find $\phi \in C_0^\infty(\mathbf{R}^+)$ such that

- (1) $\phi(x) = x$ for $0 \leq x \leq 40$;
- (2) $\phi'(x) \geq -\epsilon$ for all $x \geq 0$;
- (3) $x\phi''(x) \geq -\epsilon$ for all $x \geq 0$.

In fact, the proof given in [16] shows that ϕ' is a decreasing function on $[0, x_0]$, where $x_0 > 0$ is some number that $\phi'(x_0) = 0$. By slightly modifying this proof, we can assume that $\phi^{(n)}(x_0) = 0$ for all $n \geq 1$. We define the function $\mu(x)$ by

- (1) $\mu(x) = \phi(x)$ for $x \leq x_0$;
- (2) $\mu(x) = \phi(x_0)$ for $x_0 \leq x \leq \frac{1}{100}K$;

$$(3) \mu(x) = \phi\left(\frac{x - \frac{1}{100}K}{K} + x_0\right) \text{ for } x \geq \frac{1}{100}K.$$

If ϕ is supported in $[0, A]$, then μ is supported in $[0, K(A - x_0 + \frac{1}{100})]$. We now verify that the properties stated in the lemma are all satisfied. In fact, properties (1)–(5) are obvious from the construction, and (6) and (7) follow easily from the fact that when $x \geq \frac{1}{100}K$,

$$|\mu'(x)| = \frac{1}{K} \left| \phi' \left(\frac{x - \frac{1}{100}K}{K} + x_0 \right) \right| \leq \frac{C}{K}$$

and

$$|\mu''(x)| = \frac{1}{K^2} \left| \phi'' \left(\frac{x - \frac{1}{100}K}{K} + x_0 \right) \right| \leq \frac{C}{K^2}. \quad \square$$

LEMMA 12. *There exists a function $\chi \in C^\infty(\mathbf{R}^+)$ which satisfies the following properties:*

- (1) $\chi(x) = 0$ when $x \leq 20K$ and $\chi(x) = 1$ when $x \geq 40K$.
- (2) $\chi' \geq 0$.
- (3) $|\chi'| \leq \frac{1}{2K}$.
- (4) $|\chi''| \leq \frac{1}{10K^2}$.

Proof. First, we can find a function $\theta(t) \in C_0^\infty(\mathbf{R})$ with the following properties:

- (1) $\theta(t)$ is supported in $[20, 40]$.
- (2) $\theta(t) \geq 0$ in $[20, 30]$.
- (3) $\theta(t) \leq \frac{1}{10}$.
- (4) $\int_{20}^{25} \theta(t) dt \geq \frac{1}{10}$ and $\int_{20}^{30} \theta(t) dt \leq \frac{1}{2}$.
- (5) $\theta(t - 30) = -\theta(30 - t)$.

Next, we set $\theta_1(t) = \int_0^t \theta(y) dy$. Then it is clear that $\theta_1(t)$ has the following properties:

- (1) $\theta_1(t) = 0$ for $x \leq 20$.
- (2) $0 \leq \theta_1(t) \leq \frac{1}{2}$ and θ_1 is supported in $[20, 40]$.
- (3) $\int_0^{40} \theta_1(t) dt \geq 1$.

We may assume that $\int_0^{40} \theta_1(t) dt = 1$.

Finally, we set $\xi(x) = \int_0^x \theta_1(t) dt$, and define $\chi(x) = \xi\left(\frac{x}{K}\right)$, so that $\chi'(x) = \frac{1}{K}\xi'\left(\frac{x}{K}\right)$. It is clear that χ satisfies the required properties. \square

5. Proof of the Main Theorem

In view of Lemma 10 the following proposition implies the Main Theorem.

PROPOSITION 13. *Assume that there is a neighborhood U of the origin, a non-negative function $f \in C^\infty(U \cap \bar{\Omega})$, and a C^∞ real vector field X in $U \cap \bar{\Omega}$*

such that $X^m f(0) \neq 0$ and $X^i f(0) = 0$ for all $i \leq m-1$. Then there exists a neighborhood $V \subset U$ of 0 and a function $\tilde{\mu} \in C^\infty(V \cap \bar{\Omega})$ that satisfies

$$2^k f + X^2 \tilde{\mu} \geq C 2^{\frac{2k}{m+2}}$$

in $V \cap \bar{\Omega}$, where the constant C is independent of k .

We define

$$\tilde{\mu} = \sum_{i=1}^m c_i \mu_i \left(2^{\frac{2k}{m+2}} \left| \frac{X^{i-1} f}{X^i f} \right|^2 \right) \prod_{j=i}^{m-1} \chi_j \left(2^{\frac{2k}{m+2}} \left| \frac{X^j f}{X^{j+1} f} \right|^2 \right) = \sum_{i=1}^m c_i \phi_i,$$

where we let $X^m f = 1$ in the above definition of the functions μ_m and χ_{m-1} , the constants c_i will be chosen later, and the functions μ_i and χ_j are defined as follows:

- (1) $\mu_i(x) = \mu\left(\frac{x}{K^{i-1}}\right)$ for $1 \leq i \leq m$;
- (2) $\chi_j(x) = \chi\left(\frac{x}{K^{j-1}}\right)$ for $1 \leq j \leq m-1$.

From the properties of the functions μ and χ given in Lemmas 11 and 12 it is clear that for $i \geq 1$ we have:

- (m1) $\mu'_i(x) = \frac{1}{K^{i-1}}$ for $x \leq 40K^{i-1}$.
- (m2) μ_i is supported in $[0, MK^i]$, where M is independent of K .
- (m3) $\mu'_i(x) \geq 0$ if $x \leq \frac{1}{100}K^i$.
- (m4) $x\mu''_i(x) \geq \frac{\epsilon}{K^{i-1}}$ if $x \leq \frac{1}{100}K^i$.
- (m5) $|\mu'_i(x)| \leq \frac{C}{K^i}$ for $x \geq \frac{1}{100}K^i$.
- (m6) $|\mu''_i(x)| \leq \frac{C}{K^{2i}}$ for $x \geq \frac{1}{100}K^i$.

Also for $j \geq 1$ we have:

- (c1) $\chi_j = 0$ for $x \leq 20K^j$.
- (c2) $\chi_j = 1$ for $x \geq 40K^j$.
- (c3) $\chi'_j \geq 0$ and is supported in between $20K^j \leq x \leq 40K^j$.
- (c4) $|\chi'_j| \leq \frac{1}{2K^j}$.
- (c5) $|\chi''_j| \leq \frac{1}{10K^{2j}}$.

The number K here will be specified at the end of our estimates. In the following argument C is independent of K .

The above function $\tilde{\mu}$ is well-defined in a small neighborhood V of the origin since $X^m f(x) \neq 0$ if V is small enough. Hence if $X^i f = 0$ in some term $\mu_i \left(2^{\frac{2k}{m+2}} \left| \frac{X^{i-1} f}{X^i f} \right|^2 \right)$, then there will be a term equal to 0 in the product $\prod_{j=i}^{m-1} \chi_j \left(2^{\frac{2k}{m+2}} \left| \frac{X^j f}{X^{j+1} f} \right|^2 \right)$.

To show why this function $\tilde{\mu}$ gives the desired result, we first prove two lemmas.

LEMMA 14. *With the same notations as in Proposition 13, given any $\epsilon > 0$ there exists a neighborhood V of the origin so that when K is large enough the following inequalities are satisfied in $V \cap \overline{\Omega}$:*

- (1) $X^2\phi_m \geq \frac{C}{K^{m-1}}2^{\frac{2k}{m+2}}$ if $|X^{m-1}f|^2 \leq 40 \cdot 2^{-\frac{2k}{m+2}} \cdot K^{m-1}$;
- (2) $|X^2\phi_m| \leq \frac{\epsilon}{K^{m-1}}2^{\frac{2k}{m+2}}$ if $|X^{m-1}f|^2 \geq 40 \cdot 2^{-\frac{2k}{m+2}} \cdot K^{m-1}$.

Proof. We have

$$\begin{aligned} X^2\phi_m &= X^2\mu_m \left(2^{\frac{2k}{m+2}} |X^{m-1}f|^2 \right) \\ &= 2^{\frac{2k}{m+2}}\mu'_m \left(2^{\frac{2k}{m+2}} |X^{m-1}f|^2 \right) \left(2(X^m f)^2 + 2X^{m+1}fX^{m-1}f \right) \\ &\quad + 2^{\frac{4k}{m+2}}\mu''_m \left(2^{\frac{2k}{m+2}} |X^{m-1}f|^2 \right) (2X^m fX^{m-1}f)^2. \end{aligned}$$

It is clear that if the neighborhood V is small enough, then

$$(X^m f)^2 + X^{m+1}fX^{m-1}f \geq C > 0.$$

Hence when $|X^{m-1}f|^2 \leq 40 \cdot 2^{-\frac{2k}{m+2}} \cdot K^{m-1}$, then from property (m1) and the fact that $\mu''_m = 0$ we get $X^2\phi_m \geq \frac{C}{K^{m-1}}2^{\frac{2k}{m+2}}$.

If $|X^{m-1}f|^2 \geq 40 \cdot 2^{-\frac{2k}{m+2}} \cdot K^{m-1}$, then we distinguish two cases:

- (1) When $2^{\frac{2k}{m+2}} |X^{m-1}f|^2 \leq \frac{1}{100}K^m$, we have by (m4)

$$2^{\frac{2k}{m+2}}\mu''_m \left(2^{\frac{2k}{m+2}} |X^{m-1}f|^2 \right) (X^{m-1}f)^2 \geq \frac{-\epsilon}{K^{m-1}}$$

and $\mu'_m \geq 0$ by (m3). Hence $X^2\phi_m \geq -\frac{\epsilon C}{K^{m-1}}2^{\frac{2k}{m+2}}$.

- (2) When $\frac{1}{100}K^m \leq 2^{\frac{2k}{m+2}} |X^{m-1}f|^2 \leq MK^m$, we have by (m5) and (m6), $|\mu'_m| \leq \frac{C}{K^m}$ and $|\mu''_m| \leq \frac{C}{K^{2m}}$. Hence

$$|X^2\phi_m| \leq \frac{C}{K^m}2^{\frac{2k}{m+2}} + MK^m \frac{C}{K^{2m}}2^{\frac{2k}{m+2}}.$$

This proves (2) when K is large. □

LEMMA 15. *With the same notations as in Proposition 13, given any $\epsilon > 0$ there exists a neighborhood V of the origin so that when K is large enough the following inequalities are satisfied in $V \cap \overline{\Omega}$:*

- (1) $X^2\phi_{m-1} \geq \frac{C}{K^{m-2}}2^{\frac{2k}{m+2}}$ if $\left| \frac{X^{m-2}f}{X^{m-1}f} \right|^2 \leq 40 \cdot 2^{-\frac{2k}{m+2}} \cdot K^{m-2}$ and $|X^{m-1}f|^2 \geq 40 \cdot 2^{-\frac{2k}{m+2}} \cdot K^{m-1}$.
- (2) $|X^2\phi_{m-1}| \leq \frac{\epsilon}{K^{m-2}}2^{\frac{2k}{m+2}}$ if $\left| \frac{X^{m-2}f}{X^{m-1}f} \right|^2 \geq 40 \cdot 2^{-\frac{2k}{m+2}} \cdot K^{m-2}$.

The complete proof of this lemma follows from the proof of the proposition, so we will only give an outline. Since

$$\phi_{m-1} = \mu_{m-1} \left(2^{\frac{2k}{m+2}} \left| \frac{X^{m-2}f}{X^{m-1}f} \right|^2 \right) \chi_{m-1} \left(2^{\frac{2k}{m+2}} |X^{m-1}f|^2 \right),$$

when $|X^{m-1}f|^2 \geq 40 \cdot 2^{-\frac{2k}{m+2}} \cdot K^{m-1}$, we have $\chi_{m-1} = 1$ from (c2), and hence $\phi_{m-1} = \mu_{m-1}$ there. The rest of the proof is similar to the proof of the above lemma. Of course we need to estimate the derivatives of μ_{m-1} and χ_{m-1} . This will be carried out later.

From these two lemmas we see that the only “bad part” in $X^2 \left(\sum_{j=i}^m c_j \phi_j \right)$ is the part where $X^{i-1}f$ is ‘large’. When $i = 1$ this part is covered by the function f itself. \square

Proof of Proposition. Let us consider the first term

$$\phi_1(x) = \mu_1 \left(2^{\frac{2k}{m+2}} \left| \frac{f}{Xf} \right|^2 \right) \prod_{j=1}^{m-1} \chi_j \left(2^{\frac{2k}{m+2}} \left| \frac{X^j f}{X^{j+1} f} \right|^2 \right).$$

We need to compute $X^2 \phi_1$. We write

$$X^2 \phi_1 = 2^{\frac{2k}{m+2}} (I + II + III + IV + V + VI),$$

where

$$\begin{aligned} I &= \mu_1' \left(2^{\frac{2k}{m+2}} \left| \frac{f}{Xf} \right|^2 \right) \left(2 - \frac{6fX^2f}{(Xf)^2} + \frac{6f^2(X^2f)^2}{(Xf)^4} - \frac{2f^2X^3f}{(Xf)^3} \right) \\ &\quad \times \prod_{j=1}^{m-1} \chi_j \left(2^{\frac{2k}{m+2}} \left| \frac{X^j f}{X^{j+1} f} \right|^2 \right), \\ II &= 2^{\frac{2k}{m+2}} \mu_1'' \left(2^{\frac{2k}{m+2}} \left| \frac{f}{Xf} \right|^2 \right) \left(\frac{2f}{Xf} - \frac{2f^2X^2f}{(Xf)^3} \right)^2 \prod_{j=1}^{m-1} \chi_j \left(2^{\frac{2k}{m+2}} \left| \frac{X^j f}{X^{j+1} f} \right|^2 \right), \\ III &= \sum_{j=1}^{m-1} \chi_j' \left(2^{\frac{2k}{m+2}} \left| \frac{X^j f}{X^{j+1} f} \right|^2 \right) \left(2 - \frac{6X^j f X^{j+2} f}{(X^{j+1} f)^2} + \frac{6(X^j f)^2 (X^{j+2} f)^2}{(X^{j+1} f)^4} \right. \\ &\quad \left. - \frac{2(X^j f)^2 X^{j+3} f}{(X^{j+1} f)^3} \right) \mu_1 \left(2^{\frac{2k}{m+2}} \left| \frac{f}{Xf} \right|^2 \right) \prod_{\substack{l=1 \\ l \neq j}}^{m-1} \chi_l \left(2^{\frac{2k}{m+2}} \left| \frac{X^l f}{X^{l+1} f} \right|^2 \right), \\ IV &= 2^{\frac{2k}{m+2}} \chi_j'' \left(2^{\frac{2k}{m+2}} \left| \frac{f}{Xf} \right|^2 \right) \left(\frac{2X^j f}{X^{j+1} f} - \frac{2(X^j f)^2 X^{j+2} f}{(X^{j+1} f)^3} \right)^2 \end{aligned}$$

$$\begin{aligned}
 & \times \mu_1 \left(2^{\frac{2k}{m+2}} \left| \frac{f}{Xf} \right|^2 \right) \prod_{\substack{l=1 \\ l \neq j}}^{m-1} \chi_l \left(2^{\frac{2k}{m+2}} \left| \frac{X^l f}{X^{l+1} f} \right|^2 \right), \\
 V &= 2^{\frac{2k}{m+2}+1} \sum_{j=1}^m \mu'_1 \left(2^{\frac{2k}{m+2}} \left| \frac{f}{Xf} \right|^2 \right) \chi'_j \left(2^{\frac{2k}{m+2}} \left| \frac{X^j f}{X^{j+1} f} \right|^2 \right) \left(\frac{2f}{Xf} - \frac{2f^2 X^2 f}{(Xf)^3} \right) \\
 & \times \left(\frac{2X^j f}{X^{j+1} f} - \frac{2(X^j f)^2 X^{j+2} f}{(X^{j+1} f)^3} \right) \prod_{\substack{l=1 \\ l \neq j}}^{m-1} \chi_l \left(2^{\frac{2k}{m+2}} \left| \frac{X^l f}{X^{l+1} f} \right|^2 \right), \\
 VI &= 2^{\frac{2k}{m+2}+1} \sum_{\substack{j,l=1 \\ j < l}}^{m-1} \chi'_j \left(2^{\frac{2k}{m+2}} \left| \frac{X^j f}{X^{j+1} f} \right|^2 \right) \chi'_l \left(2^{\frac{2k}{m+2}} \left| \frac{X^l f}{X^{l+1} f} \right|^2 \right) \\
 & \times \left(\frac{2X^j f}{X^{j+1} f} - \frac{2(X^j f)^2 X^{j+2} f}{(X^{j+1} f)^3} \right) \left(\frac{2X^l f}{X^{l+1} f} - \frac{2(X^l f)^2 X^{l+2} f}{(X^{l+1} f)^3} \right) \\
 & \times \mu_1 \left(2^{\frac{2k}{m+2}} \left| \frac{f}{Xf} \right|^2 \right) \prod_{\substack{p=1 \\ p \neq l, j}}^{m-1} \chi_p \left(2^{\frac{2k}{m+2}} \left| \frac{X^p f}{X^{p+1} f} \right|^2 \right).
 \end{aligned}$$

We will estimate I - VI in different regions of $\left| \frac{X^i f}{X^{i+1} f} \right|$. First we note that by (c1) we have, in the support of χ_1 , $2^{\frac{2k}{m+2}} \left| \frac{Xf}{X^2 f} \right|^2 \geq 20K$, i.e., $\left| \frac{X^2 f}{Xf} \right| \leq \frac{1}{\sqrt{20K^{1/2}}} 2^{\frac{k}{m+2}}$. In general, in the support of χ_j we have $\left| \frac{X^{j+1} f}{X^j f} \right| \leq \frac{1}{\sqrt{20K^{j/2}}} 2^{\frac{k}{m+2}}$.

Case 1. $\left| \frac{f}{Xf} \right|^2 \leq 40 \cdot 2^{\frac{-2k}{m+2}}$ and $2^{\frac{2k}{m+2}} \left| \frac{X^i f}{X^{i+1} f} \right|^2 \geq 40K^i$ for all $1 \leq i \leq m-1$.

In this case $\mu'_1 = 1, \mu''_1 = 0$, and we have $\chi_j = 1$ for all j and $\chi'_j, \chi''_j = 0$ for $1 \leq j \leq m-1$. Hence the terms II, III, IV, V and VI are all 0, and we only need to estimate I . We have

$$\begin{aligned}
 |I| &\geq 2 - 6 \left(\sqrt{40} \cdot 2^{\frac{-k}{m+2}} \right) \left(\frac{1}{\sqrt{20K^{1/2}}} 2^{\frac{k}{m+2}} \right) - 6 \left(40 \cdot 2^{\frac{-2k}{m+2}} \right) \left(\frac{1}{20K} 2^{\frac{2k}{m+2}} \right) \\
 &\quad - 2 \left(40 \cdot 2^{\frac{-2k}{m+2}} \right) \left(\frac{1}{\sqrt{20K^{1/2}}} 2^{\frac{k}{m+2}} \right) \left(\frac{1}{\sqrt{20K}} 2^{\frac{k}{m+2}} \right) \\
 &= 2 - \frac{6\sqrt{2}}{K^{1/2}} - \frac{12}{K} - \frac{4}{K^{3/2}}.
 \end{aligned}$$

Hence, when K is large enough, we get

$$X^2 \phi_1 \geq 2^{\frac{2k}{m+2}}.$$

Case 2. $\left| \frac{f}{Xf} \right|^2 \leq 40 \cdot 2^{\frac{-2k}{m+2}}$ and $2^{\frac{2k}{m+2}} \left| \frac{X^i f}{X^{i+1} f} \right|$ arbitrary.

We want to find a lower bound for $X^2\phi_1$ in this and the following cases. As in Case 1, we have, for K large enough, $I \geq 0$ and $II = 0$.

We use (c4) and (c5), i.e., $|\chi'_j| \leq \frac{1}{2K^j}$, $|\chi''_j| \leq \frac{1}{10K^{2j}}$, and the fact that $\chi'_j(x)$ is supported in $[20K^j, 40K^j]$ to obtain, for all $j \geq 1$,

$$(7) \quad \sqrt{20}K^{j/2} \leq 2^{\frac{k}{m+2}} \left| \frac{X^j f}{X^{j+1} f} \right| \leq \sqrt{40}K^{j/2}.$$

Also note that $|\mu_1| \leq C$.

Estimate for III. Using (c4) and (7), we get

$$\begin{aligned} |III| &\leq \sum_j C \cdot \frac{1}{2K^j} \left(2 + 6\sqrt{40}K^{j/2} 2^{\frac{-k}{m+2}} \frac{1}{\sqrt{20}K^{(j+1)/2}} 2^{\frac{k}{m+2}} \right. \\ &\quad \left. + 6 \cdot 40K^j 2^{\frac{-2k}{m+2}} \frac{1}{20K^{j+1}} 2^{\frac{2k}{m+2}} \right. \\ &\quad \left. + 2 \cdot 40K^j 2^{\frac{-2k}{m+2}} \frac{1}{\sqrt{20}K^{(j+1)/2}} 2^{\frac{k}{m+2}} \frac{1}{\sqrt{20}K^{(j+2)/2}} 2^{\frac{k}{m+2}} \right) \\ &\leq \sum_j C \cdot \frac{1}{2K^j} \left(2 + \frac{6\sqrt{2}}{K^{1/2}} + \frac{12}{K} + \frac{4}{K^{3/2}} \right). \end{aligned}$$

Estimate for IV. Using (c5) and (7), we get

$$\begin{aligned} |IV| &\leq \sum_j C \cdot 2^{\frac{2k}{m+2}} \frac{1}{10K^{2j}} \cdot 2 \left(4 \cdot 40K^j 2^{\frac{-2k}{m+2}} \right. \\ &\quad \left. + 4 \cdot \left(40K^j 2^{\frac{-2k}{m+2}} \frac{1}{\sqrt{20}K^{(j+1)/2}} 2^{\frac{k}{m+2}} \right)^2 \right) \\ &\leq \sum_j C \frac{1}{5K^j} \left(160 + \frac{320}{K} \right). \end{aligned}$$

Estimate for V. Using (m1), (c4), and (7), we get

$$\begin{aligned} |V| &\leq \sum_j C \cdot 2^{\frac{2k}{m+2}} \frac{2}{2K^j} \left(2\sqrt{40} 2^{\frac{-k}{m+2}} + 2 \cdot 40 \cdot 2^{\frac{-2k}{m+2}} \frac{1}{\sqrt{20}K^{1/2}} 2^{\frac{k}{m+2}} \right) \\ &\quad \times \left(2\sqrt{40}K^{j/2} 2^{\frac{-k}{m+2}} + 2 \cdot 40K^j 2^{\frac{-2k}{m+2}} \frac{1}{\sqrt{20}K^{(j+1)/2}} 2^{\frac{k}{m+2}} \right) \\ &\leq \sum_j \frac{C}{K^j} \left(2\sqrt{40} + \frac{4\sqrt{20}}{K^{1/2}} \right) \left(2\sqrt{40}K^{j/2} + 4\sqrt{20}K^{(j-1)/2} \right) \\ &= \sum_j \frac{C}{K^{j/2}} \left(2\sqrt{40} + \frac{4\sqrt{20}}{K^{1/2}} \right)^2. \end{aligned}$$

Estimate for VI. Using (c4) and (7), we get

$$\begin{aligned} |VI| &\leq \sum_{j,l} C \cdot \frac{1}{2K^j} \frac{1}{2K^l} \left(2\sqrt{40}K^{j/2} + 4\sqrt{20}K^{(j-1)/2} \right) \\ &\quad \times \left(2\sqrt{40}K^{l/2} + 4\sqrt{20}K^{(l-1)/2} \right) \\ &\leq \sum_{j,l} C \cdot \frac{1}{K^{(j+l)/2}} \left(2\sqrt{40} + \frac{4\sqrt{20}}{K^{1/2}} \right)^2. \end{aligned}$$

Combining these estimates, it is easily seen that if K is large enough, we have

$$X^2\phi_1 \geq - \left(C \cdot K^{-1/2} \right) 2^{\frac{2k}{m+2}}.$$

Case 3. $40 \cdot 2^{\frac{-2k}{m+2}} \leq \left| \frac{f}{Xf} \right|^2 \leq x_0 2^{\frac{-2k}{m+2}} \leq \frac{1}{100} K \cdot 2^{\frac{-2k}{m+2}}.$

We first note that, in this and all subsequent cases, the estimation of the terms *III*, *IV* and *VI* is the same as in Case 2, since the terms do not involve $\frac{f}{Xf}$. Hence we only need to consider the terms *I*, *II*, and *V*.

Estimate for I. By (m3), we have $\mu'_1 \geq 0$ in this part, and hence

$$\begin{aligned} I \geq \mu'_1 &\left(2 - 6 \cdot \frac{1}{10} K^{1/2} \left(\frac{1}{\sqrt{20}K^{1/2}} \right) - 6 \left(\frac{1}{100} K \right) \left(\frac{1}{20K} \right) \right. \\ &\quad \left. - 2 \left(\frac{1}{100} K \right) \left(\frac{1}{\sqrt{20}K^{1/2}} \right) \left(\frac{1}{\sqrt{20}K} \right) \right) \geq 0. \end{aligned}$$

Estimate for II. If $\mu''_1 \geq 0$, then $II \geq 0$, for otherwise using (m4) and (7) we get

$$|II| \leq \epsilon \left(2 - \frac{2f(X^2f)}{(Xf)^2} \right)^2 \leq 8\epsilon \left(1 + \frac{1}{100} K \frac{1}{20K} \right) \leq 9\epsilon.$$

Estimate for V. Using $|\mu'_1| \leq C$ and (c4) we get

$$\begin{aligned} |V| &\leq \sum_j 2 \cdot 2^{\frac{2k}{m+2}} \cdot C \cdot \frac{1}{2K^j} \left(\sqrt{x_0} 2^{\frac{-k}{m+1}} + x_0 2^{\frac{-2k}{m+1}} \frac{1}{\sqrt{20}K^{1/2}} 2^{\frac{k}{m+2}} \right) \\ &\quad \times \left(2\sqrt{40}K^{j/2} 2^{\frac{-k}{m+2}} + 2 \cdot 40K^j 2^{\frac{-2k}{m+2}} \frac{1}{\sqrt{20}K^{(j+1)/2}} 2^{\frac{k}{m+2}} \right) \\ &\leq \sum_j \frac{C}{K^{j/2}} \left(\sqrt{x_0} + \frac{x_0}{\sqrt{20}K^{1/2}} \right) \left(2\sqrt{40} + \frac{4\sqrt{20}}{K^{1/2}} \right). \end{aligned}$$

Combining these estimates we get

$$X^2\phi_1 \geq - \left(9\epsilon + CK^{-1/2} \right) 2^{\frac{2k}{m+2}},$$

where $\epsilon > 0$ is independent of K .

Case 4. $x_0 2^{\frac{-2k}{m+2}} \leq \left| \frac{f}{Xf} \right|^2 \leq \frac{1}{100} K \cdot 2^{\frac{-2k}{m+2}}$.

In this part $\mu_1 = C$. Hence $\mu'_1 = \mu''_1 = 0$ and $I = II = III = 0$.

Case 5. $\frac{1}{100} K \cdot 2^{\frac{-2k}{m+2}} \leq \left| \frac{f}{Xf} \right|^2 \leq MK \cdot 2^{\frac{-2k}{m+2}}$. (This is the last case since $\mu_1 = 0$ when $\left| \frac{f}{Xf} \right|^2 \geq MK \cdot 2^{\frac{-2k}{m+2}}$.)

Estimate for I. Using (m5) we get

$$\begin{aligned} |I| &\leq \frac{C}{K} \left(2 + 6\sqrt{MK} 2^{\frac{-k}{m+2}} \frac{1}{\sqrt{20}K^{1/2}} 2^{\frac{k}{m+2}} + 6MK 2^{\frac{-2k}{m+2}} \frac{1}{20K} 2^{\frac{2k}{m+2}} \right. \\ &\quad \left. + 2MK 2^{\frac{-2k}{m+2}} \frac{1}{\sqrt{20}K^{1/2}} 2^{\frac{k}{m+2}} \frac{1}{\sqrt{20}K} 2^{\frac{k}{m+2}} \right) \\ &\leq \frac{C}{K} \left(2 + \frac{6\sqrt{M}}{\sqrt{20}} + \frac{3M}{10} + \frac{M}{10K^{1/2}} \right). \end{aligned}$$

Estimate for II. Using (m6) we get

$$\begin{aligned} |II| &\leq 2^{\frac{2k}{m+2}} \frac{C}{K^2} \left(8MK \cdot 2^{\frac{-2k}{m+2}} + 8M^2 K^2 2^{\frac{-4k}{m+2}} \frac{1}{20K} 2^{\frac{2k}{m+2}} \right) \\ &\leq \frac{C}{K} \left(8M + \frac{2}{5} M^2 \right). \end{aligned}$$

Estimate for V. By (m5) we have

$$\begin{aligned} |V| &\leq \sum_j 2 \cdot 2^{\frac{2k}{m+2}} \frac{C}{K} \frac{1}{2K^j} \left(\sqrt{MK} 2^{\frac{-k}{m+1}} + MK 2^{\frac{-2k}{m+1}} \frac{1}{\sqrt{20}K^{1/2}} 2^{\frac{k}{m+2}} \right) \\ &\quad \times \left(2\sqrt{40}K^{j/2} 2^{\frac{-k}{m+2}} + 2 \cdot 40K^j 2^{\frac{-2k}{m+2}} \frac{1}{\sqrt{20}K^{(j+1)/2}} 2^{\frac{k}{m+2}} \right) \\ &\leq \sum_j \frac{C}{K^{(j+1)/2}} \left(\sqrt{M} + \frac{M}{\sqrt{20}} \right) \left(2\sqrt{40} + \frac{4\sqrt{20}}{K^{1/2}} \right), \end{aligned}$$

and we get again $|X^2\phi_1| \leq C \cdot K^{-1/2} 2^{\frac{2k}{m+2}}$.

Combining Cases 2-5, we conclude that in each of these cases

$$X^2\phi_1 \geq -C(\epsilon + K^{-1/2}) 2^{\frac{2k}{m+2}}.$$

Performing the same computations for $X^2\phi_1$ it is not hard to see that for any $\phi_i, i \geq 1$, we have:

- (1) If $\left| \frac{X^{i-1}f}{X^i f} \right|^2 \leq 40K^{i-1} 2^{\frac{-2k}{m+2}}$ and $\left| \frac{X^{j-1}f}{X^j f} \right|^2 \geq 40K^{j-1}$ for $i + 1 \leq j \leq m - 1$, then $X^2\phi_i \geq \frac{1}{K^{i-1}} 2^{\frac{2k}{m+2}}$ if K is large enough.

(2) In all other cases, $X^2\phi_i \geq \frac{-C(\epsilon+K^{-1/2})}{K^{i-1}}2^{\frac{2k}{m+2}}$.

We set $c_i = K^{i-1}$. We now choose $\epsilon > 0$ small enough and $K > 0$ large enough that for all i we have $c_i X^2\phi_i \geq -\delta 2^{\frac{2k}{m+2}}$, where δ is as small as we please. We then have:

(i) If $2^{\frac{2k}{m+2}} |X^{m-1}f|^2 \leq 40K^{m-1}$, then $\mu_m = 1$, and $c_m X^2\phi_m \geq 2^{\frac{2k}{m+2}}$. From (1) and (2) above, we have

$$\sum_{i=1}^{m-1} c_i X^2\phi_i \geq -(m-1)\delta 2^{\frac{2k}{m+2}},$$

and it is clear that $X^2\tilde{\mu} \geq \frac{1}{2}2^{\frac{2k}{m+2}}$.

(ii) If (i) is not satisfied, then $2^{\frac{2k}{m+2}} |X^{m-1}f|^2 \geq 40K^{m-1}$. If in addition $2^{\frac{2k}{m+2}} \left| \frac{X^{m-2}f}{X^{m-1}f} \right|^2 \leq 40K^{m-2}$, then as in (i) we have $X^2\tilde{\mu} \geq \frac{1}{2}2^{\frac{2k}{m+2}}$.

(ii) Repeating this argument shows that if

$$\begin{aligned} 2^{\frac{2k}{m+2}} |X^{m-1}f|^2 \geq 40K^{m-1}, \quad 2^{\frac{2k}{m+2}} \left| \frac{X^{m-2}f}{X^{m-1}f} \right|^2 \geq 40K^{m-2}, \\ \dots, \quad 2^{\frac{2k}{m+2}} \left| \frac{X^{i-1}f}{X^i f} \right|^2 \leq 40K^{i-1}, \end{aligned}$$

then $X^2\tilde{\mu} \geq \frac{1}{2}2^{\frac{2k}{m+2}}$.

(iv) Finally, if

$$\begin{aligned} 2^{\frac{2k}{m+2}} |X^{m-1}f|^2 \geq 40K^{m-1}, \quad 2^{\frac{2k}{m+2}} \left| \frac{X^{m-2}f}{X^{m-1}f} \right|^2 \geq 40K^{m-2}, \\ \dots, \quad 2^{\frac{2k}{m+2}} \left| \frac{f}{Xf} \right|^2 \geq 40, \end{aligned}$$

then

$$|f|^2 \geq 2^{\frac{-2mk}{m+2}} 40^m K^{m(m-1)/2}.$$

Hence $2^k f \geq 2^{\frac{2k}{m+2}}$.

Combining Cases (i)–(iv) we get

$$2^k f + X^2\tilde{\mu} \geq C2^{\frac{2k}{m+2}},$$

which is the desired inequality. □

REFERENCES

[1] T. Bloom, *On the contact between complex manifolds and real hypersurfaces in \mathbf{C}^3* , Trans. Amer. Math. Soc. **263** (1981), 515–529.
 [2] T. Bloom and I. Graham, *A geometric characterization of points of type m on real hypersurfaces*, J. Differential Geom. **12** (1977), 171–182.
 [3] D. Catlin, *Necessary conditions for subellipticity of the $\bar{\partial}$ -Neumann problem*, Ann. of Math. (2) **117** (1983), 147–171.

- [4] ———, *Boundary invariants of pseudoconvex domains*, Ann. of Math. (2) **120** (1984), 529–586.
- [5] ———, *Subelliptic estimates for the $\bar{\partial}$ -Neumann problem on pseudoconvex domains*, Ann. of Math. (2) **126** (1987), 131–191.
- [6] ———, *Estimates of invariant metrics on pseudoconvex domains of dimension two*, Math. Z. **200** (1989), 429–466.
- [7] J. D’Angelo, *Real hypersurfaces, orders of contact, and applications*, Ann. of Math. (2) **117** (1982), 615–637.
- [8] ———, *Iterated commutators and derivatives of the Levi form*, Lecture Notes in Math., vol. 1268, Springer, Berlin, 1987, pp. 103–110.
- [9] ———, *Several complex variables and the geometry of real hypersurfaces*, Studies in Adv. Math., CRC Press, 1992.
- [10] M. Derridj, *Inégalités a priori et estimation sous-elliptique pour $\bar{\partial}$ dans ouverts non pseudoconvexes*, Math. Ann. **249** (1980), 27–48.
- [11] G. B. Folland and J. J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Ann. of Math. Stud., no. 75, Princeton University Press, Princeton, NJ, 1972.
- [12] J. E. Fornæss and N. Sibony, *Construction of P.S.H. functions on weakly pseudoconvex domains*, Duke Math. J. **58** (1989), 633–655.
- [13] P. Greiner, *Subelliptic estimates for the $\bar{\partial}$ -Neumann problem in \mathbf{C}^2* , J. Differential Geom. **9** (1974), 239–250.
- [14] L. Ho, *Subellipticity of the $\bar{\partial}$ -Neumann problem on nonpseudoconvex domains*, Trans. Amer. Math. Soc. **291** (1985), 43–73.
- [15] ———, *Subelliptic estimates for the $\bar{\partial}$ -Neumann problem for $n - 1$ forms*, Trans. Amer. Math. Soc. **325** (1991), 171–185.
- [16] ———, *Exact subelliptic estimates for $n - 1$ forms*, Math. Z. **211** (1992), 635–641.
- [17] L. Hörmander, *L^2 estimates and existence theorems for the $\bar{\partial}$ -operator*, Acta Math. **113** (1965), 89–152.
- [18] J. J. Kohn, *Boundary behavior of $\bar{\partial}$ on weakly pseudoconvex manifolds of dimension two*, J. Differential Geom. **6** (1972), 523–542.
- [19] ———, *Subellipticity of the $\bar{\partial}$ -Neumann problem on pseudoconvex domains, sufficient conditions*, Acta Math. **142** (1979), 79–122.
- [20] J. J. Kohn and L. Nirenberg, *Non-coercive boundary value problems*, Comm. Pure Appl. Math. **18** (1965), 443–492.
- [21] L. P. Rothschild and E. M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (1976), 247–320.
- [22] N. Sibony, *Une classe de domaines pseudoconvexes*, Duke Math. J. **55** (1987), 299–319.
- [23] A. Talhoui, *Conditions suffisantes de sous-ellipticité pour $\bar{\partial}$* , C. R. Acad. Sci. Paris **296** (1983), 427–429.

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