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HOMOLOGICAL PROPERTIES OF BIGRADED ALGEBRAS

TIM RÖMER

ABSTRACT. We investigate the x- and y-regularity of a bigraded K-algebra R as introduced in [2]. These notions are used to study asymptotic properties of certain finitely generated bigraded modules. As an application we get for any equigenerated graded ideal I upper bounds for the number j_0 for which $\operatorname{reg}(I^j)$ is a linear function for $j \geq j_0$. Finally, we give upper bounds for the x- and y-regularity of generalized Veronese algebras.

Introduction

Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be a standard bigraded polynomial ring with $\deg(x_i) = (1,0)$ and $\deg(y_j) = (0,1)$, and let $J \subset S$ be a bigraded ideal. In this paper we study homological properties of the bigraded algebra $R = S/J$.

First we consider the x- and the y-regularity of R. According to [2] they are defined as follows:

$$\operatorname{reg}_x^S(R) = \max\{a \in \mathbb{Z} \colon \beta^S_{i,(a+i,b)}(R) \neq 0 \text{ for some } i, b \in \mathbb{Z}\},$$
$$\operatorname{reg}_y^S(R) = \max\{b \in \mathbb{Z} \colon \beta^S_{i,(a,b+i)}(R) \neq 0 \text{ for some } i, a \in \mathbb{Z}\},$$

where $\beta^S_{i,(a,b)}(R) = \dim_K \operatorname{Tor}_i^S(K,R)_{(a,b)}$ is the i^{th} bigraded Betti number of R in bidegree (a,b). We give a homological characterization of these regularities similarly as in the graded case (see [3]). As an application we generalize a result of Trung [13] concerning d-sequences. Furthermore we prove that

$$\operatorname{reg}_x^S(S/J) = \operatorname{reg}_x^S(S/\operatorname{bigin}(J))$$

where $\operatorname{bigin}(J)$ is the bigeneric initial ideal of J with respect to the bigraded reverse lexicographic order induced by $y_1 > \cdots > y_m > x_1 > \cdots > x_n$.

It was shown in [7] (or [12]) that for $j \gg 0$, $\operatorname{reg}(I^j)$ is a linear function $cj + d$ in j, for a graded ideal I in the polynomial ring. In [12] the constant c is described in terms of invariants of I. In this paper we give, in case I is equigenerated, bounds j_0 such that for $j \geq j_0$ the function is linear, and we

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also give a bound for d. Our methods can also be applied to $\operatorname{reg}(S^j(I))$, where $S^j(I)$ is the j^{th} symmetric power of I.

In the last section we introduce a generalized Veronese algebra in the bigraded setting. For a bigraded algebra R and $\tilde{\Delta} = (s,t) \in \mathbb{N}^2$ with $(s,t) \neq (0,0)$ we set

$$R_{\tilde{\Delta}} = \bigoplus_{(a,b)\in\mathbb{N}^2} R_{(as,bt)}.$$

In the same manner as it was done in [6] for diagonal subalgebras, we prove that for these algebras

$$\operatorname{reg}_x^{S_{\tilde{\Delta}}}(R_{\tilde{\Delta}}) = 0 \text{ and } \operatorname{reg}_y^{S_{\tilde{\Delta}}}(R_{\tilde{\Delta}}) = 0 \text{ if } s \gg 0 \text{ and } t \gg 0.$$

1. Preliminaries

Throughout this paper, let K be an infinite field and let $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$ be a standard bigraded polynomial ring with $\deg(x_i) = (1,0)$ and $\deg(y_j) = (0,1)$. Let M be a finitely generated bigraded S-module. For some bihomogeneous $w \in M$ with $\deg(w) = (a,b)$ we set $\deg_x(w) = a$ and $\deg_y(w) = b$. Sometimes we will consider the $\mathbb{Z}$-graded modules $M_{(a,*)} = \bigoplus_{b\in\mathbb{Z}} M_{(a,b)}$ or $M_{(*,b)} = \bigoplus_{a\in\mathbb{Z}} M_{(a,b)}$. If in addition M is $\mathbb{N}^n \times \mathbb{N}^m$-graded, we write $M_{(u,v)}$ for the homogeneous component in bidegree (u,v), where $u \in \mathbb{N}^n$ and $v \in \mathbb{N}^m$. For $u \in \mathbb{N}^n$ we set $\operatorname{supp}(u) = \{i \colon u_i > 0\}$.

Define $\mathfrak{m}_x = (x_1, \dots, x_n) = (\mathbf{x})$, $\mathfrak{m}_y = (y_1, \dots, y_m) = (\mathbf{y})$, and $\mathfrak{m} = \mathfrak{m}_x + \mathfrak{m}_y$. Let $S_x = K[x_1, \dots, x_n]$ and $S_y = K[y_1, \dots, y_m]$ be the polynomial rings with respect to the x-variables and the y-variables.

For any $u = (u_1, \dots, u_n) \in \mathbb{N}^n$ and $v = (v_1, \dots, v_m) \in \mathbb{N}^m$ we write $x^u y^v$ for the monomial $x_1^{u_1} \dots x_n^{u_n} y_1^{v_1} \dots y_m^{v_m}$. For $u, u' \in \mathbb{N}^n$ we let $u \preceq u'$ if $u_i \leq u'_i$ for all i. Furthermore we set $|u| = u_1 + \dots + u_n$. Let $\varepsilon_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^n$, where the entry 1 is at the i^{th} position. For $t \in \mathbb{N}$ define $[t] = \{1, \dots, t\}$.

We consider bigraded algebras $R = S/J$, which are quotients of S by some bigraded ideal J. For a finitely generated bigraded R-module M and $a, b \in \mathbb{N}$ let $\beta^R_{i,(a,b)}(M) = \dim_K \operatorname{Tor}_i^R(M,K)_{(a,b)}$ be the i^{th} bigraded Betti number in bidegree (a,b). We recall from [2] that

$$\operatorname{reg}_x^R(M) = \sup\{a \in \mathbb{Z} \colon \beta^R_{i,(a+i,b)}(M) \neq 0 \text{ for some } i, b \in \mathbb{Z}\},$$

$$\operatorname{reg}_y^R(M) = \sup\{b \in \mathbb{Z} \colon \beta^R_{i,(a,b+i)}(M) \neq 0 \text{ for some } i, a \in \mathbb{Z}\}$$

is the x- and y-regularity of M. In the case $R = S$ we set $\operatorname{reg}_x(M) = \operatorname{reg}_x^S(M)$ and $\operatorname{reg}_y(M) = \operatorname{reg}_y^S(M)$.

Let $K_\bullet(k,l;M)$ denote the Koszul complex of M and $H_\bullet(k,l;M)$ the Koszul homology of M with respect to $x_1, \dots, x_k$ and $y_1, \dots, y_l$ (see [5] for details). If it is clear from the context, we write $K_\bullet(k,l)$ and $H_\bullet(k,l)$ instead of $K_\bullet(k,l;M)$ and $H_\bullet(k,l;M)$. Note that $K_\bullet(k,l;M) = K_\bullet(k,l;S) \otimes_S M$ where $K_\bullet(k,l;S)$ is the exterior algebra on $e_1, \dots, e_k$ and $f_1, \dots, f_l$ with

$\deg(e_i) = (1,0)$ and $\deg(f_j) = (0,1)$ together with a differential ∂ induced by $\partial(e_i) = x_i$ and $\partial(f_j) = y_j$. For a cycle $z \in K_\bullet(k,l;M)$ we denote by $[z] \in H_\bullet(k,l;M)$ the corresponding homology class. There are two long exact sequences relating the homology groups:

$$\begin{gathered} \ldots \longrightarrow H_i(k,l;M)(-1,0) \stackrel{x_{k+1}}{\longrightarrow} H_i(k,l;M) \\ \longrightarrow H_i(k+1,l;M) \longrightarrow H_{i-1}(k,l;M)(-1,0) \stackrel{x_{k+1}}{\longrightarrow} \ldots \\ \longrightarrow H_0(k,l;M)(-1,0) \stackrel{x_{k+1}}{\longrightarrow} H_0(k,l;M) \longrightarrow H_0(k+1,l;M) \longrightarrow 0 \end{gathered}$$

and

$$\begin{gathered} \ldots \longrightarrow H_i(k,l;M)(0,-1) \stackrel{y_{l+1}}{\longrightarrow} H_i(k,l;M) \\ \longrightarrow H_i(k,l+1;M) \longrightarrow H_{i-1}(k,l;M)(0,-1) \stackrel{y_{l+1}}{\longrightarrow} \ldots \\ \longrightarrow H_0(k,l;M)(0,-1) \stackrel{y_{l+1}}{\longrightarrow} H_0(k,l;M) \longrightarrow H_0(k,l+1;M) \longrightarrow 0. \end{gathered}$$

The map $H_i(k,l;M) \longrightarrow H_i(k+1,l;M)$ is induced by the inclusion of the corresponding Koszul complexes. Every homogeneous element $z \in K_\bullet(k+1,l;M)$ can be uniquely written as $e_{k+1} \wedge z' + z''$ with $z', z'' \in K_\bullet(k,l;M)$. Then $H_i(k+1,l;M) \longrightarrow H_{i-1}(k,l;M)(-1,0)$ is given by sending $[z]$ to $[z']$. Furthermore $H_i(k,l;M)(-1,0) \stackrel{x_{k+1}}{\longrightarrow} H_i(k,l;M)$ is just the multiplication with x_{k+1}. The maps in the other exact sequence are defined analogously.

2. Regularity

Let R be a bigraded algebra. To simplify the notation we do not distinguish between the polynomial ring variables x_i or y_j and the corresponding residue classes in R. Following [3] (or [13] under the name filter regular element) we call an element $x \in R_{(1,0)}$ *an almost regular element for R* (with respect to the x-degree) if

$$(0 :_R x)_{(a,*)} = 0 \text{ for } a \gg 0.$$

A sequence $x_1, \ldots, x_t \in R_{(1,0)}$ is *an almost regular sequence* (with respect to the x-degree) if for all $i \in [t]$ the element x_i is almost regular for $R/(x_1, \ldots, x_{i-1})R$.

Analogously we call an element $y \in R_{(0,1)}$ *an almost regular element for R* (with respect to the y-degree) if

$$(0 :_R y)_{(*,b)} = 0 \text{ for } b \gg 0.$$

A sequence $y_1, \ldots, y_t \in R_{(0,1)}$ is *an almost regular sequence* (with respect to the y-degree) if for all $i \in [t]$ the element y_i is almost regular for $R/(y_1, \ldots, y_{i-1})R$.

It is well-known that, provided $|K| = \infty$, by a generic choice of coordinates we can ensure that a K-basis of $R_{(1,0)}$ is almost regular for R with respect to the x-degree and a K-basis of $R_{(0,1)}$ is almost regular for R with respect to the y-degree. For the convenience of the reader we give a proof of this fact, which follows from the following lemma (see also [13]).

LEMMA 2.1. *Let R be a bigraded algebra. If $\dim_K R_{(1,0)} > 0$ (resp. $\dim_K R_{(0,1)} > 0$), then there exists an element $x \in R_{(1,0)}$ (resp. $y \in R_{(0,1)}$) which is almost regular for R.*

Proof. By symmetry it is enough to prove the existence of x. We claim that it is possible to choose $0 \neq x \in R_{(1,0)}$ such that for all $Q \in \operatorname{Ass}_S(0 :_R x)$ one has $Q \supseteq \mathfrak{m}_x$. It follows that $\operatorname{Rad}_S(\operatorname{Ann}_S(0 :_R x)) \supseteq \mathfrak{m}_x$. Hence there exists an integer i such that $\mathfrak{m}_x^i(0 :_R x) = 0$ and this proves the lemma.

It remains to show the claim. If $P \supseteq \mathfrak{m}_x$ for all $P \in \operatorname{Ass}_S(R)$, then we may choose $0 \neq x \in R_{(1,0)}$ arbitrarily because $\operatorname{Ass}_S(0 :_R x) \subseteq \operatorname{Ass}_S(R)$. Otherwise there exists an ideal $P \in \operatorname{Ass}_S(R)$ with $P \not\supseteq \mathfrak{m}_x$. In this case we may choose $x \in R_{(1,0)}$ such that

$$x \notin \bigcup_{P \in \operatorname{Ass}_S(R),\, P \not\supseteq \mathfrak{m}_x} P$$

since $|K| = \infty$. Let $Q \in \operatorname{Ass}_S(0 :_R x)$ be arbitrary. Then $x \in Q$ because $x \in \operatorname{Ann}_S(0 :_R x)$. We also have that $Q \in \operatorname{Ass}_S(R)$, and this implies $Q \supseteq \mathfrak{m}_x$ by the choice of x, as claimed. $\square$

Let $\mathbf{x}$ and $\mathbf{y}$ be almost regular for R with respect to the x- and y-degree. Define

$$\begin{aligned} s_i^x &= \max(\{a \colon (0 :_{R/(x_1,\dots,x_{i-1})R} x_i)_{(a,*)} \neq 0\} \cup \{0\}), \\ s^x &= \max\{s_1^x, \dots, s_n^x\} \end{aligned}$$

and

$$\begin{aligned} s_i^y &= \max(\{b \colon (0 :_{R/(y_1,\dots,y_{i-1})R} y_i)_{(*,b)} \neq 0\} \cup \{0\}), \\ s^y &= \max\{s_1^y, \dots, s_m^y\}. \end{aligned}$$

The following theorem characterizes the x- and y-regularity. It is the analogue of the corresponding graded version proved in [3].

For its proof we consider $\tilde{H}_0(k-1,0) = (0 :_{R/(x_1,\dots,x_{k-1})R} x_k)$ for $k \in [n]$ and $\tilde{H}_0(n,k-1) = (0 :_{R/(\mathfrak{m}_x+y_1,\dots,y_{k-1})R} y_k)$ for $k \in [m]$. Then the beginning of the long exact Koszul sequence of the Koszul homology groups of R for $k \in [n]$ is modified to

$$\begin{aligned} \dots \longrightarrow H_1(k-1,0)(-1,0) &\xrightarrow{x_k} H_1(k-1,0) \\ \longrightarrow H_1(k,0) \longrightarrow &\tilde{H}_0(k-1,0)(-1,0) \longrightarrow 0, \end{aligned}$$

and for $k \in [m]$ to

$$\begin{aligned} \dots \longrightarrow H_1(n,k-1)(0,-1) &\xrightarrow{y_k} H_1(n,k-1) \\ \longrightarrow H_1(n,k) \longrightarrow &\tilde{H}_0(n,k-1)(0,-1) \longrightarrow 0. \end{aligned}$$

Note that for $k \in [n]$ and $i \geq 1$ one has $H_i(k,0)_{(a,*)} = 0$ for $a \gg 0$. Similarly for $k \in [m]$ and $i \geq 1$ one has $H_i(n,k)_{(*,b)} = 0$ for $b \gg 0$.

THEOREM 2.2. *Let R be a bigraded algebra, $\mathbf{x}$ almost regular for R with respect to the x-degree and $\mathbf{y}$ almost regular for R with respect to the y-degree. Then*

$$\operatorname{reg}_x(R) = s^x \text{ and } \operatorname{reg}_y(R) = s^y.$$

Proof. By symmetry it is enough to show this theorem only for $\mathbf{x}$. Let

$$r_{(k,0)} = \max(\{a\colon H_i(k,0)_{(a+i,*)} \neq 0 \text{ for } i \in [k]\} \cup \{0\})$$

for $k \in [n]$ and

$$r_{(n,k)} = \max(\{a\colon H_i(n,k)_{(a+i,*)} \neq 0 \text{ for } i \in [n+k]\} \cup \{0\})$$

for $k \in [m]$. Then $r_{(n,m)} = \operatorname{reg}_x(R)$ because $H_0(n,m) = K$. We will prove:

(i) For $k \in [n]$ one has $r_{(k,0)} = \max\{s^x_1, \ldots, s^x_k\}$.
(ii) For $k \in [m]$ one has $r_{(n,k)} = \max\{s^x_1, \ldots, s^x_n\}$.

These two assertions yield the theorem.

We show (i) by induction on $k \in [n]$. For $k = 1$ we have the following exact sequence

$$0 \longrightarrow H_1(1,0) \longrightarrow \tilde{H}_0(0,0)(-1,0) \longrightarrow 0,$$

which proves this case. Let $k > 1$. Since

$$\ldots \longrightarrow H_1(k,0) \longrightarrow \tilde{H}_0(k-1,0)(-1,0) \longrightarrow 0,$$

we get $r_{(k,0)} \geq s^x_k$. If $r_{(k-1,0)} = 0$, then $r_{(k,0)} \geq r_{(k-1,0)}$. Assume that $r_{(k-1,0)} > 0$. There exists an integer i such that $H_i(k-1)_{(r_{(k-1,0)}+i,*)} \neq 0$. Since $H_i(k-1,0)_{(r_{(k-1,0)}+i+1,*)} = 0$ and since we have the exact sequence

$$\ldots \longrightarrow H_{i+1}(k,0)_{(r_{(k-1,0)}+i+1,*)} \longrightarrow H_i(k-1,0)_{(r_{(k-1,0)}+i,*)}$$
$$\longrightarrow H_i(k-1,0)_{(r_{(k-1,0)}+i+1,*)} \longrightarrow \ldots,$$

it follows that $H_{i+1}(k,0)_{(r_{(k-1,0)}+i+1,*)} \neq 0$. This gives again $r_{(k,0)} \geq r_{(k-1,0)}$. On the other hand, let $a > \max\{r_{(k-1,0)}, s^x_k\}$. If $i \geq 2$, then by the exact sequence

$$\longrightarrow H_i(k-1,0)_{(a+i,*)} \longrightarrow H_i(k,0)_{(a+i,*)} \longrightarrow H_{i-1}(k-1,0)_{(a+i-1,*)} \longrightarrow$$

we get $H_i(k,0)_{(a+i,*)} = 0$ because $H_i(k-1,0)_{(a+i,*)} = H_{i-1}(k-1,0)_{(a+i-1,*)} = 0$. Similarly $H_1(k,0)_{(a+1,*)} = 0$. Therefore, by the induction hypothesis, we obtain $r_{(k,0)} = \max\{r_{(k-1,0)}, s^x_k\} = \max\{s^x_1, \ldots, s^x_k\}$.

We prove (ii) also by induction on $k \in \{0, \ldots, m\}$. The case $k = 0$ was shown in (i), so let $k > 0$. Assume that $a > s^x$. For $i \geq 2$ one has

$$\longrightarrow H_i(n,k-1)_{(a+i,*)} \longrightarrow H_i(n,k)_{(a+i,*)} \longrightarrow H_{i-1}(n,k-1)_{(a+i,*)} \longrightarrow$$

Then $H_i(n,k)_{(a+i,*)} = 0$ because $H_i(n,k-1)_{(a+i,*)} = H_{i-1}(n,k-1)_{(a+i,*)} = 0$. Similarly, $H_1(n,k)_{(a+1,*)} = 0$ and therefore $r_{(n,k)} \leq s^x$. If $s^x = 0$, then

$r_{(n,k)} = s^x$. Assume that $0 < s^x = r_{(n,k-1)}$. There exists an integer i such that $H_i(n, k-1)_{(s^x+i,*)} \neq 0$. Consider the sequence

$$\longrightarrow H_i(n,k-1)_{(s^x+i,*)} \xrightarrow{y_k} H_i(n,k-1)_{(s^x+i,*)} \longrightarrow H_i(n,k)_{(s^x+i,*)} \longrightarrow$$

If $H_i(n,k)_{(s^x+i,*)} = 0$, then $H_i(n,k-1)_{(s^x+i,*)} = y_k H_i(n,k-1)_{(s^x+i,*)}$. This is a contradiction by Nakayama's lemma because $H_i(n,k-1)_{(s^x+i,*)}$ is a finitely generated S_y-module. Hence we see that $H_i(n,k)_{(s^x+i,*)} \neq 0$ and thus $r_{(n,k)} = s^x$. $\square$

3. d-sequences and s-sequences

The concept of a d-sequence was introduced by Huneke [11]. Recall that a sequence of elements $f_1, \ldots, f_r$ in a ring is called a *d-sequence*, if

(i) $f_1, \ldots, f_r$ is a minimal system of generators of the ideal $I = (f_1, \ldots, f_r)$.
(ii) $(f_1, \ldots, f_{i-1}) : f_i \cap I = (f_1, \ldots, f_{i-1})$.

A result in [13] motivated the following theorem. For a bigraded algebra R let n_x denote the ideal generated by the $(1,0)$-forms of R and let n_y denote the ideal generated by the $(0,1)$-forms of R.

PROPOSITION 3.1. *Let R be a bigraded algebra. Then:*

(i) $\operatorname{reg}_x(R) = 0$ *if and only if a generic minimal system of generators of* $(1,0)$*-forms for* n_x *is a d-sequence.*
(ii) $\operatorname{reg}_y(R) = 0$ *if and only if a generic minimal system of generators of* $(0,1)$*-forms for* n_y *is a d-sequence.*

Proof. By symmetry we only have to prove (i). Without loss of generality $\mathbf{x} = x_1, \ldots, x_n$ is an almost regular sequence for R with respect to the x-degree because a generic minimal system of generators of $(1,0)$-forms for n_x has this property.

By Theorem 2.2 one has $\operatorname{reg}_x(R) = 0$ if and only if $s^x = 0$. By the definition of s^x this is equivalent to the fact that, for all $i \in [n]$ and all $a > 0$, we have

$$\left(\frac{(x_1, \ldots, x_{i-1}) :_R x_i}{(x_1, \ldots, x_{i-1})} \right)_{(a,*)} = 0.$$

Equivalently, for all $i \in [n]$ we obtain $(x_1, \ldots, x_{i-1}) :_R x_i \cap n_x = (x_1, \ldots, x_{i-1})$. This concludes the proof. $\square$

If n_x (resp. n_y) can be generated by a d-sequence (not necessarily generic), then the proof of Proposition 3.1 shows that $\operatorname{reg}_x(R) = 0$ (resp. $\operatorname{reg}_y(R) = 0$).

For an application we recall some further definitions. Let $I = (f_1, \ldots, f_m) \subset S_x$ be a graded ideal generated in degree d. Let $R(I)$ denote the Rees algebra of I and let $S(I)$ denote the symmetric algebra of I. It is well known that both algebras are bigraded and have a presentation S/J for a bigraded ideal $J \subset S$. For example, let $R(I) = S_x[It] \subset S_x[t]$. Define

$$\varphi : S \longrightarrow R(I),\ x_i \mapsto x_i,\ y_j \mapsto f_j t,$$

and let $J = \mathrm{Ker}(\varphi)$. Under the assumption that I is generated in one degree we have that J is a bigraded ideal. We will always assume that $R(I) = S/J$. Note that then $I^j \cong (S/J)_{(*,j)}(-jd)$ for all $j \in \mathbb{N}$. Similarly we may assume that $S(I) = S/J$ for a bigraded ideal $J \subset S$. We also consider the finitely generated S_x-module $S^j(I) = (S/J)_{(*,j)}(-jd)$, which we call the j^{th} symmetric power of I.

For the notion of an s-sequence see [10]. The following results were shown in [10] and [13].

COROLLARY 3.2. *Let $I = (f_1, \ldots, f_m) \subset S_x$ be a graded ideal generated in degree d. Then:*

(i) *I can be generated by an s-sequence (with respect to the reverse lexicographic order) if and only if* $\mathrm{reg}_y(S(I)) = 0$.

(ii) *I can be generated by a d-sequence if and only if* $\mathrm{reg}_y(R(I)) = 0$.

Proof. In [10] and [13] it was shown that

(i) I can be generated by an s-sequence (with respect to the reverse lexicographic order) if and only if $n_y \subseteq S(I)$ can be generated by a d-sequence;

(ii) I can be generated by a d-sequence if and only if $n_y \subseteq R(I)$ can be generated by a d-sequence.

Combining these results with Proposition 3.1 concludes the proof. □

4. Bigeneric initial ideals

We recall the following definitions from [2]. For a monomial $x^u y^v \in S$ we set

$$m_x(x^u y^v) = m(u) = \max\{0, i \text{ with } u_i > 0\},$$
$$m_y(x^u y^v) = m(v) = \max\{0, i \text{ with } v_i > 0\}.$$

Similarly we define for $L \subseteq [n]$

$$m(L) = \max\{0, i \text{ with } i \in L\}.$$

Let $J \subset S$ be a monomial ideal. Let $G(J)$ denote the unique minimal system of generators of J. If $G(J) = \{z_1, \ldots, z_t\}$ with $\deg(z_i) = (u^i, v^i) \in \mathbb{N}^n \times \mathbb{N}^m$, then we set $m_x(J) = \max\{|u^i|\}$ and $m_y(J) = \max\{|v^i|\}$.

J is called *bistable* if for all monomials $z \in J$, all $i \le m_x(z)$, all $j \le m_y(z)$ one has $x_i z / x_{m_x(z)} \in J$ and $y_j z / y_{m_y(z)} \in J$. J is called *strongly bistable* if for all monomials $z \in J$, all $i \le s$ with x_s divides z, all $j \le t$ with y_t divides z one has $x_i z / x_s \in J$ and $y_j z / y_t \in J$.

LEMMA 4.1. *Let $J \subset S$ be a bistable ideal and $R = S/J$. Then:*

(i) $x_n, \dots, x_1$ *is an almost regular sequence for R with respect to the x-degree.*

(ii) $y_m, \dots, y_1$ *is an almost regular sequence for R with respect to the y-degree.*

Proof. This follows easily from the fact that J is bistable. $\square$

We fix a term order $>$ on S by defining $x^u y^v > x^{u'} y^{v'}$ if either $(|u| + |v|, |v|, |u|) > (|u'| + |v'|, |v'|, |u'|)$ lexicographically or $(|u| + |v|, |v|, |u|) = (|u'| + |v'|, |v'|, |u'|)$ and $x^u y^v > x^{u'} y^{v'}$ reverse lexicographically with respect to the order induced by $y_1 > \dots > y_m > x_1 > \dots > x_n$. (See [8] for details on monomial orders.) For a bigraded ideal J let $\operatorname{in}(J)$ denote the monomial ideal generated by $\operatorname{in}(f)$ for all $f \in J$. In [2] the bigeneric initial ideal $\operatorname{bigin}(J)$ was constructed in the following way: For $t \in \mathbb{N}$ let $\operatorname{GL}(t, K)$ be the general linear group of the $t \times t$-matrices with entries in K. Let $G = \operatorname{GL}(n, K) \times \operatorname{GL}(m, K)$ and $g = (d_{ij}, e_{kl}) \in G$. Then g defines an S automorphism by extending $g(x_j) = \sum_i d_{ij} x_i$ and $g(y_l) = \sum_k e_{kl} y_k$. There exists a non-empty Zariski open set $U \subset G$ such that for all $g \in U$ we have $\operatorname{bigin}(J) = \operatorname{in}(gJ)$. We call these $g \in U$ *generic* for J. If $\operatorname{char}(K) = 0$, then $\operatorname{bigin}(J)$ is strongly bistable for every bigraded ideal J. See, for example, [3] for similar results in the graded case.

PROPOSITION 4.2. *Let $J \subset S$ be a bigraded ideal. If* $\operatorname{char}(K) = 0$*, then*

$$\operatorname{reg}_x(S/J) = \operatorname{reg}_x(S/\operatorname{bigin}(J)).$$

Proof. Set $\mathbf{x} = x_n, \dots, x_1$, choose $g \in G$ generic for J and let $\tilde{\mathbf{x}} = \tilde{x}_n, \dots, \tilde{x}_1$ such that $x_i = g(\tilde{x}_i)$. We may assume that the sequence $\tilde{\mathbf{x}}$ is almost regular for S/J with respect to the x-degree. Furthermore, by Lemma 4.1 the sequence $\mathbf{x}$ is almost regular for $S/\operatorname{bigin}(J)$ with respect to the x-degree. We have

$$(0 :_{S/((\tilde{x}_n, \dots, \tilde{x}_{i+1}) + J)} \tilde{x_i}) \cong (0 :_{S/((x_n, \dots, x_{i+1}) + g(J))} x_i).$$

It follows from [8, 15.12] that

$$(0 :_{S/((x_n, \dots, x_{i+1}) + g(J))} x_i) \cong (0 :_{S/((x_n, \dots, x_{i+1}) + \operatorname{bigin}(J))} x_i).$$

By Theorem 2.2 we get the desired result. $\square$

REMARK 4.3.

(i) In general it is not true that

$$\operatorname{reg}_y(S/J) = \operatorname{reg}_y(S/\operatorname{bigin}(J)).$$

For example, let $S = K[x_1, \dots, x_3, y_1, \dots, y_3]$ and $J = (y_2 x_2 - y_1 x_3, y_3 x_1 - y_1 x_3)$. Then the minimal bigraded free resolution of S/J is given by

$$0 \longrightarrow S(-2,-2) \longrightarrow S(-1,-1) \oplus S(-1,-1) \longrightarrow S \longrightarrow 0.$$

Therefore $\operatorname{reg}_x(S/J) = 0$ and $\operatorname{reg}_y(S/J) = 0$. On the other hand, $\operatorname{bigin}(J) = (y_2x_1, y_1x_1, y_1^2x_2)$ with the minimal bigraded free resolution of $S/\operatorname{bigin}(J)$

$$0 \longrightarrow S(-2,-2) \oplus S(-1,-2)$$
$$\longrightarrow S(-1,-1) \oplus S(-1,-1) \oplus S(-1,-2) \longrightarrow S \longrightarrow 0.$$

Hence $\operatorname{reg}_x(S/\operatorname{bigin}(J)) = 0$ and $\operatorname{reg}_y(S/\operatorname{bigin}(J)) = 1$.

(ii) It is easy to calculate the x- and the y-regularity of bistable ideals. In fact, in [2] it was shown that for a bistable ideal $J \subset S$ we have

$$\operatorname{reg}_x(J) = m_x(J) \text{ and } \operatorname{reg}_y(J) = m_y(J).$$

5. Regularity of powers and symmetric powers of ideals

Consider a bigraded algebra $R = S/J$ where J is a bistable ideal. Note that by Lemma 4.1 the sequence $x_n, \dots, x_1$ is almost regular for R with respect to the x-degree. For $i \in [n]$ and $j \geq 0$ we define

$$m_j^i = \max(\{a \in \mathbb{N} \colon (0 :_{R/(x_n,\dots,x_{i+1})R} x_i)_{(a,j)} \neq 0\} \cup \{0\}).$$

Furthermore, for a bistable ideal J and $v \in \mathbb{N}^n$ we set $J_{(*,v)} = I_v y^v$ where $I_v \subset S_x$ is again a monomial ideal, which is stable in the usual sense, that is, if $x^u \in I_v$, then $x_i x^u / x_{m(u)} \in I_v$ for $i \leq m(u)$.

PROPOSITION 5.1. *Let $J \subset S$ be a bistable ideal and $R = S/J$. Then:*

(i) *For every $i \in [n]$ and for $j \geq 0$ we have $m_j^i \leq \max\{m_x(J) - 1, 0\}$.*

(ii) *For every $i \in [n]$ and for $j \geq m_y(J)$ we have $m_j^i = m_{m_y(J)}^i$.*

Proof. If $G(J) = \{x^{u^k} y^{v^k} \colon k = 1, \dots, r\}$, then $I_v = (x^{u^k} \colon v^k \preceq v)$ for $v \in \mathbb{N}^n$. This means that for all $x^u \in G(I_v)$ one has $|u| \leq m_x(J)$. For fixed v with $|v| = j$ we have

$$(0 :_{R/(x_n,\dots,x_{i+1})R} x_i)_{(*,v)} = \frac{((x_n,\dots,x_{i+1}) + I_v :_{S_x} x_i)}{(x_n,\dots,x_{i+1}) + I_v} y^v.$$

As a K-vector space

$$\frac{((x_n,\dots,x_{i+1}) + I_v :_{S_x} x_i)}{(x_n,\dots,x_{i+1}) + I_v} y^v = \bigoplus_{x^u \in G(I_v), m(u) = i} K(x^u / x_{m(u)}) y^v$$

because I_v is stable. Thus $m_j^i \leq \max\{m_x(J) - 1, 0\}$, which is (i).

To prove (ii) we replace J by $J_{(*, \geq m_y(J))}$ and may assume that J is generated in y-degree $t = m_y(J)$. Then $G(J) = \{x^{u^k} y^{v^k} \colon k = 1, \dots, r\}$ where $|v^k| = t$ for all $k \in [r]$. Let $|u^k|$ be maximal with $m(u^k) = i$ and define $c^i = \max\{|u^k| - 1, 0\}$. We will show that $m_j^i = c^i$ for $j \geq t$. This gives (ii).

By a similar argument as in (i) we have $m_{s+t}^i \leq c^i$ for all $s \geq 0$. If $c^i = 0$. Then $m_{s+t}^i = 0$. Assume that $c^i \neq 0$. We claim that

$$(*) \quad 0 \neq [(x^{u^k}/x_i) y^{v^k} y_n^s] \in (0 :_{R/(x_n,\dots,x_{i+1})R} x_i)_{(*,s+t)} \text{ for } s \geq 0.$$

Assume this is not the case, then either

$$(x^{u^k}/x_i)y^{v^k}y_n^s = x_l x^{u'} y^{v'}$$

for some u', v' and $l \geq i+1$, which contradicts to $m(u^k) = i$, or

$$(x^{u^k}/x_i)y^{v^k}y_n^s = x^{u^{k'}}y^{v^{k'}}x^{u'}y^{v'}$$

for $x^{u^{k'}}y^{v^{k'}} \in G(J)$. It follows that $|v'| = s$. Let k_1 be the largest integer l such that $y_n^l | y^{v^{k'}}$. Then

$$(x^{u^k}/x_i)y^{v^k} = ((x^{u^{k'}}y^{v^{k'}}x^{u'})/y_n^{k_1})y^{v'}/y_n^{s-k_1} \in J$$

because J is bistable, and this is again a contradiction. Therefore $(*)$ is true and we get $m^i_{s+t} \geq c^i$ for $s \geq 0$. This concludes the proof. $\square$

Remark 5.2. Proposition 5.1 can also be formulated with the roles of $\mathbf{x}$ and $\mathbf{y}$ interchanged.

Let A be a standard graded K-algebra. For a finitely generated graded A-module M the usual Castelnuovo-Mumford regularity is defined as

$$\operatorname{reg}^A(M) = \sup\{r \in \mathbb{Z}\colon \beta^A_{i,i+r}(M) \neq 0 \text{ for some integer } i\}.$$

In [7] and [12] it was shown that for a graded ideal $I \subset S_x$ the function $\operatorname{reg}^{S_x}(I^j)$ is a linear function $pj + c$ for $j \gg 0$. In the case that I is generated in one degree we give an upper bound for c and find an integer j_0 such that $\operatorname{reg}^{S_x}(I^j)$ is a linear function for all $j \geq j_0$.

Theorem 5.3. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$. Let $R(I) = S/J$ for a bigraded ideal J. Then:*

(i) $\operatorname{reg}^{S_x}(I^j) \leq jd + \operatorname{reg}^S_x(R(I))$.
(ii) $\operatorname{reg}^{S_x}(I^j) = jd + c$ *for* $j \geq m_y(\operatorname{bigin}(J))$ *and some constant* $0 \leq c \leq \operatorname{reg}^S_x(R(I))$.

Proof. We choose an almost regular sequence $\tilde{\mathbf{x}} = \tilde{x}_n, \dots, \tilde{x}_1$ for $R(I)$ over S with respect to the x-degree. For all $j \in \mathbb{N}$ the sequence $\tilde{\mathbf{x}}$ is almost regular for I^j over S_x in the sense of [3] because $R(I)_{(*,j)}(-dj) \cong I^j$ as graded S_x-modules and

$$(0 :_{R(I)/(\tilde{x}_n,\dots,\tilde{x}_{i+1})R(I)} \tilde{x}_i)_{(*,j)}(-dj) = (0 :_{I^j/(\tilde{x}_n,\dots,\tilde{x}_{i+1})I^j} \tilde{x}_i).$$

Define m^i_j for $\operatorname{bigin}(J)$ as in Proposition 5.1. Since

$$(0 :_{R(I)/(\tilde{x}_n,\dots,\tilde{x}_{i+1})R(I)} \tilde{x}_i) \cong (0 :_{S/((x_n,\dots,x_{i+1})+\operatorname{bigin}(J))} x_i),$$

it follows that

$$jd + m^i_j = r^i_j = \max(\{l\colon (0 :_{I^j/(\tilde{x}_n,\dots,\tilde{x}_{i+1})I^j} \tilde{x}_i)_l \neq 0\} \cup \{0\}).$$

By a characterization of the regularity of graded modules in [3] we have

$$\operatorname{reg}^{S_x}(I^j) = \max\{jd, r_j^1, \dots, r_j^n\}.$$

Hence the assertion follows from Proposition 4.2, Remark 4.3(ii) and Proposition 5.1. □

Analogously to Theorem 5.3 one has:

THEOREM 5.4. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$. Let $S(I) = S/J$ for a bigraded ideal J. Then:*

(i) $\operatorname{reg}^{S_x}(S^j(I)) \leq jd + \operatorname{reg}_x^S(S(I))$.

(ii) $\operatorname{reg}^{S_x}(S^j(I)) = jd + c$ *for* $j \geq m_y(\operatorname{bigin}(J))$ *and some constant* $0 \leq c \leq \operatorname{reg}_x^S(S(I))$.

Blum [4] proved the following result with different methods.

COROLLARY 5.5. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$.*

(i) *If* $\operatorname{reg}_x(R(I)) = 0$, *then* $\operatorname{reg}^{S_x}(I^j) = jd$ *for* $j \geq 1$.

(ii) *If* $\operatorname{reg}_x(S(I)) = 0$, *then* $\operatorname{reg}^{S_x}(S^j(I)) = jd$ *for* $j \geq 1$.

Proof. This follows from Theorems 5.3 and 5.4. □

Next we give a more theoretical bound for the regularity function becoming linear. Consider a bigraded algebra R. Let y be almost regular for all $\operatorname{Tor}_i^S(S/\mathfrak{m}_x, R)$ with respect to the y-degree. Define

$$w(R) = \max\{b \colon (0 :_{\operatorname{Tor}_i^S(S/\mathfrak{m}_x, R)} y)_{(*,b)} \neq 0 \text{ for some } i \in [n]\}.$$

LEMMA 5.6. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$.*

(i) *For* $j > w(R(I))$ *we have* $\operatorname{reg}^{S_x}(I^{j+1}) \geq \operatorname{reg}^{S_x}(I^j) + d$.

(ii) *For* $j > w(S(I))$ *we have* $\operatorname{reg}^{S_x}(S^{j+1}(I)) \geq \operatorname{reg}^{S_x}(S^j(I)) + d$.

Proof. We prove the case $R = R(I)$. For $j > w(R)$ one has the exact sequence

$$0 \longrightarrow \operatorname{Tor}_i^S(S/\mathfrak{m}_x, R)_{(*,j)} \xrightarrow{y} \operatorname{Tor}_i^S(S/\mathfrak{m}_x, R)_{(*,j+1)}.$$

In [7, 3.3] it was shown that

$$\operatorname{Tor}_i^S(S/\mathfrak{m}_x, R)_{(a,j)} \cong \operatorname{Tor}_i^{S_x}(K, I^j)_{a+jd},$$

and this concludes the proof. □

LEMMA 5.7. *Let R be a bigraded algebra. Then*

$$H_\bullet(0,m)_{(*,j)} = 0 \text{ for } j > \operatorname{reg}_y(R) + m.$$

Proof. We know that

$$H_\bullet(0,m) \cong \operatorname{Tor}_\bullet^S(S/\mathfrak{m}_y, R) \cong H_\bullet(S/\mathfrak{m}_y \otimes_S F_\bullet),$$

where $F_\bullet$ is the minimal bigraded free resolution of R over S. Let

$$F_i = \bigoplus S(-a,-b)^{\beta^S_{i,(a,b)}(R)}.$$

Then, by the definition of y-regularity, we have $b \leq \operatorname{reg}_y(R) + m$ for all $\beta^S_{i,(a,b)}(R) \neq 0$. Thus $(S/(\mathbf{y}) \otimes_S F_i)_{(*,j)} = 0$ for $j > \operatorname{reg}_y(R) + m$. The assertion now follows. $\square$

We obtain the following exact sequences.

COROLLARY 5.8. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$.*

(i) *For $j > \operatorname{reg}_y(R(I)) + m$ we have the exact sequence*

$$0 \longrightarrow I^{j-m}(-md) \longrightarrow \bigoplus_m I^{j-m+1}(-(m-1)d) \longrightarrow \dots$$
$$\longrightarrow \bigoplus_m I^{j-1}(-d) \longrightarrow I^j \longrightarrow 0.$$

(ii) *For $j > \operatorname{reg}_y(S(I)) + m$ we have the exact sequence*

$$0 \longrightarrow S^{j-m}(I)(-md) \longrightarrow \bigoplus_m S^{j-m+1}(I)(-(m-1)d) \longrightarrow$$
$$\dots \longrightarrow \bigoplus_m S^{j-1}(I)(-d) \longrightarrow S^j(I) \longrightarrow 0.$$

Proof. This statement follows from Lemma 5.7 since $R(I)_{(*,j)}(-jd) \cong I^j$ or $S(I)_{(*,j)}(-jd) \cong S^j(I)$, respectively. $\square$

COROLLARY 5.9. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$. Then:*

(i) *For $j \geq \max\{\operatorname{reg}_y(R(I)) + m, w(R(I)) + m\}$ we have*

$$\operatorname{reg}^{S_x}(I^{j+1}) = d + \operatorname{reg}^{S_x}(I^j).$$

(ii) *For $j \geq \max\{\operatorname{reg}_y(S(I)) + m, w(S(I)) + m\}$ we have*

$$\operatorname{reg}^{S_x}(S^{j+1}(I)) = d + \operatorname{reg}^{S_x}(S^j(I)).$$

Proof. We prove the corollary for $R(I)$. By Corollary 5.8 and standard arguments (see Lemma 6.1 for the bigraded case) we obtain for $j \geq \operatorname{reg}_y(R(I)) + m$

$$\operatorname{reg}^{S_x}(I^{j+1}) \leq \max\{\operatorname{reg}^{S_x}(I^{j+1-i}) + id - i + 1 \colon i \in [m]\}.$$

Since $j+1-i > w(R(I))$, it follows from Lemma 5.6 that

$$\operatorname{reg}^{S_x}(I^{j+1-i}) \le \operatorname{reg}^{S_x}(I^{j+1-i+1}) - d \le \cdots \le \operatorname{reg}^{S_x}(I^{j+1}) - id.$$

Hence $\operatorname{reg}^{S_x}(I^{j+1}) = \operatorname{reg}^{S_x}(I^j) + d$. □

We now consider a special case where $\operatorname{reg}^{S_x}(I^j)$ can be computed precisely.

PROPOSITION 5.10. *Let $R = S/J$ be a bigraded algebra which is a complete intersection. Let $\{z_1, \ldots, z_t\}$ be a homogeneous minimal system of generators of J which is a regular sequence. Assume that $\deg_x(z_t) \ge \cdots \ge \deg_x(z_1) > 0$ and $\deg_y(z_k) = 1$ for all $k \in [t]$. Then for all $j \ge t$*

$$\operatorname{reg}^{S_x}(R_{(*,j+1)}) = \operatorname{reg}^{S_x}(R_{(*,j)}).$$

If in addition $\deg_x(z_k) = 1$ for all $k \in [t]$, then for $j \ge 1$

$$\operatorname{reg}^{S_x}(R_{(*,j)}) = 0.$$

Proof. The Koszul $K_\bullet(\mathbf{z})$ complex with respect to $\{z_1, \ldots, z_t\}$ provides a minimal bigraded free resolution of R because these elements form a regular sequence. Observe that $(*, j)$ is an exact functor on complexes of bigraded modules. Note that $K_\bullet(\mathbf{z})_{(*,j)}$ is a complex of free S_x-modules because

$$K_i(\mathbf{z}) \cong \bigoplus_{\{j_1,\ldots,j_i\}\subseteq[t]} S(-\deg(z_{j_1}) - \cdots - \deg(z_{j_i})),$$

and

$$S(-a,-b)_{(*,j)} \cong \bigoplus_{|v|=j-b} S_x(-a)y^v \text{ as graded } S_x\text{-modules.}$$

Furthermore $K_\bullet(\mathbf{z})_{(*,j)}$ is minimal by the additional assumption $\deg_x(z_k) > 0$. We have for $j \ge t$

$$\operatorname{reg}^{S_x}(R_{(*,j)}) = \max\{\deg_x(z_t) + \cdots + \deg_x(z_{t-i+1}) - i \colon i \in [t]\},$$

and this is independent of j. If in addition $\deg_x(z_k) = 1$ for all k, then we obtain

$$\operatorname{reg}^{S_x}(R_{(*,j)}) = 0 \text{ for } j \ge 1.$$ □

Recall that a graded ideal I is said to be of linear type, if $R(I) = S(I)$. For example, ideals generated by d-sequences are of linear type. Let $I = (f_1, \ldots, f_m) \subset S_x$ be a graded ideal, which is Cohen-Macaulay of codim 2. By the Hilbert-Burch theorem S_x/I has a minimal graded free resolution

$$0 \longrightarrow \bigoplus_{i=1}^{m-1} S_x(-b_i) \xrightarrow{B} \bigoplus_{i=1}^{m} S_x(-a_i) \longrightarrow S_x \longrightarrow S_x/I \longrightarrow 0,$$

where $B = (b_{ij})$ is a $m \times m-1$-matrix with $b_{ij} \in \mathfrak{m}$ and we may assume that the ideal I is generated by the maximal minors of B. The matrix B is said to be the Hilbert-Burch matrix of I. If I is generated in degree d, then $S(I) = S/J$ where J is the bigraded ideal $(\sum_{i=1}^{m} b_{ij} y_i \colon j = 1, \dots, m-1)$.

Corollary 5.11. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$, which is Cohen-Macaulay of codim 2 with $m \times m-1$ Hilbert-Burch matrix $B = (b_{ij})$ and of linear type. Then for $j \geq m-1$*

$$\operatorname{reg}^{S_x}(I^{j+1}) = \operatorname{reg}^{S_x}(I^j) + d.$$

If additionally $\deg_x(b_{ij}) = 1$ for $b_{ij} \neq 0$, then the equality holds for $j \geq 1$.

Proof. Since I is of linear type, we have $R(I) = S(I) = S/J$ with the ideal $J = (\sum_{i=1}^{m} b_{ij} y_i \colon j = 1, \dots, m-1)$. One knows that (Krull-) $\dim(R(I)) = n+1$. Since J is defined by $m-1$ equations, we conclude that $R(I)$ is a complete intersection. Now apply Proposition 5.10. □

6. Bigraded Veronese algebras

Let R be a bigraded algebra and fix $\tilde{\Delta} = (s,t) \in \mathbb{N}^2$ with $(s,t) \neq (0,0)$. We call

$$R_{\tilde{\Delta}} = \bigoplus_{(a,b) \in \mathbb{N}^2} R_{(as,bt)}$$

the *bigraded Veronese algebra of* R with respect to $\tilde{\Delta}$ (see, for example, [9] for the graded case and [6] for similar constructions in the bigraded case). Note that $R_{\tilde{\Delta}}$ is again a bigraded algebra. We want to relate $\operatorname{reg}_x^{S_{\tilde{\Delta}}}(R_{\tilde{\Delta}})$ and $\operatorname{reg}_y^{S_{\tilde{\Delta}}}(R_{\tilde{\Delta}})$ to $\operatorname{reg}_x^S(R)$ and $\operatorname{reg}_y^S(R)$. We follow the presentation in [6] for the case of diagonals.

Lemma 6.1. *Let R be a bigraded algebra and let*

$$0 \longrightarrow M_r \longrightarrow \dots \longrightarrow M_0 \longrightarrow N \longrightarrow 0$$

be an exact complex of finitely generated bigraded R-modules. Then

$$\operatorname{reg}_x^R(N) \leq \sup\{\operatorname{reg}_x^R(M_k) - k \colon 0 \leq k \leq r\}$$

and

$$\operatorname{reg}_y^R(N) \leq \sup\{\operatorname{reg}_y^R(M_k) - k \colon 0 \leq k \leq r\}.$$

Proof. We prove by induction on $r \in \mathbb{N}$ the above inequality for $\operatorname{reg}_x^R(N)$. The case $r = 0$ is trivial. Now let $r > 0$, and consider

$$0 \longrightarrow N' \longrightarrow M_0 \longrightarrow N \longrightarrow 0,$$

where N' is the kernel of $M_0 \longrightarrow N$. Then for every integer a we have the exact sequence

$$\dots \longrightarrow \operatorname{Tor}_i^R(M_0, K)_{(a+i,*)} \longrightarrow \operatorname{Tor}_i^R(N, K)_{(a+i,*)}$$
$$\longrightarrow \operatorname{Tor}_{i-1}^R(N', K)_{(a+1+i-1,*)} \longrightarrow \dots$$

We get

$$\begin{aligned}\operatorname{reg}_x^R(N) &\le \sup\{\operatorname{reg}_x^R(M_0),\ \operatorname{reg}_x^R(N')-1\}\\ &\le \sup\{\operatorname{reg}_x^R(M_k)-k\colon 0\le k\le r\},\end{aligned}$$

where the last inequality follows from the induction hypothesis. Analogously we obtain the inequality for $\operatorname{reg}_y^R(N)$. □

Lemma 6.2. *Let A and B be graded K-algebras, let M be a finitely generated graded A-module and let N be a finitely generated graded B-module. Then $M\otimes_K N$ is a finitely generated bigraded $A\otimes_K B$-module with*

$$\operatorname{reg}_x^{A\otimes_K B}(M\otimes_K N)=\operatorname{reg}^A(M) \text{ and } \operatorname{reg}_y^{A\otimes_K B}(M\otimes_K N)=\operatorname{reg}^B(N).$$

Proof. Let $F_\bullet$ be the minimal graded free resolution of M over A and $G_\bullet$ be the minimal graded free resolution of N over B. Then $H_\bullet = F_\bullet \otimes_K G_\bullet$ is the minimal bigraded free resolution of $M\otimes_K N$ over $A\otimes_K B$ with $H_i=\bigoplus_{k+l=i}F_k\otimes G_l$. Since $A(-a)\otimes_K B(-b)=(A\otimes_K B)(-a,-b)$, the assertion follows. □

Theorem 6.3. *Let R be a bigraded algebra, $\tilde\Delta=(s,t)\in\mathbb{N}^2$ with $(s,t)\neq(0,0)$. Then*

$$\operatorname{reg}_x^{S_{\tilde\Delta}}(R_{\tilde\Delta})\le \max\{c\colon c=\lceil a/s\rceil-i,\beta^S_{i,(a,b)}(R)\neq 0 \text{ for } i,b\in\mathbb{N}\}$$

and

$$\operatorname{reg}_y^{S_{\tilde\Delta}}(R_{\tilde\Delta})\le \max\{c\colon c=\lceil b/t\rceil-i,\beta^S_{i,(a,b)}(R)\neq 0 \text{ for } i,a\in\mathbb{N}\}.$$

Proof. It suffices to show the inequality for $\operatorname{reg}_x^{S_{\tilde\Delta}}(R_{\tilde\Delta})$. Let

$$0\longrightarrow F_r\longrightarrow\dots\longrightarrow F_0\longrightarrow R\longrightarrow 0$$

be the minimal bigraded free resolution of R over S. Since $()_{\tilde\Delta}$ is an exact functor, we obtain the exact complex of finitely generated $S_{\tilde\Delta}$-modules

$$0\longrightarrow (F_r)_{\tilde\Delta}\longrightarrow\dots\longrightarrow (F_0)_{\tilde\Delta}\longrightarrow R_{\tilde\Delta}\longrightarrow 0.$$

By Lemma 6.1 we have

$$\operatorname{reg}_x^{S_{\tilde\Delta}}(R_{\tilde\Delta})\le\max\{\operatorname{reg}_x^{S_{\tilde\Delta}}((F_i)_{\tilde\Delta})-i\}.$$

Since

$$F_i=\bigoplus_{(a,b)\in\mathbb{N}^2}S(-a,-b)^{\beta^S_{i,(a,b)}(R)},$$

one has

$$\operatorname{reg}_x^{S_{\tilde\Delta}}((F_i)_{\tilde\Delta})=\max\{\operatorname{reg}_x^{S_{\tilde\Delta}}(S(-a,-b)_{\tilde\Delta})\colon \beta^S_{i,(a,b)}(R)\neq 0\}.$$

We have to compute $\operatorname{reg}_x^{S_{\tilde\Delta}}(S(-a,-b)_{\tilde\Delta})$. Let $M_0,\dots,M_{s-1}$ the relative Veronese modules of S_x and $N_0,\dots,N_{t-1}$ be the relative Veronese modules of

S_y. That is, $M_j = \bigoplus_{k\in\mathbb{N}}(S_x)_{ks+j}$ for $j = 0, \dots, s-1$ and $N_j = \bigoplus_{k\in\mathbb{N}}(S_y)_{kt+j}$ for $j = 0, \dots, t-1$. Then

$$\begin{aligned} S(-a,-b)_{\tilde{\Delta}} &= \bigoplus_{(k,l)\in\mathbb{N}^2} (S_x)_{ks-a} \otimes_K (S_y)_{lt-b} \\ &= M_i(-\lceil a/s\rceil) \otimes_K N_j(-\lceil b/t\rceil), \end{aligned}$$

where $i \equiv -a \bmod s$ for $0 \le i \le s-1$ and $j \equiv -b \bmod t$ for $0 \le j \le t-1$.

By [1] the relative Veronese modules over a polynomial ring have a linear resolution over the Veronese algebra. Hence Lemma 6.2 implies that $\operatorname{reg}_x^{S_{\tilde{\Delta}}}(S(-a,-b)_{\tilde{\Delta}}) = \lceil a/s\rceil$. This concludes the proof. □

Corollary 6.4. *Let R be a bigraded algebra.*

(i) *For $s \gg 0, t \in \mathbb{N}$ and $\tilde{\Delta} = (s,t)$ one has $\operatorname{reg}_x^{S_{\tilde{\Delta}}}(R_{\tilde{\Delta}}) = 0$.*
(ii) *For $t \gg 0, s \in \mathbb{N}$ and $\tilde{\Delta} = (s,t)$ one has $\operatorname{reg}_y^{S_{\tilde{\Delta}}}(R_{\tilde{\Delta}}) = 0$.*

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FB6 Mathematik und Informatik, Universität Essen, 45117 Essen, Germany
E-mail address: `tim.roemer@uni-essen.de, tim.roemer@gmx.de`