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PSEUDODIFFERENTIAL OPERATORS ASSOCIATED TO LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. We investigate connections between pseudodifferential operators and linear ordinary differential equations via their respective links to automorphic forms. We also introduce Hecke operators on the space of pseudodifferential operators as well as on the space of certain meromorphic functions associated to ordinary differential equations and prove that the actions of those Hecke operators are compatible with the correspondence between pseudodifferential operators and linear ordinary differential equations.

1. Introduction

Automorphic forms for discrete subgroups of $SL(2, \mathbb{R})$ or for those of more general semisimple Lie groups play an important role in modern number theory, and they are closely linked to various other areas of pure and applied mathematics. One of the examples of such links can be found in the theory of linear ordinary differential equations. Indeed, certain types of second order linear ordinary differential equations with regular singular points on a Riemann surface determine meromorphic automorphic forms of weight three for the monodromy groups of the given equations. Meromorphic automorphic forms of weight $k \geq 3$ can also be obtained from differential equations of order $k-1$ associated to such second order equations.

Pseudodifferential operators are formal Laurent series in the formal inverse ∂^{-1} of the differential operator $\partial = d/dz$ on the complex plane $\mathbb{C}$. One of the most widely known applications of such operators can be found in the theory of integrable nonlinear partial differential equations, also known as soliton equations. Soliton equations have been the subject of numerous studies for the past few decades, and they include many well-known equations in mathematical physics such as the nonlinear Schrödinger equation, the Sine-Gordon equation, the Korteweg-de Vries (KdV) equation, and the Katomtsev-Petviashvili (KP) equation (see e.g. [3], [4]).

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In a recent paper [2], Cohen, Manin and Zagier studied relations between pseudodifferential operators and automorphic forms. Given a discrete subgroup Γ of $SL(2,\mathbb{R})$, they constructed a Γ-invariant pseudodifferential operator associated to an automorphic form for Γ, which determines a lifting for the symbol map of pseudodifferential operators.

Hecke operators are an important tool for the study of automorphic forms, and the connections between automorphic forms and differential equations allow us to consider Hecke operators for differential equations (see [5]). The goal of this paper is to investigate connections between pseudodifferential operators and linear ordinary differential equations via their respective links to automorphic forms. We also introduce Hecke operators on the space of pseudodifferential operators as well as on the space of certain meromorphic functions associated to ordinary differential equations and prove that the actions of those Hecke operators are compatible with the correspondence between pseudodifferential operators and linear ordinary differential equations.

2. Differential equations and automorphic forms

In this section we review connections between meromorphic automorphic forms of one variable and a certain class of linear ordinary differential equations, following closely the work of Stiller in [6].

Let $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ be the Poincaré upper half plane on which the group $SL(2,\mathbb{R})$ acts by linear fractional transformations. Thus we have

$$\gamma z = \frac{az+b}{cz+d} \in \mathcal{H} \tag{2.1}$$

for all $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$. Given such a matrix γ, a function $h : \mathcal{H} \to \mathbb{C}$, and an integer ℓ, we set

$$(h|_\ell \gamma)(z) = (cz+d)^{-\ell} h(\gamma z) \tag{2.2}$$

for all $z \in \mathcal{H}$. Let $\Gamma \subset SL(2,\mathbb{R})$ be a Fuchsian group of the first kind, that is, a discrete subgroup such that the quotient space $\Gamma\backslash\mathcal{H}^*$ is compact, where $\mathcal{H}^*$ denotes the union of $\mathcal{H}$ and the set of cusps of Γ.

Definition 2.1. A *meromorphic automorphic form of weight k for Γ* is a meromorphic function $f : \mathcal{H} \to \mathbb{C}$ which is meromorphic at the cusps of Γ and satisfies

$$f|_k \gamma = f$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. We denote by $M_k(\Gamma)$ the space of all meromorphic automorphic forms of weight k for Γ.

Throughout the rest of this paper we fix a meromorphic automorphic form $\varphi \in M_1(\Gamma)$ of weight one for the Fuchsian group Γ of the first kind. Then the associated compact Riemann surface $X = \Gamma\backslash\mathcal{H}^*$ may be considered as an algebraic curve over $\mathbb{C}$. We denote by $K(X)$ the function field of the algebraic curve X, and choose a nonconstant element x of $K(X)$. If the functions $\varphi(z)$ and $z\varphi(z)$ on $\mathcal{H}$ are regarded as functions on X, they satisfy a second order homogeneous linear ordinary differential equation $\mathcal{D}_{\varphi,X} f = 0$ on X with

$$\mathcal{D}_{\varphi,X} = \frac{d^2}{dx^2} + P_X(x)\frac{d}{dx} + Q_X(x) \tag{2.3}$$

that has regular singular points, where $P_X(x)$ and $Q_X(x)$ are elements of $K(X)$. Given an element $f \in K(X)$, we have $df/dx = (df/dz)(dz/dx)$ and

$$\begin{aligned}\frac{d^2 f}{dx^2} = \frac{d}{dz}\Big(\frac{df}{dz}\cdot\frac{dz}{dx}\Big)\cdot\frac{dz}{dx} &= \Big[\frac{d^2 f}{dz^2}\cdot\frac{dz}{dx} + \frac{df}{dz}\cdot\frac{d}{dz}\Big(\Big(\frac{dx}{dz}\Big)^{-1}\Big)\Big]\cdot\frac{dz}{dx}\\ &= \Big[\frac{d^2 f}{dz^2} - \frac{df}{dz}\cdot\Big(\frac{dx}{dz}\Big)^{-1}\cdot\frac{d^2 x}{dz^2}\Big]\cdot\Big(\frac{dz}{dx}\Big)^2\\ &= \Big[\frac{d^2 f}{dz^2} - \frac{df}{dz}\cdot\frac{d}{dz}\log\frac{dx}{dz}\Big]\cdot\Big(\frac{dz}{dx}\Big)^2,\end{aligned}$$

where z is the standard coordinate in $\mathbb{C}$. Using this, we can pull the differential operator (2.3) back via the natural projection $\mathcal{H}^* \to X = \Gamma\backslash\mathcal{H}^*$. Thus the homogeneous equation $\mathcal{D}_{\varphi,X} f = 0$ on X is equivalent to the equation $\mathcal{D}_\varphi f = 0$ on $\mathcal{H}$ with

$$\mathcal{D}_\varphi = \frac{d^2}{dz^2} + P(z)\frac{d}{dz} + Q(z), \tag{2.4}$$

where $P(z)$ and $Q(z)$ are meromorphic functions on $\mathcal{H}$ given by

$$P(z) = P_X(x(z))\frac{dx}{dz} - \frac{d}{dz}\log\frac{dx}{dz}, \qquad Q(z) = Q_X(x(z))\Big(\frac{dx}{dz}\Big)^2$$

(cf. [6, p. 63]). Thus the functions $z\varphi(z)$ and $\varphi(z)$ for $z \in \mathcal{H}$ are linearly independent solutions of the associated homogeneous equation $\mathcal{D}_\varphi f = 0$, and the regular singular points of $\mathcal{D}_\varphi$ coincide with the cusps of Γ (see [6] for details).

Given a positive integer m, let $S^m\mathcal{D}_\varphi$ be the linear ordinary differential operator of order $m+1$ such that the solutions of the corresponding homogeneous equation $S^m\mathcal{D}_\varphi f = 0$ are of the form

$$\sum_{i=0}^{m} C_i (z\varphi(z))^{m-i}(\varphi(z))^i = \sum_{i=0}^{m} C_i z^{m-i}\varphi(z)^m \tag{2.5}$$

for some constants $C_i \in \mathbb{C}$.

We now consider a more general ordinary differential operator of order n of the form

$$\mathcal{D} = \frac{d^n}{dx^n} + P_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \cdots + P_1\frac{d}{dx} + P_0,$$

where $P_i \in K(X)$ for $0 \le i \le n-1$. Let $S \subset X$ be the set of singular points of $P_0, \dots, P_{n-1}$, and let $X_0 = X - S$. We choose a base point $x_0 \in X_0$ and let $\omega_1, \ldots, \omega_n$ be a basis for the space of local solutions of $\mathcal{D}f = 0$ near x_0. Then the Wronskian

$$W_{\mathcal{D}} = \det M_{\mathcal{D}} \tag{2.6}$$

is the determinant of the $n \times n$ matrix $M_{\mathcal{D}} = (d^{j-1}\omega_i/dx^{j-1})$ whose (i,j) entry is $d^{j-1}\omega_i/dx^{j-1}$ for $1 \leq i,j \leq n$. Given $x \in X$, let $\eta = \{\eta_1, \ldots, \eta_{n-1}\}$ be the set of $n-1$ local solutions of $\mathcal{D}f = 0$ near x, and let A_η be the $(n-1)\times(n-1)$ matrix whose (i,j) entry is $d^{j-1}\eta_i/dx^{j-1}$ for $1 \leq i,j \leq n-1$. Then a function $\psi \in K(X)$ is said to satisfy the *residue conditions with respect to* $\mathcal{D}$ if the differential $(A_\eta\psi/W)dx$ has zero residue at every $x \in X_0 = X - S$ for each set η of $n-1$ local solutions of $\mathcal{D}f = 0$ near x.

DEFINITION 2.2. An element $\psi \in K(X)$ is said to satisfy the *parabolic residue conditions with respect to* $\mathcal{D}$ if it satisfies the residue conditions and if for each η the differential $(A_\eta\psi/W)dx$ has zero residue at every singular point $x \in S$ whenever A_η is single-valued.

Let ν be a positive integer, and let $\mathfrak{P}_{\nu,\varphi}$ be the set of meromorphic functions ψ on $\mathcal{H}$ whose associated elements ψ_X in $K(X)$ satisfy the parabolic residue conditions with respect to $S^{2\nu}\mathcal{D}_\varphi$. Given $\psi \in \mathfrak{P}_{\nu,\varphi}$, we denote by $\mathfrak{S}(\psi)$ a solution of the differential equation $S^{2\nu}\mathcal{D}_\varphi f = \psi$, and set

$$\rho_{\nu,\varphi}(\psi) = \frac{d^{2\nu+1}}{dz^{2\nu+1}}\Big(\frac{\mathfrak{S}(\psi)}{\varphi^{2\nu}}\Big). \tag{2.7}$$

Note that $\rho_{\nu,\varphi}(\psi)$ is independent of the choice of the solution $\mathfrak{S}(\psi)$ because we have

$$\frac{d^{2\nu+1}}{dz^{2\nu+1}}\Big(\frac{1}{\varphi^{2\nu}}\sum_{i=0}^{2\nu} C_i z^{2\nu-i}\varphi^{2\nu}\Big) = \frac{d^{2\nu+1}}{dz^{2\nu+1}}\Big(\sum_{i=0}^{2\nu} C_i z^{2\nu+i}\Big) = 0$$

for any constants $C_i \in \mathbb{C}$.

LEMMA 2.3. *The function $\rho_{\nu,\varphi}(\psi)$ on $\mathcal{H}$ given by (2.7) is a meromorphic automorphic form for Γ of weight $2\nu+2$, and the associated map $\rho_{\nu,\varphi} : \mathfrak{P}_{\nu,\varphi} \to M_{2\nu+2}(\Gamma)$ is a one-to-one linear map of complex vector spaces.*

Proof. The fact that $\rho_{\nu,\varphi}(\psi)$ is an element of $M_{2\nu+2}(\Gamma)$ follows from results in [6, p. 32]. Since the map $\rho_{\nu,\varphi}$ is clearly complex linear, it suffices to show that its kernel is zero. Suppose $\rho_{\nu,\varphi}(\psi) = 0$ for some $\psi \in \mathfrak{P}_{\nu,\varphi}$. Then by (2.7) we see that

$$\mathfrak{S}(\psi) = \varphi(z)^{2\nu}\sum_{i=0}^{2\nu} C_i z^i$$

for some constants $C_i \in \mathbb{C}$. Since $\mathfrak{S}(\psi)$ is a solution of the differential equation $S^{2\nu}\mathcal{D}_\varphi f = \psi$, we have

$$\psi = S^{2\nu}\mathcal{D}_\varphi \mathfrak{S}(\psi) = S^{2\nu}\mathcal{D}_\varphi \Big(\varphi(z)^{2\nu} \sum_{i=0}^{2\nu} C_i z^i\Big).$$

However, by (2.5) the functions $\varphi(z)^{2\nu}\sum_{i=0}^{2\nu} C_i z^i$ are solutions of the homogeneous equation $S^{2\nu}\mathcal{D}_\varphi f = 0$, and therefore it follows that $\psi = 0$. □

3. Pseudodifferential operators

In this section we review pseudodifferential operators with coefficients in the space of meromorphic functions on the Poincaré upper half plane $\mathcal{H} \subset \mathbb{C}$ and discuss their connections with meromorphic automorphic forms.

Let z be the standard coordinate for $\mathbb{C}$, and let ∂ be the differential operator d/dz. We denote by $\mathcal{F}$ the ring of meromorphic functions on $\mathcal{H}$. A pseudodifferential operator L over $\mathcal{F}$ is a formal Laurent series in the formal inverse ∂^{-1} of ∂ with coefficients in $\mathcal{F}$, that is, a formal series of the form

$$L = \sum_{n=-\infty}^{n_0} \xi_n(z)\partial^n \tag{3.1}$$

for some $n_0 \in \mathbb{Z}$ with $\xi_n \in \mathcal{F}$ for each n. We denote by $\Psi\,\mathrm{DO} = \Psi\,\mathrm{DO}(\mathcal{F})$ the set of all pseudodifferential operators over $\mathcal{F}$. Then $\Psi\,\mathrm{DO}$ is a ring whose multiplication operation is given by

$$\Big(\sum_n \xi_n(z)\partial^n\Big)\Big(\sum_m \eta_m(z)\partial^m\Big) = \sum_{n,m}\sum_{r\geq 0}\binom{n}{r}\xi_n(z)\eta_m^{(r)}(z)\partial^{n+m-r},$$

where

$$\binom{n}{0} = 1, \quad \binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!}$$

for $n \in \mathbb{Z}$ and $r \geq 1$.

Let $\Gamma \subset SL(2,\mathbb{R})$ be a Fuchsian group of the first kind. Then Γ acts on $\Psi\,\mathrm{DO}$ by

$$\gamma \cdot L = \sum_{n=-\infty}^{n_0} \xi_n(\gamma z)\Big(\frac{d(\gamma z)}{dz}\partial\Big)^n$$

for each $\gamma \in \Gamma$ if L is given by (3.1).

Proposition 3.1. *An element*

$$\Phi(z) = \sum_{k=1}^{\infty}(-1)^k k!(k-1)!\widetilde{\phi}_k(z)\partial^{-k}$$

of $\Psi\,\mathrm{DO}$ *is* Γ*-invariant if and only if there is a meromorphic automorphic form* $h_j \in M_{2j}(\Gamma)$ *for* Γ *for each* $j \geq 1$ *such that*

$$\widetilde{\phi}_k(z) = \sum_{r=0}^{k-1} \frac{1}{r!(2k-r-1)!} h_{k-r}^{(r)}(z) \tag{3.2}$$

for all $z \in \mathcal{H}$ *and* $k \geq 1$*. Furthermore, the formula* (3.2) *is equivalent to the relation*

$$h_m(z) = (2m-1) \sum_{r=0}^{m-1} \frac{(-1)^r (2m-r-2)!}{r!} \widetilde{\phi}_{m-r}(z)$$

for each $m \geq 1$*.*

Proof. This follows from Proposition 2 in [2]. □

Let $\prod_{\nu=1}^{\infty} \mathfrak{P}_{\nu,\varphi}$ be the set of sequences $(\psi_\nu)_{\nu=1}^{\infty}$ of meromorphic functions on $\mathcal{H}$, which has the natural structure of a complex vector space. Given $\boldsymbol{\psi} = (\psi_\nu(z))_{\nu=1}^{\infty} \in \prod_{\nu=1}^{\infty} \mathfrak{P}_{\nu,\varphi}$, we set

$$\Xi_\varphi(\boldsymbol{\psi}) = \sum_{k=2}^{\infty} \sum_{r=0}^{k-1} \frac{(-1)^k k!(k-1)!}{r!(2k-r-1)!} \Big(\frac{\mathfrak{S}(\psi_{k-r-1})}{\varphi^{2k-2r-2}} \Big)^{(2k-r-1)} \partial^{-k}. \tag{3.3}$$

Theorem 3.2. *The formula* (3.3) *determines a linear map*

$$\Xi_\varphi : \prod_{\nu=1}^{\infty} \mathfrak{P}_{\nu,\varphi} \to \Psi\,\mathrm{DO}^\Gamma$$

of complex vector spaces, where $\Psi\,\mathrm{DO}^\Gamma$ *denotes the space of* Γ*-invariant elements of* $\Psi\,\mathrm{DO}$*.*

Proof. Given a sequence $\boldsymbol{\psi} = (\psi_\nu)_{\nu=1}^{\infty} \in \prod_{\nu=1}^{\infty} \mathfrak{P}_{\nu,\varphi}$, we set $\xi_1 = 0$ and

$$\xi_\ell = \Big(\frac{\mathfrak{S}(\psi_{\ell-1})}{\varphi^{2\ell-2}} \Big)^{(2\ell-1)} \tag{3.4}$$

for integers $\ell \geq 2$. Then by Lemma 2.3 we see that $\xi_\ell \in M_{2\ell}(\Gamma)$ for all $\ell \geq 1$. Thus by Proposition 3.1 the pseudodifferential operator

$$\sum_{k=1}^{\infty} \sum_{r=0}^{k-1} \frac{(-1)^k k!(k-1)!}{r!(2k-r-1)!} \xi_{k-r}^{(r)} \partial^{-k}$$
$$= \sum_{k=2}^{\infty} \sum_{r=0}^{k-1} \frac{(-1)^k k!(k-1)!}{r!(2k-r-1)!} \Big(\frac{\mathfrak{S}(\psi_{k-r-1})}{\varphi^{2k-2r-2}} \Big)^{(2k-r-1)} \partial^{-k}$$

is Γ-invariant, where we used (3.4) and the fact that $\xi_1 = 0$. Hence we have $\Xi_\varphi(\boldsymbol{\psi}) \in \Psi\,\mathrm{DO}^\Gamma$. Since the linearity of Ξ_φ is clear, the proof of the theorem is complete. □

Given a positive integer ν, we define the linear map

$$\mathcal{T}_\nu : \mathfrak{P}_{\nu,\varphi} \to \Psi\,\mathrm{DO}_{-\nu-1}$$

of complex vector spaces by

$$\mathcal{T}_\nu(\psi) = \sum_{n=\nu+1}^{\infty} \frac{(-1)^n n!(n-1)!}{(n-\nu-1)!(n+\nu)!} \Big(\frac{\mathfrak{S}(\psi_\nu)}{\varphi^{2\nu}}\Big)^{(n+\nu)} \partial^{-n} \tag{3.5}$$

for all $\psi \in \mathfrak{P}_{\nu,\varphi}$.

Theorem 3.3. *For each $\psi \in \mathfrak{P}_{\nu,\varphi}$ the pseudodifferential operator $\mathcal{T}_\nu(\psi) \in \Psi\,\mathrm{DO}_{-\nu-1}$ is invariant under the action of Γ, and it can be written in the form*

$$\mathcal{T}_\nu(\psi) = \sum_{n=\nu+1}^{\infty} \frac{(-1)^{n+1} n!(n-1)!}{(n-\nu-1)!(n+\nu)!} \bigg(\frac{\varphi^{2\nu+2}\psi}{\widetilde{W}_{\mathcal{D}_\varphi}^{2\nu+1}}\bigg)^{(n-\nu-1)} \partial^{-n}, \tag{3.6}$$

where $\widetilde{W}_{\mathcal{D}_\varphi}(z)$ is the pullback of the Wronskian $W_{\mathcal{D}_\varphi}(x)$ defined as in (2.6) *via the natural projection map $\mathcal{H} \to X_0 = \Gamma\backslash\mathcal{H}$.*

Proof. Given $\psi \in \mathfrak{P}_{\nu,\varphi}$, let $\boldsymbol{\psi} = (\psi_r)_{r=1}^\infty \in \prod_{\nu=1}^\infty \mathfrak{P}_{\nu,\varphi}$ be a sequence defined by

$$\psi_r = \begin{cases} \psi & \text{if } r = \nu, \\ 0 & \text{if } r \neq \nu, \end{cases}$$

and let Ξ_φ be the map given by (3.3). Using (3.3) and (3.5), we obtain

$$\begin{aligned} \Xi_\varphi(\boldsymbol{\psi}) &= \sum_{k=\nu+1}^{\infty} \sum_{r=0}^{k-1} \frac{(-1)^k k!(k-1)!}{r!(2k-r-1)!} \Big(\frac{\mathfrak{S}(\psi_{k-r-1})}{\varphi^{2k-2r-2}}\Big)^{(2k-r-1)} \partial^{-k} \\ &= \sum_{k=\nu+1}^{\infty} \frac{(-1)^k k!(k-1)!}{(k-\nu-1)!(k+\nu)!} \Big(\frac{\mathfrak{S}(\psi)}{\varphi^{2\nu}}\Big)^{(k+\nu)} \partial^{-k} = \mathcal{T}_\nu(\psi). \end{aligned}$$

Hence by Theorem 3.2 it follows that the pseudodifferential operator $\mathcal{T}_\nu(\psi) = \Xi_\varphi(\boldsymbol{\psi})$ is Γ-invariant. On the other hand, using [6, Theorem 3 bis. 5], we see that

$$\Big(\frac{\mathfrak{S}(\psi_\nu)}{\varphi^{2\nu}}\Big)^{(2\nu+1)} = (-1)^{2\nu+1} \frac{\varphi^{2\nu+2}\psi}{\widetilde{W}_{\mathcal{D}_\varphi}^{2\nu+1}} = -\frac{\varphi^{2\nu+2}\psi}{\widetilde{W}_{\mathcal{D}_\varphi}^{2\nu+1}}. \tag{3.7}$$

Therefore (3.6) follows from this and (3.5). □

4. Hecke operators

In this section we consider Hecke operators on the spaces $\mathfrak{P}_{\nu,\varphi}$, $\prod_{\nu=1}^\infty \mathfrak{P}_{\nu,\varphi}$ and $\Psi\,\mathrm{DO}^\Gamma$ and discuss the compatibility of such operators with the usual Hecke operators on the spaces of automorphic forms.

We first extend the action of $SL(2,\mathbb{R})$ on the Poincaré upper half plane $\mathcal{H}$ given by (2.1) to an action of the multiplicative group $GL^+(2,\mathbb{R})$ of 2×2 real matrices of positive determinant. Given an integer ℓ, if f is a function on $\mathcal{H}$ and if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL^+(2, \mathbb{R})$, we set

$$(f|_\ell \gamma)(z) = (\det \gamma)^{\ell/2} (cz + d)^{-\ell} f(\gamma z) \tag{4.1}$$

for all $z \in \mathcal{H}$. This definition reduces to (2.2) if $\gamma \in SL(2, \mathbb{R})$.

Two subgroups Γ_1 and Γ_2 of $GL^+(2, \mathbb{R})$ are said to be *commensurable* if $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2, in which case we write $\Gamma_1 \sim \Gamma_2$. Let $\Gamma \subset SL(2, \mathbb{R})$ be a Fuchsian group of the first kind, and let $\widetilde{\Gamma} \subset GL^+(2, \mathbb{R})$ be its commensurator, that is, the set of elements of $g \in GL^+(2, \mathbb{R})$ such that $g\Gamma g^{-1}$ and Γ are commensurable. Let Γ_0 be another Fuchsian group of the first kind, and such that its commensurator coincides with $\widetilde{\Gamma}$. Then for each $\alpha \in \widetilde{\Gamma}$ the double coset $\Gamma \alpha \Gamma_0$ has a decomposition of the form

$$\Gamma \alpha \Gamma_0 = \coprod_{\ell=1}^{d} \Gamma \alpha_\ell \tag{4.2}$$

for some $\alpha_1, \ldots, \alpha_d \in GL^+(2, \mathbb{R})$. The Hecke operator on $M_k(\Gamma)$ associated to the double coset $\Gamma \alpha \Gamma_0$ is the linear map $T_k(\alpha) : M_k(\Gamma) \to M_k(\Gamma_0)$ defined by

$$T_k(\alpha) f = \det(\alpha)^{k/2-1} \sum_{\ell=1}^{d} (f|_k \alpha_\ell) \tag{4.3}$$

for all $f \in M_k(\Gamma)$, where $f|_k \alpha_\ell$ is as in (4.1). In particular, if $\Gamma_0 = \Gamma$, then $T_k(\alpha)$ is a linear endomorphism of $M_k(\Gamma)$.

Let $\Gamma, \Gamma_0 \subset SL(2, \mathbb{R})$ be as above with $\Gamma \sim \Gamma_0$, and let $\varphi \in M_1(\Gamma)$ and $S^m \mathcal{D}_\varphi$ be as in Section 2. Let $\varphi_0 : \mathcal{H} \to \mathbb{C}$ be a nonzero meromorphic automorphic form of weight one for Γ_0. Then, as in (2.4), we can consider the associated differential operator

$$\mathcal{D}_{\varphi_0} = \frac{d^2}{dz^2} + P_0(z) \frac{d}{dz} + Q_0(z),$$

where $P_0(z)$ and $Q_0(z)$ are meromorphic functions on $\mathcal{H}$, the functions $z\varphi_0(z)$ and $\varphi_0(z)$ are linearly independent solutions of the associated homogeneous equation $\mathcal{D}_{\varphi_0} f = 0$, and the regular singular points of $\mathcal{D}_{\varphi_0}$ coincide with the cusps of Γ_0. Thus $S^m \mathcal{D}_{\varphi_0}$ is the differential operator such that $\{z^{m-i} \varphi_0(z)^m \mid 0 \le i \le m\}$ is the set of linearly independent solutions of the homogeneous equation $S^m \mathcal{D}_{\varphi_0} f = 0$. We also consider the associated space $\mathfrak{P}_{\nu, \varphi_0}$ and the complex linear map

$$\rho_{\nu, \varphi_0} : \mathfrak{P}_{\nu, \varphi_0} \to M_{2\nu+2}(\Gamma_0)$$

using (2.7) with φ replaced with φ_0.

Let α be an element of the commensurator $\widetilde{\Gamma}$ of Γ such that $\Gamma \alpha \Gamma_0$ is as in (4.2), and let $T_{2\nu+2}(\alpha) : M_{2\nu+2}(\Gamma) \to M_{2\nu+2}(\Gamma_0)$ be the associated Hecke operator in (4.3) with $k = 2\nu + 2$. Given $\psi \in \mathfrak{P}_{\nu,\varphi}$, we set

$$F_{\nu,\alpha}(z) = \varphi_0(z)^{2\nu} \int \cdots \int (T_{2\nu+2}(\alpha))(\rho_{\nu,\varphi}(\psi))\, dz \cdots dz, \tag{4.4}$$

$$\begin{aligned} (T_\nu^{\mathfrak{P}}(\alpha)\psi)(z) = {} & \det(\alpha)^\nu \sum_{\ell=1}^{d} \frac{\det(\alpha_\ell)^{\nu+1}\psi(\alpha_\ell z)}{j(\alpha_\ell, z)^{2\nu+2}} \\ & \times \left(\frac{\varphi(\alpha_\ell z)}{\varphi_0(z)}\right)^{2\nu+2} \left(\frac{W_{\mathcal{D}_{\varphi_0}}(z)}{W_{\mathcal{D}_\varphi}(\alpha_\ell z)}\right)^{2\nu+1} \end{aligned} \tag{4.5}$$

for all $z \in \mathcal{H}$, where $\rho_{\nu,\varphi}(\psi)$ is as in (2.7) and $\int \cdots \int dz \cdots dz$ denotes the $(2\nu+1)$-fold indefinite integral with respect to z.

Theorem 4.1. *The formula* (4.5) *determines a linear operator*

$$T_\nu^{\mathfrak{P}}(\alpha) : \mathfrak{P}_{\nu,\varphi} \to \mathfrak{P}_{\nu,\varphi_0}$$

on the space $\mathfrak{P}_{\nu,\varphi}$ *satisfying*

$$T_\nu^{\mathfrak{P}}(\alpha)(\psi) = S^{2\nu}\mathcal{D}_{\varphi_0}(F_{\nu,\alpha}) \tag{4.6}$$

for all $\psi \in \mathfrak{P}_{\nu,\varphi}$, *where* $S^{2\nu}\mathcal{D}_{\varphi_0}(F_{\nu,\alpha})$ *is the function obtained by applying the differential operator* $S^{2\nu}\mathcal{D}_{\varphi_0}$ *to the function* $F_{\nu,\alpha}$ *given by* (4.4).

Proof. Given $\psi \in \mathfrak{P}_{\nu,\varphi}$, by (3.7) and Lemma 2.3 the function

$$\rho_{\nu,\varphi}(\psi) = -\frac{\varphi^{2\nu+2}\psi}{\widetilde{W}_{\mathcal{D}_\varphi}^{2\nu+1}}$$

on $\mathcal{H}$ is an element of $M_{2\nu+2}(\Gamma)$. Thus, using (4.3) and (4.5), we obtain

$$\begin{aligned} T_{2\nu+2}(\alpha)(\rho_{\nu,\varphi}(\psi))(z) &= -\det(\alpha)^\nu \sum_{\ell=1}^{d} \frac{\det(\alpha_\ell)^{\nu+1}\varphi^{2\nu+2}(\alpha_\ell z)\psi(\alpha_\ell z)}{j(\alpha_\ell, z)^{2\nu+2}\widetilde{W}_{\mathcal{D}_\varphi}^{2\nu+1}(\alpha_\ell z)} \\ &= -\frac{\varphi_0(z)^{2\nu+2}}{\widetilde{W}_{\mathcal{D}_{\varphi_0}}(z)} (T_\nu^{\mathfrak{P}}(\alpha)\psi)(z) \\ &= \rho_{\nu,\varphi_0}(T_\nu^{\mathfrak{P}}(\alpha)(\psi))(z) \end{aligned} \tag{4.7}$$

for all $z \in \mathcal{H}$. On the other hand, since $F_{\nu,\alpha}$ is a solution of the differential equation $S^{2\nu}\mathcal{D}_{\varphi_0} f = S^{2\nu}\mathcal{D}_{\varphi_0}(F_{\nu,\alpha})$, it follows from (2.7), (4.4) and (4.7) that

$$\begin{aligned} \rho_{\nu,\varphi_0}(S^{2\nu}\mathcal{D}_{\varphi_0}(F_{\nu,\alpha})) &= \frac{d^{2\nu+1}}{dz^{2\nu+1}}\Big(\frac{\mathfrak{S}(S^{2\nu}\mathcal{D}_{\varphi_0}(F_{\nu,\alpha}))}{\varphi^{2\nu}}\Big) = \frac{d^{2\nu+1}}{dz^{2\nu+1}}\Big(\frac{F_{\nu,\alpha}}{\varphi^{2\nu}}\Big) \\ &= T_{2\nu+2}(\alpha)(\rho_{\nu,\varphi}(\psi)) = \rho_{\nu,\varphi_0}(T_\nu^{\mathfrak{P}}(\alpha)(\psi)). \end{aligned}$$

Since ρ_{ν,φ_0} is injective by Lemma 2.3, we obtain (4.6), and therefore the proof of the theorem is complete. □

The linear map $T_\nu^{\mathfrak{P}}(\alpha)$ in Theorem 4.1 may be regarded as the Hecke operator on $\mathfrak{P}_{\nu,\varphi}$ associated to α, and such operators allow us to define the Hecke operator

$$T^{\mathfrak{P}}(\alpha) : \prod_{\nu=1}^{\infty} \mathfrak{P}_{\nu,\varphi} \to \prod_{\nu=1}^{\infty} \mathfrak{P}_{\nu,\varphi_0}$$

on $\prod_{\nu=1}^{\infty} \mathfrak{P}_{\nu,\varphi}$ associated to α by setting

$$T^{\mathfrak{P}}(\alpha)\boldsymbol{\psi} = (T_\nu^{\mathfrak{P}}(\alpha)(\psi_\nu))$$

for each sequence $\boldsymbol{\psi} = (\psi_\nu)_{\nu=1}^{\infty} \in \prod_{\nu=1}^{\infty} \mathfrak{P}_{\nu,\varphi}$.

We now discuss Hecke operators $T^{\Psi}(\alpha)$ on the space $\Psi\,\mathrm{DO}^{\Gamma}$ of Γ-invariant pseudodifferential operators (see [1] for another description of Hecke operators on pseudodifferential operators). Given an element $\Phi(z) = \sum_{\ell=1}^{\infty} \phi_\ell(z)\partial^{-\ell}$ of $\Psi\,\mathrm{DO}^{\Gamma}$ and a positive integer k, we set

$$\mathcal{A}(\Phi)_k(z) = \sum_{r=0}^{k-1} \frac{(-1)^k (2k-r-2)!}{r!(k-r)!(k-r-1)!} \phi_{k-r}(z) \tag{4.8}$$

for all $z \in \mathcal{H}$.

Lemma 4.2. *If $\Phi \in \Psi\,\mathrm{DO}^{\Gamma}$, then for each positive integer k, the function $\mathcal{A}(\Phi)_k : \mathcal{H} \to \mathbb{C}$ is a meromorphic automorphic form of weight $2k$ for Γ.*

Proof. Since Φ is Γ-invariant, using Proposition 3.1, we have

$$\frac{(-1)^j \phi_j}{j!(j-1)!} = \sum_{r=0}^{j-1} \frac{1}{r!(2j-r-1)!} h_{j-r}^{(r)}$$

for each $j \geq 1$, where, for each $k \geq 1$, h_k is a meromorphic automorphic form of weight $2k$ for Γ given by

$$\begin{aligned} h_k &= (2k-1) \sum_{\ell=0}^{k-1} \frac{(-1)^\ell (2k-\ell-2)!}{\ell!} \cdot \frac{(-1)^{k-\ell} \phi_{k-\ell}}{(k-\ell)!(k-\ell-1)!} \\ &= (2k-1) \sum_{\ell=0}^{k-1} \frac{(-1)^k (2k-\ell-2)!}{\ell!(k-\ell)!(k-\ell-1)!} \phi_{k-\ell}. \end{aligned}$$

Thus we see that $\mathcal{A}(\Phi)_k = (2k-1)^{-1} h_k$, and therefore the lemma follows. □

Given $\alpha \in \widetilde{\Gamma}$ and $\Phi \in \Psi\,\mathrm{DO}$, we set

$$\begin{aligned} T^{\Psi}(\alpha)\Phi = \sum_{k=1}^{\infty} \sum_{r=0}^{k-1} (-1)^k \frac{k!(k-1)!(2k-2r-1)}{r!(2k-r-1)!} \\ \times (T_{2k-2r}(\alpha)\mathcal{A}(\Phi)_{k-r})^{(r)} \partial^{-k}. \end{aligned} \tag{4.9}$$

We note that

$$T_{2k}(\alpha)\mathcal{A}(\Phi)_k \in M_{2k}(\Gamma_0)$$

for each $k \geq 1$, since $\mathcal{A}(\Phi)_k \in M_{2k}(\Gamma)$ by Lemma 4.2. The following theorem shows that $T_{2k}(\alpha)$ is an operator on $\Psi\,\mathrm{DO}^\Gamma$, which we call the Hecke operator on $\Psi\,\mathrm{DO}^\Gamma$ associated to α, and that it is compatible with the Hecke operator $T^{\mathfrak{P}}(\alpha)$ on $\mathfrak{P}_{\nu,\varphi}$.

THEOREM 4.3. *For each $\Phi \in \Psi\,\mathrm{DO}^\Gamma$, the pseudodifferential operator $T^\Psi(\alpha)\Phi$ given by (4.9) is Γ-invariant, and the diagram*

$$\begin{array}{ccc} \prod_{\nu=1}^\infty \mathfrak{P}_{\nu,\varphi} & \xrightarrow{\Xi_\varphi} & \Psi\,\mathrm{DO}^\Gamma \\ {\scriptstyle T^{\mathfrak{P}}(\alpha)}\big\downarrow & & \big\downarrow{\scriptstyle T^\Psi(\alpha)} \\ \prod_{\nu=1}^\infty \mathfrak{P}_{\nu,\varphi_0} & \xrightarrow{\Xi_{\varphi_0}} & \Psi\,\mathrm{DO}^\Gamma \end{array}$$

is commutative.

Proof. Let $\boldsymbol{\psi} = (\psi_\nu)_{\nu=1}^\infty \in \prod_{\nu=1}^\infty \mathfrak{P}_{\nu,\varphi}$. Then by (2.7) and (3.3) we have

$$\Xi_\varphi(\boldsymbol{\psi}) = \sum_{k=1}^\infty (-1)^k k!(k-1)!\widetilde{\phi}_k \partial^{-k} \tag{4.10}$$

with $\widetilde{\phi}_1 = 0$ and

$$\widetilde{\phi}_k = \sum_{r=0}^{k-1} \frac{1}{r!(2k-r-1)!}(\rho_{k-r-1,\varphi}(\psi_{k-r-1}))^{(r)} \tag{4.11}$$

for $k \geq 2$ with $\psi_0 = 0$. Using Proposition 3.1, we see that (4.11) is equivalent to the relation

$$\rho_{k-1,\varphi}(\psi_{k-1}) = (2k-1)\sum_{r=0}^{k-1} \frac{(-1)^r(2k-r-2)!}{r!}\widetilde{\phi}_{k-r}.$$

On the other hand, by (4.8) and (4.10) we have

$$\mathcal{A}(\Xi_\varphi(\boldsymbol{\psi}))_k = \sum_{r=0}^{k-1} \frac{(-1)^r(2k-r-2)!}{r!(k-r)!(k-r-1)!} \cdot (-1)^{k-r}(k-r)!(k-r-1)!\widetilde{\phi}_{k-r}.$$

Thus we obtain

$$\rho_{k-1,\varphi}(\psi_{k-1}) = (2k-1)\mathcal{A}(\Xi_\varphi(\boldsymbol{\psi}))_k.$$

Hence by (4.9) we have

$$\begin{aligned} T^\Psi(\alpha)(\Xi_\varphi(\boldsymbol{\psi})) = \sum_{k=1}^\infty \sum_{r=0}^{k-1} (-1)^k \frac{k!(k-1)!}{r!(2k-r-1)!} \\ \times (T_{2k-2r}(\alpha)(\rho_{k-r-1,\varphi}(\psi_{k-r-1}))^{(r)}\partial^{-k}. \end{aligned}$$

On the other hand, since $T^{\mathfrak{P}}(\alpha)\boldsymbol{\psi} = (T^{\mathfrak{P}}_\nu(\alpha)(\psi_\nu))$, by (4.10) we obtain

$$\Xi_{\varphi_0}(T^{\mathfrak{P}}(\alpha)\boldsymbol{\psi}) = \sum_{k=1}^{\infty}\sum_{r=0}^{k-1} \frac{(-1)^k k!(k-1)!}{r!(2k-r-1)!} \times (\rho_{k-r-1,\varphi_0}(T^{\mathfrak{P}}_{k-r-1}(\alpha)(\psi_{k-r-1})))^{(r)}\partial^{-k}.$$

However, by (4.7) we have

$$\rho_{k-r-1,\varphi_0}(T^{\mathfrak{P}}_{k-r-1}(\alpha)(\psi_{k-r-1})) = T_{2k-2r}(\alpha)(\rho_{k-r-1,\varphi}(\psi_{k-r-1})).$$

Hence we see that

$$T^{\Psi}(\alpha)(\Xi_\varphi(\boldsymbol{\psi})) = \Xi_{\varphi_0}(T^{\mathfrak{P}}(\alpha)\boldsymbol{\psi}),$$

and therefore the theorem follows. □

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