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HARMONIC MAPS FROM FINSLER MANIFOLDS

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ABSTRACT. A Finsler manifold is a Riemannian manifold without the quadratic restriction. In this paper we introduce the energy functional, the Euler-Lagrange operator, and the stress-energy tensor for a smooth map ϕ from a Finsler manifold to a Riemannian manifold. We show that ϕ is an extremal of the energy functional if and only if ϕ satisfies the corresponding Euler-Lagrange equation. We also characterize weak Landsberg manifolds in terms of harmonicity and horizontal conservativity. Using the representation of a tension field in terms of geodesic coefficients, we construct new examples of harmonic maps from Berwald manifolds which are neither Riemannian nor Minkowskian.

1. Introduction

Harmonic maps between Riemannian manifolds are mappings $\phi\colon (M, g) \to (N, h)$ for which the "Dirichlet energy functional"

$$\frac{1}{2}\int_M \|d\phi\|^2 \, dv$$

is extremal. Harmonic maps are solutions to an elliptic system of partial differential equations, which in general is non-linear. They are very important in both classical and modern differential geometry.

On the other hand, Riemannian manifolds are a special case of Finsler manifolds [9], namely Finsler manifolds with the quadratic restriction [8].

Let $\phi\colon (M, F) \to (N, h)$ be a smooth map from a Finsler manifold to a Riemannian manifold. In this paper we introduce a natural energy functional and the Euler-Lagrange operator of ϕ. We show that ϕ is an extremal of the energy functional if and only if ϕ satisfies the corresponding Euler-Lagrange equation.

The weak Landsberg manifolds (see Definition 4.2) are special Finsler manifolds. They have constant volume of Finsler spheres and therefore satisfy the Gauss-Bonnet formula [4][13]. The weak Landsberg spaces have the following

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interesting geometric characterization: a Finsler manifold is of weak Landsberg type if and only if all projective spheres in the projective sphere bundle are minimal [6].

In this paper we introduce a notion of stress-energy tensor for maps from a Finsler manifold to a Riemannian manifold and characterize weak Landsberg manifolds in terms of harmonicity and horizontal conservativity. Here horizontal conservativity means that the stress-energy tensor is divergence-free with respect to the horizontal subbundle of the projective sphere bundle. We refer to [15] for the relation between horizontally conservative maps and harmonic morphisms.

2. Preliminaries

Let M be a C^∞ m-dimensional manifold and $\widetilde{TM} = TM\backslash\{0\}$. A function $F\colon TM \to [0,\infty)$ is called a *Finsler structure* on M if F has the following properties:

(i) $F(tY) = tF(Y)$ for all $t \in R^+$.
(ii) F is C^∞ on $\widetilde{TM}$.
(iii) For every non-zero $Y \in T_xM$, the induced quadratic form g_Y given by

$$g_Y(U,V) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\left(F^2(Y+sU+tV)\right)\Big|_{s=t=0}$$

is an inner product in T_xM.

A *Finsler manifold* is a C^∞ manifold M with a Finsler structure F.

Important examples of Finsler manifolds are Riemannian manifolds and Minkowski manifolds. Let (M,F) be a Finsler manifold, SM the projective sphere bundle of M, with canonical projection map $\pi\colon SM \to M$ given by $(x,[y]) \to x$, and let $S_xM := \pi^{-1}(x)$ be the projective sphere at x. We denote the pull-backs of TM and T^*M by π^*TM and π^*T^*M, respectively, and consider these as vector bundles (with m-dimensional fibres) over the $(2m-1)$-dimensional base SM.

Given local coordinates (x^i) on M, we can write any $y \in T_xM$ as $y^i\frac{\partial}{\partial x^i}$. This generates local coordinates $(x^i;y^i)$ on SM.

At each point of SM, the fiber of π^*TM has a basis $\{\frac{\partial}{\partial x^i}\}$. Hence F inherits the *Hilbert form*, the *fundamental tensor*, and the *Cartan tensor* as follows:

$$\omega := \frac{\partial F}{\partial y^i}\,dx^i, \qquad g := g_{ij}(x,y)\,dx^i\otimes dx^j, \qquad g_{ij} = \left[\frac{1}{2}F^2\right]_{y^iy^j},$$

$$A := H_{ijk}\,dx^i\otimes dx^j\otimes dx^k, \qquad H_{ijk} := F\frac{\partial g_{ij}}{\partial y^k},$$

where $1 \leq i, j, k \cdots \leq m = \dim M$. From the Cartan tensor one can construct the Cartan form by setting

$$\eta = \sum_{i,j,k} H_{ijk} g^{jk}\, dx^i, \quad \left(g^{jk}\right) = \left(g_{jk}\right)^{-1}.$$

This is a global section of $\pi^* T^* M \subset T^* SM$. We introduce a dual adapted orthonormal frame $\{e_i\}$ on the Riemannian vector bundle $(\pi^* TM, g)$ and a coframe $\{\omega_i\}$ with $\omega_n = \omega$. Putting $\omega_i = \sum v_{ij}\, dx^j$, we have $\det(v_{ij}) = \sqrt{\det(g_{kl})}$ and $g = \sum \omega_i^2 \in \Gamma(\odot^2 \pi^* T^* M)$.

Taking the exterior derivative of ω yields the Chern connection on $\pi^* TM$ described by an $m \times m$ matrix of 1-forms (ω_{ij}) on SM. These connection forms determine horizontal and vertical derivatives, Riemannian and Minkowski curvature, the Riemannian metric on SM and its Riemannian connection [11]. Notice that

$$\omega_1 \wedge \cdots \wedge \omega_m \wedge \omega_{m1} \wedge \cdots \wedge \omega_{m,m-1}$$

is the volume form with respect to the Riemannian metric on SM. We denote it by Π.

The following lemmas will be used later.

LEMMA 2.1. *If M is a compact Finsler manifold, then for any function $f\colon SM \to R$*

$$\int_{SM} f\Pi = \int_M dx \int_{S_x M} f \sqrt{\det\left(g_{ij}\right)}\chi,$$

where $dx = dx^1 \wedge \cdots \wedge dx^m$ and

$$\chi \equiv \omega_{m1} \wedge \cdots \wedge \omega_{m,m-1} \qquad \bmod \quad dx^j.$$

In particular, if M is Riemannian and f is defined on M, then

$$\int_{SM} f\Pi = \operatorname{Vol}\left(S^{m-1}\right) \int_M f\, dv,$$

where $\operatorname{Vol}(S^{m-1})$ *is the volume of standard* $(m-1)$*-dimensional sphere.*

Proof. Obvious. ☐

Set

$$1 \leq \lambda, \mu, \tau, \ldots \leq m-1, \quad \bar{\lambda} = m + \lambda, \quad 1 \leq a, b, c, \ldots \leq 2m-1.$$

The first structure equation for (M, F) can be written as

$$d\omega_i = \sum \omega_j \wedge \omega_{ji}, \qquad \omega_{ij} + \omega_{ji} = -2 \sum A_{ij\lambda} \omega_{m\lambda}, \tag{2.1}$$

where $A_{ijk} = A(e_i, e_j, e_k)$. Taking the exterior derivative of (2.1) we see that the curvature 2-forms $\Omega_{ij} := d\omega_{ij} - \sum \omega_{ik} \wedge \omega_{kj}$ can be expressed in the form

$$\Omega_{ij} = \frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l + \sum P_{ijk\lambda} \omega_k \wedge \omega_{m\lambda},$$

where $R_{ijkl} = -R_{ijlk}$. Set

$$P_{\lambda\mu\nu} := P_{m\lambda\mu\nu}.$$

$P_{\lambda\mu\nu}$ is called the *Landsberg curvature* [14]. From [4] we have

$$P_{m\lambda\mu} = 0, \quad \sum_{\lambda} P_{\lambda\lambda\mu} = -\sum_{\lambda} \dot{A}_{\lambda\lambda\mu}, \tag{2.2}$$

where the dot denotes the covariant derivative along the Hilbert form.

Denote the Riemannian metric on SM by G (cf. [11]). The divergence of a form Ψ on SM with respect to G is defined by

$$\operatorname{div}\Psi := \sum_{a} \left(D_{\epsilon_a}\Psi\right)\left(\quad, \epsilon_a\right),$$

where $\{\epsilon_a\}$ is the dual basis of $\{\omega_1, \ldots, \omega_m, \omega_{m1}, \ldots, \omega_{m,m-1}\}$ on $T(SM)$ and D_{ϵ_a} is the covariant derivative induced by G along ϵ_a.

Lemma 2.2.

(i) *For* $S = \sum S_i\omega_i \in \Gamma(\pi^*T^*M)$, $\operatorname{div} S = \sum S_{i|i} + \sum S_\mu P_{\lambda\lambda\mu}$.

(ii) *For* $T = \sum T_{ij}\omega_i\omega_j \in \Gamma(\odot^2\pi^*T^*M)$, $\operatorname{div} T\,(\epsilon_i) = \sum T_{ij|j} + \sum T_{i\mu}P_{\lambda\lambda\mu}$.

Proof. With the abbreviations

$$\psi_i = \omega_i, \qquad \psi_{\bar{\lambda}} = \omega_{m\lambda},$$

denote the Levi-Civita connection with respect to $\{\psi_a\}$ by $\{\psi_{ab}\}$. Then (cf. [11])

$$\psi_{ij} \equiv \omega_{ji} \mod \psi_{\bar{\lambda}}, \quad \psi_{i\bar{\lambda}} \equiv -\sum P_{i\lambda\mu}\psi_{\bar{\mu}} \mod \psi_j,$$

and

$$\begin{aligned}
\operatorname{div} S &= \sum_{a} \left(DS_a\right)\left(\epsilon_a\right) \\
&= \sum \Big(dS_i - \sum S_j\psi_{ji}\Big)\left(\epsilon_i\right) - \sum S_j\psi_{j\bar{\lambda}}\left(\epsilon_{\bar{\lambda}}\right) \\
&= \sum \Big(dS_i - \sum S_j\omega_{ij}\Big)\left(\epsilon_i\right) - \sum S_\mu\left(-P_{\lambda\lambda\mu}\right) \\
&= \sum_{i} S_{i|i} + \sum S_\mu P_{\lambda\lambda\mu},
\end{aligned}$$

where the covariant derivative of S is defined by

$$DS_i = dS_i - \sum S_j\omega_{ij} = \sum S_{i|j}\omega + \sum S_{i;\lambda}\omega_{m\lambda}.$$

This proves (i).

Similarly, for T we have

$$\begin{aligned}(\operatorname{div} T)(\epsilon_i) &= \sum \left(dT_{ib} - \sum T_{cb}\psi_{ci} - \sum T_{ic}\psi_{cb}\right)(\epsilon_b)\\ &= \sum \left(dT_{ij} - T_{kj}\omega_{ik} - \sum T_{ik}\omega_{jk}\right)(\epsilon_j) + \sum T_{i\mu}P_{\lambda\lambda\mu}\\ &= \sum T_{ij|j} + \sum T_{i\mu}P_{\lambda\lambda\mu},\end{aligned}$$

which proves (ii). $\square$

The *energy density* of a map $\phi\colon (MF) \to (N,h)$ from a Finsler manifold to a Riemannian manifold is the function $e(\phi)\colon SM \to R_{\geq 0}$ defined by

$$e(\phi)(x,[y]) = \frac{1}{2}\sum_j h\left(\phi_* e_j, \phi_* e_j\right), \tag{2.3}$$

where $\{e_j\}$ is the orthonormal basis with respect to g (the fundamental tensor of F) at $(x,[y])$.

If Ω is a compact domain in M, we use the canonical volume element Π associated with F to define the *energy* of $\phi\colon (\Omega,F) \to (N,h)$ by

$$E(\phi,\Omega) = \frac{1}{c}\int_{S\Omega} e(\phi)\Pi,$$

where $c := \operatorname{Vol}(S^{m-1})$ is the volume of the standard $(m-1)$-dimensional sphere and $S\Omega$ the projective sphere bundle of Ω. If M is compact, we write $E(\phi) = E(\phi, M)$.

Remark. By Lemma 2.1, our notion of energy reduces to the usual notion of energy if M is a compact Riemannian manifold.

A smooth map $\phi\colon (M,F) \to (N,h)$ from a Finsler manifold to a Riemannian manifold is said to be *harmonic* if it is an extremal of the restriction of E on every compact subdomain of (M,F).

3. The first variation

Let (M,F) be a smooth Finsler manifold and g the fundamental tensor of F. Let (N,h) be a Riemannian manifold. Let $\phi\colon (M,F) \to (N,h)$ be a smooth map. Set

$$h = \sum \theta_\alpha^2 \in \Gamma\left(\odot^2 T^*N\right), \quad 1 \leq \alpha,\beta,\gamma,\ldots \leq n. \tag{3.1}$$

The first structure equation for (N,h) is

$$d\theta_\alpha = \sum \theta_\beta \wedge \theta_{\beta\alpha}, \quad \theta_{\alpha\beta} + \theta_{\beta\alpha} = 0. \tag{3.2}$$

A vector field v along ϕ determines a variation ϕ_t by $\phi_t(x) = \exp_{\phi(x)}[tv(x)]$, where $t \in I := (-\varepsilon,\varepsilon)$ for some $\varepsilon > 0$. Noting that

$$\phi_t^*\theta_\alpha \in \Gamma\left(T^*M\right) \subset \Gamma\left(\pi^*T^*M\right),$$

we put

$$\phi_t^*\theta_\alpha = \sum a_{\alpha i}\omega_i, \tag{3.3}$$

where $a_{\alpha i} = a_{\alpha i}(t)$. It follows that

$$\phi_t^*(h) = \phi_t^*\left(\sum \theta_\alpha^2\right) = \sum \left[\phi_t^*\theta_\alpha\right]^2 = \sum a_{\alpha i}a_{\alpha j}\omega_i\omega_j. \tag{3.4}$$

Since $\{e_i\}$ is the dual frame field of $\{\omega_i\}$, from (2.3) and (3.4) we obtain

$$2e\left(\phi_t\right) = \sum_i \left(\sum a_{\alpha j}a_{\alpha k}\omega_j\omega_k\right)\left(e_i, e_i\right) = \sum a_{\alpha i}^2.$$

If v has compact support $\Omega \subset M$, then

$$c \cdot \frac{d}{dt}E\left(\phi_t, \Omega\right)\bigg|_{t=0} = \int_{SM} \sum a_{\alpha i}\ \frac{\partial a_{\alpha i}}{\partial t}\bigg|_{t=0} \Pi. \tag{3.5}$$

Define $\Phi\colon M \times I \to N$ by

$$(x, t) \xrightarrow{\Phi} \phi_t(x).$$

It is easy to see that

$$\Phi^*\theta_\alpha \equiv \phi_t^*\theta_\alpha, \quad \Phi^*\theta_{\alpha\beta} \equiv \phi_t^*\theta_{\alpha\beta} \quad \mathrm{mod} \quad dt.$$

Put

$$\Phi^*\theta_\alpha = \phi_t^*\theta_\alpha + b_\alpha\, dt, \tag{3.6}$$

$$\Phi^*\theta_{\alpha\beta} = \phi_t^*\theta_{\alpha\beta} + B_{\alpha\beta}\, dt. \tag{3.7}$$

Then $\sum b_\alpha v_\alpha|_{t=0} = b$ is the deformation vector field, where $\{v_\alpha\}$ is the dual frame field of $\{\theta_\alpha\}$, and $B_{\alpha\beta}$ satisfies

$$B_{\alpha\beta} = -B_{\beta\alpha}. \tag{3.8}$$

Using (3.2), (3.3), (3.6), and (3.7), we obtain

$$\begin{aligned} d\left(\Phi^*\theta_\alpha\right) &= \Phi^*\ \left(d\theta_\alpha\right) \\ &= \Phi^*\left(\sum \theta_\beta \wedge \theta_{\beta\alpha}\right) \\ &= \sum \Phi^*\theta_\beta \wedge \Phi^*\theta_{\beta\alpha} \\ &= \sum \left(\sum_i a_{\beta i}\omega_i + b_\beta\, dt\right) \wedge \left(\phi_t^*\theta_{\beta\alpha} + B_{\beta\alpha}\, dt\right). \end{aligned} \tag{3.9}$$

On the other hand, from (3.3) and (3.6), we have

(3.10)

$$\begin{aligned} d\left(\Phi^*\theta_\alpha\right) &= d\left[\sum a_{\alpha i}\omega_i + b_\alpha\, dt\right] \\ &= \sum \left(da_{\alpha i}\right)\wedge\omega_i + \sum a_{\alpha i}\, d\omega_i + db_\alpha \wedge dt \\ &= \sum\left(d_{SM}a_{\alpha i} + \frac{\partial a_{\alpha i}}{\partial t}\, dt\right)\wedge\omega_i + \sum a_{\alpha i}\, d\omega_i + d_{SM}b_\alpha\wedge dt. \end{aligned}$$

Comparing the coefficients of dt in (3.9) and (3.10), we obtain

$$\sum \frac{\partial a_{\alpha i}}{\partial t}\omega_i - d_{SM}b_\alpha = \sum_\beta\left(b_\beta\phi_t^*\theta_{\beta\alpha} - \sum_i B_{\beta\alpha}a_{\beta i}\omega_i\right). \tag{3.11}$$

Define the covariant derivative of $\{b_\alpha\}$ by

$$\begin{aligned} Db_\alpha &= d_{SM}b_\alpha - \sum_\beta b_\beta\phi_t^*\theta_{\alpha\beta} \\ &= \sum b_{\alpha|i}\omega_i + \sum b_{\alpha;\lambda}\omega_{m\lambda}. \end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.11) we obtain

$$\frac{\partial a_{\alpha i}}{\partial t} = b_{\alpha|i} - \sum B_{\beta\alpha}a_{\beta i}, \quad b_{\alpha;\lambda} = 0. \tag{3.13}$$

From (3.5), (3.8), and (3.13) we have

$$\begin{aligned} c\cdot\frac{d}{dt}E\left(\phi_t\right)\Big|_{t=0} &= \int \sum a_{\alpha i}\left(b_{\alpha|i} - \sum B_{\beta\alpha}a_{\beta i}\right)\Pi \\ &= \int \left(\sum a_{\alpha i}b_{\alpha|i} - \sum a_{\alpha i}B_{\beta\alpha}a_{\beta i}\right)\Pi \\ &= \int \sum_i\left(\sum a_{\alpha i}b_\alpha\right)_{|i}\Pi - \int\sum a_{\alpha i|i}b_\alpha\Pi, \end{aligned}$$

where

$$a_{\alpha i|j} := \left[da_{\alpha i} - \sum a_{\alpha k}\omega_{ik} - \sum a_{\beta i}\phi^*\theta_{\alpha\beta}\right]\left(e_j\right)$$

and

$$\sum_i\left(\sum a_{\alpha i}b_\alpha\right)\omega_i = \langle d\phi, b\rangle$$

is a global section on the dual Finsler bundle π^*T^*M. Using (2.2) and Lemma 2.2 we get

$$\begin{aligned} c\cdot\frac{d}{dt}E\left(\phi_t\right)\Big|_{t=0} &= \int_{SM}\operatorname{div}\langle d\phi, b\rangle\Pi - \int_{SM}\langle\tau(\phi), b\rangle\Pi \\ &= -\int_{SM}\langle\tau(\phi), b\rangle\Pi, \end{aligned}$$

where

$$\tau(\phi) := -\langle d\phi, \dot{\eta}\rangle + \operatorname{Tr} D\, d\phi \in \Gamma\left((\phi\circ\pi)^* TN\right) \tag{3.14}$$

and η (resp. $D\, d\phi$) denotes the Cartan form (resp. the *second fundamental form*) of ϕ. The field $\tau(\phi)$ is called the *tension field* of ϕ.

THEOREM 3.1. *Let ϕ be a smooth map from a Finsler manifold M to a Riemannian manifold N. Then ϕ is harmonic if and only if it has vanishing tension field.*

Let us now express the tension field in local coordinates (x^i) on M and (u^α) on N. We denote by g_{ij} and ${}^M\Gamma^i_{jk}$ the components of the fundamental tensor and the Christoffel symbols of the Chern connection on (M, F), and by $h_{\alpha\beta}$ and ${}^N\Gamma^\alpha_{\beta\gamma}$ the corresponding objects on (N, h). Note that ${}^N\Gamma^\alpha_{\beta\gamma}$ are just the Christoffel symbols of Levi-Civita on N because $h_{\alpha\beta}$ are Riemannian.

Let D denote the covariant differentiation (of sections of tensor products of π^*TM and π^*T^*M) on SM with respective to the Chern connection. Then, by (2.46) and (2.47a) in [3], we have

$$D\frac{\partial}{\partial x^k} = {}^M\Gamma^i_{kl}\, dx^l \otimes \frac{\partial}{\partial x^i},$$

where

$$ {}^M\Gamma^i_{kl} = g^{ij}\,{}^M\Gamma_{jkl}, \tag{3.15}$$

$$ {}^M\Gamma_{jkl} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^l} - \frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k}\right) + \frac{1}{2}\left(M_{jkl} - M_{klj} + M_{ljk}\right), \tag{3.16}$$

$$ M_{ijk} = -\frac{\partial g_{ij}}{\partial y^l}\frac{\partial G^l}{\partial y^k}, \tag{3.17}$$

and G^l are the *geodesic coefficients* of (M, F) (cf. [12]). Using the Leibniz rule, we obtain (cf. [5, p. 41])

$$ D\, dx^i = -{}^M\Gamma^i_{kl}\, dx^k \otimes dx^l. \tag{3.18}$$

Suppose that $\phi\colon (M, F) \to (N, h)$ is a smooth map. Locally, we can write $\phi = (\phi^\alpha)$ where each ϕ^α is a smooth function defined an open subset in M. Let D denote the covariant differentiation on $\pi^*T^*M \otimes (\phi\circ\pi)^*TN$. Then (cf. [10])

$$\begin{aligned} D_{\partial/\partial x^i}(d\phi) &= D_{\partial/\partial x^i}\left(\phi^\alpha_j\, dx^j \frac{\partial}{\partial u^\alpha}\right) \\ &= \phi^\alpha_{ij}\, dx^j \frac{\partial}{\partial u^\alpha} + \phi^\alpha_j D^{\pi^*T^*M}_{\partial/\partial x^i}\, dx^j \frac{\partial}{\partial u^\alpha} + \phi^\alpha_j\, dx^j D^{(\phi\circ\pi)^*TN}_{\partial/\partial x^i} \frac{\partial}{\partial u^\alpha}, \end{aligned}$$

where

$$\phi^\alpha_i = \frac{\partial\phi^\alpha}{\partial x^i}, \quad \phi^\alpha_{ij} = \frac{\partial^2\phi^\alpha}{\partial x^i \partial x^j}.$$

Now

$$D^{\pi^*T^*M}_{\partial/\partial x^i}\, dx^j = -{}^M\Gamma^j_{ki}\, dx^k$$

and

$$D^{\pi^*\phi^*TN}_{\partial/\partial x^i} \frac{\partial}{\partial u^\alpha} = \phi^\beta_i {}^N\Gamma^\gamma_{\alpha\beta} \frac{\partial}{\partial u^\gamma},$$

so that

$$D_{\partial/\partial x^i}(d\phi) = \left(\phi^\alpha_{ij} - {}^M\Gamma^k_{ij}\phi^\alpha_k + {}^N\Gamma^\alpha_{\beta\gamma}\phi^\beta_i\phi^\gamma_j\right) dx^j \frac{\partial}{\partial u^\alpha},$$

where we have used the fact that (cf. [3])

$${}^M\Gamma^k_{ij} = {}^M\Gamma^k_{ji}.$$

It follows that the components of the second fundamental form $D\, d\phi$ satisfy

$$(D\, d\phi)^\alpha_{ij} = \phi^\alpha_{ij} - {}^M\Gamma^k_{ij}\phi^\alpha_k + {}^N\Gamma^\alpha_{\beta\gamma}\phi^\beta_i\phi^\gamma_j.$$

Now consider a smooth function f defined on an open subset in M. Set

$$f_j = \frac{\partial f}{\partial x^j}, \qquad f_{ij} = \frac{\partial f_j}{\partial x^i},$$

and

$$D_{\partial/\partial x^i}(df) = (D\, df)_{ij}\, dx^j.$$

Then

$$df = f_j\, dx^j,$$
$$(D\, df)_{ij} = (D\, df)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = f_{ij} - {}^M\Gamma^k_{ij} f_k.$$

Thus (3.14) reduces to

$$\begin{aligned} \tau(f) &= -\langle df, \dot\eta\rangle + \operatorname{Tr} D\, df \\ &= g^{ij}\left[f_{ij} - {}^M\Gamma^k_{ij} f_k - \xi_i f_j\right], \end{aligned} \tag{3.19}$$

where

$$\xi_j = \dot\eta\left(\frac{\partial}{\partial x^j}\right). \tag{3.20}$$

Suppose that $\phi\colon (M,F) \to (N,h)$ is a smooth map. By (3.19) we have

$$\tau(\phi^\alpha) = g^{ij}\left[\phi^\alpha_{ij} - {}^M\Gamma^k_{ij}\phi^\alpha_k - \xi_i\phi^\alpha_j\right]. \tag{3.21}$$

Hence the tension field of ϕ is

$$\begin{aligned} \tau^\alpha_\phi &= du^\alpha\left(-\langle d\phi, \dot\eta\rangle + \operatorname{Tr} D\, d\phi\right) \\ &= g^{ij}\left[-\xi_i\phi^\alpha_j + \phi^\alpha_{ij} - {}^M\Gamma^k_{ij}\phi^\alpha_k + {}^N\Gamma^\alpha_{\beta\gamma}\phi^\beta_i\phi^\gamma_j\right] \\ &= \tau(\phi^\alpha) + g^{ij}\, {}^N\Gamma^\alpha_{\beta\gamma}\phi^\beta_i\phi^\gamma_j. \end{aligned} \tag{3.22}$$

A direct calculation using (2.2), (3.15), (3.16), and (3.17) yields (cf. [5, (3.3.3)])

$$\xi_i = -y^j \frac{\partial ^M\Gamma^k_{jk}}{\partial y^i}$$

and

$${}^M\Gamma^i_{ki} = \left(\frac{\partial}{\partial x^k} - \frac{\partial G^i}{\partial y^k}\frac{\partial}{\partial y^i}\right) \log \sqrt{\det (g_{jl})}.$$

4. The stress-energy tensor

Let $\phi\colon (M,F) \to (N,h)$ be a smooth map from a Finsler manifold (M,F) to a Riemannian manifold (N,h). The *stress-energy tensor* S_ϕ is a tensor on SM defined by

$$S_\phi := e(\phi)g - \phi^* h,$$

where $e(\phi)$ (resp. g) denotes the energy density (resp. the fundamental tensor) of ϕ and ϕ^*h denotes the pull back of the tensor h to a tensor on SM. We say that S_ϕ is *horizontally divergence-free* if $\sum_{i=1}^m (D_{\epsilon_i} S_\phi)(\epsilon_i, Y) = 0$ for all $Y \in H_p$, where $\{\epsilon_i\}$ is any orthonormal basis for the horizontal space H_p and $H_p := \{X \in T_pSM, \omega_{m\lambda}(X)\} = 0$ (cf. [11]).

Let $\phi\colon (M,F) \to (N,h)$ be a smooth map. Define on M

$$\langle \theta_\alpha, d\phi\rangle = \sum a_{\alpha i}\omega_i,$$

where $\{\theta_\alpha\}$ is an orthonormal coframe of h. Then

$$\begin{aligned} d\left(\sum_i a_{\alpha i}\omega_i\right) &= d\left(\phi^*\theta_\alpha\right) \\ &= \phi^*\, d\theta_\alpha \\ &= \phi^*\left(\sum \theta_\beta \wedge \theta_{\beta\alpha}\right) \\ &= \sum \phi^*\theta_\beta \wedge \phi^*\theta_{\beta\alpha}, \end{aligned} \tag{4.1}$$

by (3.2). Consider (4.1) as a two-form defined on the projective sphere bundle SM. We have

$$\begin{aligned} d\left(\sum_i a_{\alpha i}\omega_i\right) &= \sum da_{\alpha i}\wedge \omega_i + \sum a_{\alpha i}\, d\omega_i \\ &= \sum da_{\alpha i}\wedge \omega_i + \sum a_{\alpha i}\omega_j \wedge \omega_{ji} \\ &= \sum a_{\beta i}\omega_i \wedge \phi^*\theta_{\beta\alpha}. \end{aligned}$$

It follows that

$$\sum_i Da_{\alpha i}\wedge \omega_i = 0, \tag{4.2}$$

where

$$Da_{\alpha i} := da_{\alpha i} - \sum a_{\alpha j}\omega_{ij} + \sum a_{\beta i}\phi^*\theta_{\beta\alpha} \\ := \sum a_{\alpha i|j}\omega_j + \sum a_{\alpha i;\lambda}\omega_{m\lambda}. \tag{4.3}$$

Substituting (4.3) into (4.2) yields the following result.

PROPOSITION 4.1. *The second fundamental form of* $\phi\colon (M,F) \to (N,g)$ *satisfies* $a_{\alpha i|j} = a_{\alpha j|i}$ *and* $a_{\alpha i;\lambda} = 0$.

Denote the stress-energy S_ϕ of ϕ by

$$S_\phi = \sum S_{ij}\omega_i \otimes \omega_j.$$

Then

$$S_{ij} = e(\phi)\delta_{ij} - \sum a_{\alpha i}a_{\alpha j}, \tag{4.4}$$

where $e(\phi)$ is the energy density of ϕ. Then the horizontal divergence of S_ϕ is

$$\begin{aligned} \operatorname{div}_H S_\phi &= \sum S_{ijj}\omega_i \\ &= \sum_i \Big(\sum S_{ij|j} + \sum S_{i\mu}P_{\lambda\lambda\mu}\Big)\omega_i \\ &= \sum_i \Bigg\{ \sum_j \Big[e(\phi)\delta_{ij} - \sum a_{\alpha i}a_{\alpha j}\Big]_{|j} \\ &\qquad + \sum \Big[e(\phi)\delta_{i\mu} - \sum a_{\alpha i}a_{\alpha\mu}\Big] P_{\lambda\lambda\mu}\Bigg\}\omega_i \\ &= \sum \Big[e(\phi)_{|i} - \sum a_{\alpha i|j}a_{\alpha j} - \sum a_{\alpha i}a_{\alpha j|j} \\ &\qquad + \sum e(\phi)P_{\lambda\lambda i} - \sum a_{\alpha i}a_{\alpha\mu}P_{\lambda\lambda\mu}\Big]\omega_i \\ &= \sum_i \Big[\sum a_{\alpha j}a_{\alpha j|i} - \sum a_{\alpha i|j}a_{\alpha j} \\ &\qquad - \sum a_{\alpha i}a_{\alpha j|j} + \sum e(\phi)P_{\lambda\lambda i} - \sum a_{\alpha i}a_{\alpha\mu}P_{\lambda\lambda\mu}\Big]\omega_i \\ &= -\sum_i \Big[\sum a_{\alpha i}a_{\alpha j|j} \\ &\qquad + \sum a_{\alpha i}a_{\alpha\mu}P_{\lambda\lambda\mu}\Big]\omega_i + e(\phi)\sum P_{\lambda\lambda\mu}\omega_\mu \\ &= -\langle \tau(\phi), d\phi\rangle - e(\phi)\dot{\eta}, \end{aligned} \tag{4.5}$$

where $\dot{\eta}$ denotes the covariant derivative of the Cartan form along the Hilbert form, and where we have used (2.2) and (ii) of Lemma 2.2.

Definition 4.2. A Finsler manifold is said to be *of weak Landsberg type* if $\dot{\eta} = 0$.

The following theorems are immediate consequences of (4.5).

Theorem 4.3. *Let $\phi\colon (M,F) \to (N,h)$ be a non-constant harmonic map from a Finsler manifold to a Riemannian manifold. Then S_ϕ is horizontally divergence-free if and only if (M,F) is of weak Landsberg type.*

Combining this with Shen's theorem ([6], [11]) (see the Introduction) we obtain the following Wood type result (cf. [15, Theorem 2.9]).

Theorem 4.4. *Let $\phi\colon (M,F) \to (N,h)$ be a submersion from a Finsler manifold to a Riemannian manifold. Then any two of the following conditions imply the third condition:*

(i) *ϕ is harmonic;*
(ii) *S_ϕ is horizontally divergence-free;*
(iii) *$\pi\colon SM \to M$ has minimal fibers.*

5. Harmonicity of the identity map

In this section we present the harmonic equation of the identity map from a Finsler manifold to a Riemannian manifold in terms of their geodesic coefficients, and we construct harmonic maps from Berwald manifolds which are neither Riemannian nor Minkowskian to Riemannian manifolds.

Let $I\colon (M,F) \to (M,h)$ be the identity map from a Finsler manifold (M,F) to a Riemannian manifold (M,h). As usual we put

$$I = \left(I^i\right) : U(\subset M) \to R,$$

where locally $I^i(x^1,\dots,x^m) = x^i$. It follows that

$$I^i_j = \frac{\partial I^i}{\partial x^j} = \delta^i_j, \quad I^i_{jk} = 0.$$

Using (3.21) and (3.22) we have

$$\tau\left(I^k\right) = -g^{ij}\,{}^F\Gamma^k_{ij} - g^{kj}\xi_j,$$

and hence

$$\tau^k_I = g^{ij}\left[{}^h\Gamma^k_{ij} - {}^F\Gamma^k_{ij}\right] - g^{kj}\xi_j. \tag{5.1}$$

Denote the geodesic coefficients of (M,F) and (M,h) by ${}^FG^i$ and ${}^hG^i$, respectively. By [5, (3.8.3)] we have

$$\frac{1}{2}\left({}^FG^i\right)_{y^jy^k} = {}^F\Gamma^i_{jk} + \dot{H}^i{}_{jk},$$

where $\dot{H}^i{}_{jk}$ is the covariant derivative of the Cartan tensor along the Hilbert form. It follows that

$$g^{ij}\,{}^F\Gamma^k_{ij} + g^{ki}\xi_i = g^{ij}\left[\frac{1}{2}\left({}^FG^k\right)_{y^iy^j} - \dot{H}^k_{ij}\right] + g^{ki}\dot{H}_i \tag{5.2}$$
$$= \frac{1}{2}g^{ij}\left({}^FG^k\right)_{y^iy^j} - \dot{H}^k + \dot{H}^k = \frac{1}{2}g^{ij}\left({}^FG^k\right)_{y^iy^j},$$

where in the second step we used [5, (2.5.11)]. Similarly, for the Riemannian metric h we have

$$\frac{1}{2}\left({}^hG^i\right)_{y^jy^k} = {}^h\Gamma^i_{jk}. \tag{5.3}$$

Substituting (5.2) and (5.3) into the harmonic equation (5.1) gives

$$\tau^k_I = \frac{1}{2}g^{ij}\left({}^hG^k - {}^FG^k\right)_{y^iy^j}. \tag{5.4}$$

Thus we have the following result.

Proposition 5.1. *Let (M,h) be a flat Riemannian space. Then, for any local Minkowski structure F on M, the identity map*

$$I\colon (M,F) \to (M,h)$$

is harmonic.

Proof. By [12, (3.23)] we have

$${}^FG^j = \frac{1}{2}\sum_{i,k,l} g^{jl}\left[2\frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^l}\right]y^iy^k. \tag{5.5}$$

On the other hand, F is *local Minkowskian* if and only if

$$g_{ij}(x,y) = g_{ij}(y). \tag{5.6}$$

The conclusion is now immediate from (5.3)–(5.5). □

Definition 5.2. A Finsler manifold (M,F) is said to be of

(i) *Randers type* if $F = \alpha + \beta$, where α is a Riemannian metric on M and $\beta = \beta_i\,dx^i$ is a 1-form;
(ii) *Berwald type* if F has vanishing Minkowski curvature, i.e., if $P_{ijk\lambda} = 0$ for all i,j,k,λ (cf. [7]).

It is easy to see that a Randers manifold $(M,\alpha+\beta)$ is a Berwald manifold if and only if $\beta_{i|j} = 0$, where $\beta_{i|j}$ is the covariant derivative of β with respect to the Riemannian metric α, that is, if the 1-form β is parallel with respect to α. In this case, the Randers metric $\alpha+\beta$ and Riemannian metric α have same geodesic coefficients (cf. [5, 11.3.11]). Combining this with (5.4) we obtain:

PROPOSITION 5.3. *Let $(M, \alpha+\beta)$ be a Randers manifold. If β is parallel with respect to the Riemannian metric α, then the identity*

$$I\colon (M, \alpha+\beta) \to (M, \alpha)$$

is harmonic.

Antonelli, Ingarden, and Matsumoto [1] showed that Berwald manifolds which are neither Riemannian nor Minkowskian can be constructed using certain Randers metrics. In view of this, our results give examples of harmonic maps from Berwald manifolds which are neither Riemannian nor Minkowskian.

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