Illinois Journal of Mathematics
Volume 45, Number 4, Winter 2001, Pages 1315–1330
S 0019-2082

ON THE EXTENSION PROBLEM FOR CONTINUOUS POSITIVE DEFINITE GENERALIZED TOEPLITZ KERNELS

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ABSTRACT. The extension problem for positive definite generalized Toeplitz Kernels defined on the positive semi-axis is discussed, and criteria for the uniqueness of the extension are given. These results are applied to the generalized Nehari problem.

1. Introduction

Let $\mathcal{K}(t,s)$ be a matrix-valued kernel of the form

$$\mathcal{K}(t,s) = \begin{bmatrix} T_1(t-s) & \Gamma^*(t+s) \\ \Gamma(t+s) & T_2(t-s) \end{bmatrix}, \tag{1.1}$$

where T_i, $i = 1,2$, are matrix functions of orders $m \times m$ and $n \times n$, respectively, and Γ is a matrix function of order $n \times m$, and the arguments t, s belong to a set Δ. The set Δ can be the set $\mathbb{Z}_N^+$ ($N \leq \infty$) (discrete kernels) or an interval $[0, l)$ ($l \leq \infty$) of the real axis (continuous kernels). Such kernels are called generalized Toeplitz kernels (GTKs). It is easy to see that a kernel $\mathcal{K}(t,s)$ is a GTK if and only if it satisfies (in the sense of distributions) the identity

$$J\frac{\partial \mathcal{K}(t,s)}{\partial t} + \frac{\partial \mathcal{K}(t,s)}{\partial s} J = 0, \tag{1.2}$$

where

$$J = \begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix}. \tag{1.3}$$

The study of GTKs was initiated by M. Cotlar, and numerous papers by M. Cotlar and his collaborators, especially C. Sadosky and R. Arocena, are devoted to the study of GTKs and their applications; see, e.g., [CS1], [CS2], [CS3], [Ar1], [Ar2], [Sa], and the references in these papers.

In this paper we consider continuous positive definite GTKs, defined on the nonnegative semi-axis, so that the arguments t, s are in $\mathbb{R}_+$. A kernel

Received October 23, 2000; received in final form March 29, 2001.
2000 *Mathematics Subject Classification.* Primary 47A57, 42A70. Secondary 42A38, 30E05.

$\mathcal{K}$ is called positive definite if it satisfies, for any $t_\nu \in \mathbb{R}_+$, $\nu = 1, 2, \dots, M$, $M = 1, 2, \dots$, and any set of vectors $\xi_\nu, \nu = 1, 2, \dots, M$, of the appropriate dimension

$$\sum_{\mu,\nu=1}^{M} \xi_\mu^* \mathcal{K}(t_\mu, t_\nu)\xi_\nu \geq 0. \tag{1.4}$$

For continuous kernels, this condition is equivalent to the following condition: For any smooth vector function $f(t)$ of the appropriate dimension with compact support, one has

$$\int_0^\infty \int_0^\infty f^*(t)\mathcal{K}(t,s)f(s)dtds \geq 0. \tag{1.4'}$$

Positive definite GTKs arise naturally in problems of harmonic analysis (in connection with the Hilbert transform and the Theorem of Helson and Szege), in scattering theory, and in the theory of generalized stationary stochastic processes. Also, many classical interpolation problems can be reduced to the study of positive definite GTKs.

V. M. Adamjan, D. Z. Arov, and M. G. Krein, in their remarkable papers [AAK1][AAK2] on the Nehari problem and its generalization, studied a special class of discrete GTKs. In [AAK2] these authors remarked that some of the results and methods can also be applied to the case of continuous GTKs. Krein and F. E. Melik-Adamjan [KM-A1][KM-A2] considered the special class of continuous GTKs defined on the semi-axis. The author [Bek2] obtained some results on positive definite GTKs defined on semi-axis. Positive definite GTKs defined on a finite interval of the real axis were studied by the author [Bek3] and, via a different method, by R. Bruzual [Br]. Interesting results for kernels of this type were also obtained by L. A. Sakhnovich [Sak].

In Section 2 we prove that positive definite GTKs admit integral representations similar to the Bochner-Krein representation for positive definite matrix functions. In Section 3 we consider conditions which guarantee the uniqueness of the integral representation, and in Section 4 we give a parameterization of the set of integral representations. In Section 5 we apply these results to the generalized Nehari Problem, which can be formulated as follows:

Given an $m \times m$ matrix function $\varphi(t)$ from $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ find an $m \times m$ matrix function $h(t)$ from $H^1(\mathbb{R}) \cap H^\infty(\mathbb{R})$ such that $\|\varphi(t) - h(t)\| \leq \|\psi(t)\|$ for almost all $t \in \mathbb{R}$, where $\psi(t) \geq 0$ is another given $m \times m$ matrix function from $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

We show that this problem is equivalent to the extension problem for the corresponding positive definite GTKs.

2. Integral representations of positive definite GTKs

1. The following theorem was announced by V. Katznelson [Ka] and was proved by the author [Bek1] as a particular case of a more general statement. Its proof is particularly adapted to the study of positive definite GTKs, defined on the semi-axis.

THEOREM 1. *In order that a matrix-valued kernel* $\mathcal{K}(t,s), t,s \geq 0$, *admits an integral representation*

$$\mathcal{K}(t,s) = \int_{-\infty}^{\infty} \exp(-iJ\lambda t) d\Sigma(\lambda) \exp(iJ\lambda s), \tag{2.1}$$

where $\Sigma(\lambda)$ *is a monotone non-decreasing and bounded matrix function, it is necessary and sufficient that* $\mathcal{K}(t,s)$ *be a continuous positive definite generalized Toeplitz kernel.*

Proof. The necessity of the given condition can be proved easily. Indeed, from (2.1) it follows that $\mathcal{K}(t,s)$ has the structure of a GTK. The continuity of $\mathcal{K}(t,s)$ follows from the boundedness of the function $\Sigma(\lambda)$. For any smooth vector function $f(t) = \{f_\nu(t)\}_{\nu=1}^{m+n}, t \geq 0$, with compact support we obtain from (2.1)

$$\begin{aligned}&\int_0^\infty \int_0^\infty f^*(t)\mathcal{K}(t,s)f(s)dtds \\ &= \int_{-\infty}^{\infty} \left[\int_0^\infty \exp(iJ\lambda t)f(t)dt\right]^* d\Sigma(\lambda) \left[\int_0^\infty \exp(iJ\lambda s)f(s)ds\right] \geq 0,\end{aligned}$$

using the monotonicity of $\Sigma(\lambda)$. Therefore, by (1.4′), the kernel $\mathcal{K}$ is positive definite.

In order to prove the sufficiency of the condition, we use Krein's method of directing functionals [Kr1][Kr2][Kr3].

Let $\mathcal{L}$ be the set of all vector functions $f(t) = \{f_\nu(t)\}_{\nu=1}^{m+n} \in C^\infty$, $t \geq 0$, with compact support. We introduce in $\mathcal{L}$ a (possibly degenerate) scalar product $(\cdot,\cdot)_-$ by putting

$$(f,g)_- = \int_0^\infty \int_0^\infty g^*(t)\mathcal{K}(t,s)f(s)w(t)w(s)dtds, \tag{2.2}$$

where $w(t) = (1+t^2)^{-1}$. From the positive definiteness of the kernel $\mathcal{K}$ it follows that

$$(f,f)_- \geq 0.$$

We define a linear operator A in $\mathcal{L}$ by

$$(Af)(t) = iJ\frac{1}{w(t)}\frac{d}{dt}(w(t)f(t)). \tag{2.3}$$

The domain $\mathfrak{D}(A)$ of this operator consists of the functions $f(t) \in \mathcal{L}$ satisfying $f(0) = 0$. It is obvious that the set $\mathfrak{D}(A)$ is dense in $\mathcal{L}$ in the following sense: For any function $f \in \mathcal{L}$ there exists a sequence f_n, $n = 1, 2, \ldots$, with $f_n \in \mathfrak{D}(A)$, such that

$$(f - f_n, f - f_n)_- \to 0.$$

Using (1.2) it is easy to verify that the operator A is Hermitian, that is,

$$(Af, g)_- = (f, Ag)_-, \quad f, g \in \mathcal{D}(A). \tag{2.4}$$

Consider the map $\Phi : (f, \lambda) \to \Phi(f; \lambda)$ from $\mathcal{L} \times \mathbb{R}$ to $\mathbb{C}^{m+n}$, defined by

$$\Phi(f; \lambda) = \int_0^\infty \exp(iJ\lambda t) f(t) w(t) dt. \tag{2.5}$$

This map has the following properties:

(i) For any $\lambda \in \mathbb{R}$ the map $f \to \Phi(f; \lambda)$ is linear.

(ii) For any $f \in \mathcal{L}$ the map $\lambda \to \Phi(f; \lambda)$ is holomorphic on $\mathbb{R}$. (Actually, $\Phi(f; \lambda)$ is an entire function for any $f \in \mathcal{L}$.)

(iii) For $\lambda = 0$ the range of $\Phi(\cdot; 0)$ is the whole space $\mathbb{C}^{m+n}$.

(iv) The equality $\Phi(g; \lambda) = 0$ holds if and only if there exists a vector $f \in \mathfrak{D}(A)$ such that

$$Af - \lambda f = g. \tag{2.6}$$

These properties mean that $\Phi(f; \lambda)$ is a directing functional for the operator A (see [Kr1], [Kr2], [Kr3]). By Krein's theorem it follows that there exists a non-decreasing left continuous matrix function $\Sigma(\lambda)$, with $\Sigma(0) = 0$, such that for any $f, g \in \mathcal{L}$

$$(f, g)_- = \int_{-\infty}^\infty \Phi^*(g; \lambda) d\Sigma(\lambda) \Phi(f; \lambda). \tag{2.7a}$$

Writing out explicitly the left and right hand sides of this relation, we obtain

$$\int_0^\infty \int_0^\infty g^*(t) \mathcal{K}(t, s) f(s) w(t) w(s) dt ds \tag{2.7b}$$
$$= \int_{-\infty}^\infty \left[\int_0^\infty \exp(iJ\lambda t) g(t) w(t) dt \right]^* d\Sigma(\lambda) \left[\int_0^\infty \exp(iJ\lambda s) f(s) w(s) ds \right].$$

From the second relation it is not hard to derive (2.1). Since

$$\mathcal{K}(0, 0) = \int_{-\infty}^\infty d\Sigma(\lambda)$$

and $\mathcal{K}$ is a continuous kernel, the last integral is finite, which means that $\Sigma(\lambda)$ is a bounded matrix function. This completes the proof of Theorem 1. $\square$

Remark. The right-hand side of (2.1) defines a positive definite GTK which coincides with $\mathcal{K}$ on nonnegative values of the arguments. Therefore, from Theorem 1 it follows that a positive definite GTK, defined on $\mathbb{R}_+$, can be extended to the entire real axis, while preserving its structure and the positive definiteness.

3. Conditions guaranteeing uniqueness of integral representations of GTKs

Let $v(\mathcal{K})$ denote the set of all matrix functions $\Sigma(\lambda)$ of order $(m+n) \times (m+n)$ that have the integral representation (2.1). In the previous section we have shown that the set $v(\mathcal{K})$ is not empty. In this section we will obtain conditions under which the set $v(\mathcal{K})$ contains only one element, that is, conditions guaranteeing the uniqueness of the integral representation (2.1). In Section 4, a parameterization of the set $v(\mathcal{K})$ will be obtained, in the case these conditions are not fulfilled.

In our proofs we will use the techniques of equipped Hilbert spaces. Here we briefly state, without proofs, the facts we shall need; details can be found in the book by Yu. M. Berezansky [Ber, Chapters I and VIII].

1. Let $\mathfrak{h}_0$ denote the Hilbert space consisting of vector functions $f(t) = \{f_\nu(t)\}_{\nu=1}^{m+n}$, defined on the positive semi-axis, for which

$$\|f\|_0^2 = \sum_{\nu=1}^{m+n} \int_0^\infty |f_\nu(t)|^2 w(t) dt < \infty. \tag{3.1}$$

The scalar product in $\mathfrak{h}_0$ is

$$(f,g)_0 = \int_0^\infty (f(t), g(t)) w(t) dt = \sum_{\nu=1}^{m+n} \int_0^\infty f_\nu(t) \overline{g_\nu(t)} w(t) dt, \tag{3.2}$$

where $(\cdot,\cdot)$ denotes the scalar product in the $\mathbb{C}^{m+n}$.

The kernel $\mathcal{K}(t,s)$ generates a positive operator $\mathbf{K}$ in $\mathfrak{H}_0$ via the formula

$$\mathbf{K} f(t) = \int_0^\infty \mathcal{K}(t,s) f(s) w(s) ds. \tag{3.3}$$

Because the diagonal elements of the kernels $\mathcal{K}(t,t)$ take constant values and

$$\int_0^\infty \mathrm{tr}[\mathcal{K}(t,t)] w(t) dt < \infty,$$

the operator $\mathbf{K}$ is nuclear (belongs to the trace class) (see [GK, Chapter 3]), that is, $\mathbf{K}$ is a compact operator and its eigenvalues $\rho_k (\geq 0)$ satisfy

$$\sum_k \rho_k < \infty.$$

We will suppose that for any vector $f \in \mathfrak{H}_0$, $f \neq 0$, the condition

$$0 < (\mathbf{K} f, f)_0 \leq \|f\|_0^2$$

holds, i.e., that the operator $\mathbf{K}$ does not annihilate a non-zero vector.

The negative space $\mathfrak{H}_-$ is obtained by completing the space $\mathfrak{H}_0$ with respect to the norm generated by the scalar product

$$(3.4)\quad (f,g)_- = (\mathbf{K}f,g)_0 = \int_0^\infty \int_0^\infty g^*(t)\mathcal{K}(t,s)f(s)w(t)w(s)dtds.$$

The positive space $\mathfrak{H}_+$ is defined by completing the lineal $\mathcal{R}(\mathbf{K})$ (the range of the operator $\mathbf{K}$) with respect to the norm generated by the scalar product

$$(3.5)\qquad (u,v)_+ = (\mathbf{K}^{-1}u,v)_0, \qquad (u,v \in \mathcal{R}(\mathbf{K})).$$

These three spaces satisfy

$$\mathfrak{H}_+ \subset \mathfrak{H}_0 \subset \mathfrak{H}_-;$$

$\mathfrak{H}_+$ is dense in $\mathfrak{H}_0$ with respect to the norm $\|\cdot\|_0$, and $\mathfrak{H}_0$ is dense in $\mathfrak{H}_-$ with respect to the norm $\|\cdot\|_-$.

Since $\mathbf{K}$ is a compact operator, its eigenvectors h_j, $j = 1, 2, \dots$, form an orthonormal basis in the space $\mathfrak{H}_0$. The system $\{\rho_j^{-1/2}h_j\}$ forms an orthonormal basis in the space $\mathfrak{H}_-$, and the system $\{\rho_j^{1/2}h_j\}$ forms an orthonormal basis in the space $\mathfrak{H}_+$.

The operator $\hat{\mathbf{K}}$ is defined as the continuous extension of the operator $\mathbf{K}$ to the space $\mathcal{H}_-$. Its range is the whole space $\mathcal{H}_+$. Any vector $\omega \in \mathfrak{H}_-$ satisfies

$$\|\hat{\mathbf{K}}\omega\|_+ = \|\omega\|_-,$$

and for any vector $\omega \in \mathfrak{H}_-$ and any function $f \in \mathfrak{H}_0$ we have

$$(3.6)\qquad (\omega, f)_- = (\hat{\mathbf{K}}\omega, f)_0 = (\hat{\mathbf{K}}\omega, \hat{\mathbf{K}}f)_+.$$

The space $\mathfrak{H}_-$ contains δ-functions $\delta_\nu(t), \nu = 1, \dots, p+q$, such that

$$(3.7)\qquad (\delta_\nu(s), \delta_\mu(t))_- = K_{\mu\nu}(t,s),$$

and for any continuous vector function $f(s)$ a bilinear form is defined by

$$(3.8)\qquad (f, \delta_\nu(t))_0 = f_\nu(t).$$

2. In Section 2 an operator A was defined in order to prove Theorem 1. Let A_- be the Hermitian operator obtained by taking the closure of this operator in the space $\mathfrak{H}_-$, and let A_0 be the closure of A in the space gH_0. By Krein's theorem the integral representation (2.1) is unique if and only if A_- is a maximal (and, in particular, selfadjoint) operator. Therefore, we have to determine the defect numbers of the operator A_-.

LEMMA 1. *Suppose that the following conditions hold:*

(i) $m = n$;
(ii) $T_1(s) = T_2(-s)$;
(iii) $\Gamma^\tau(s) = \Gamma(s)$, *where* Γ^τ *denotes the transpose matrix.*

Then the defect numbers of the operator A_- *are equal.*

Proof. Consider the map $\mathbf{I}$: $\mathfrak{H}_- \to \mathfrak{H}_-$, defined on the dense lineal $\mathfrak{H}_0$ by the formula

$$\mathbf{I}\begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} = \begin{pmatrix} \bar{f}_2(s) \\ \bar{f}_1(s) \end{pmatrix},$$

where f_1 and f_2 are m-dimensional components of the vector $f \in \mathfrak{H}_0$ and the bar denotes complex conjugation. Evidently, we have $\mathbf{I}(\lambda f + \mu g) = \bar{\lambda}\mathbf{I}f + \bar{\mu}\mathbf{I}g$, $(\mathbf{I}f, \mathbf{I}g)_- = \overline{(f,g)_-}$, and $\mathbf{I}(\mathbf{I}f) = f$. Thus, $\mathbf{I}$ is an involution and can be extended to the space $\mathfrak{H}_-$.

Since the domain $\mathcal{D}(A)$ of the operator A is invariant under the involution $\mathbf{I}$, and $\mathbf{I}(Af) = A(\mathbf{I}f)$, the operator A is real with respect to this involution. Therefore, the defect numbers of the operator A_- are equal. □

In order to determine the defect numbers of the operator A_-, consider the equation

$$A_-^* \omega(t;z) - z\omega(t;z) = 0, \qquad \Im z \neq 0. \tag{3.9}$$

The defect number n_- is equal to the number of linearly independent solutions of this equation in the space $\mathfrak{H}_-$ for z belonging to the upper half-plane $\mathbb{C}_+$. Analogously, the defect number n_+ is equal to the number of linearly independent solutions of the equation (3.9) in the space $\mathfrak{H}_-$ for z belonging to the lower half-plane $\mathbb{C}_-$.

The operator A_-^* is given by

$$A_-^* = \hat{\mathbf{K}}^{-1} A_0^* \hat{\mathbf{K}}.$$

The domain $\mathcal{D}(A_0^*)$ of the operator A_0^* consists of all vector functions $g(s)$ which are absolutely continuous and satisfy $g' \in \mathfrak{H}_0$. The operator A_0^* acts via the formula

$$A_0^* g(s) = iJg'(s).$$

Therefore, equation (3.9) takes the form

$$iJ\frac{du}{dt} - zu = 0, \tag{3.10}$$

where $u = \hat{\mathbf{K}}\omega$. Equation (3.9) has a solution ω in the space $\mathfrak{H}_-$ if and only if equation (3.10) has a solution in the space $\mathfrak{H}_+$.

The linearly independent solutions of equation (3.10) are proportional to the columns of the matrix function

$$U(t;z) = \exp(-iztJ) = \begin{bmatrix} e^{-izt}I_m & 0 \\ 0 & e^{izt}I_n \end{bmatrix} = [u_1(t;z), \ldots, u_{m+n}(t;z)]. \tag{3.11}$$

If a vector $u_\nu(t;z)$ belongs to the space $\mathfrak{H}_+$, then there exists a vector $\omega_\nu \in \mathfrak{H}_-$ such that $\hat{\mathbf{K}}\omega_\nu = u_\nu(t;z)$, and the corresponding defect number is not equal to zero.

For $z \in \mathbb{C}_+$ and $\nu = 1, \ldots, m$,

$$\int_0^\infty (u_\nu(t;z), u_\nu(t;z))w(t)dt = \infty.$$

Hence, $u_\nu(t;z)$ is not in the space $\mathfrak{H}_0$, and therefore not in the space $\mathfrak{H}_+$. Thus, the defect number n_- satisfies $n_- \le n$. Similarly, for $z \in \mathbb{C}_-$ and $\nu = m+1, \ldots, n$, the vector $u_\nu(t;z)$ does not belong to the space $\mathfrak{H}_+$ and we therefore have $n_+ \le m$.

The map Φ from $\mathcal{L} \times \mathbb{R}$ to $\mathbb{C}^{m+n}$ was defined by (2.5). We extend the definition of this map to $\mathcal{L} \times \mathbb{C}$ by setting

$$\Phi(f;z) = \int_0^\infty \exp(iJzt)f(t)w(t)dt, \qquad f \in \mathcal{L}.$$

For any vector $f \in \mathcal{L}$, the function $\Phi(f;z) = \{\Phi_\nu(f;z)\}_{\nu=1}^{m+n}$ is an entire vector function. (Recall that $\mathcal{L}$ consists of functions with compact support.)

Theorem 2. *In order that the set $V(\mathcal{K})$ contain more than one element it is necessary that there exist numbers ν_- $(m+1 \le \nu_- \le m+n)$ and ν_+ $(1 \le \nu_+ \le m)$ and numbers C_z^- and C_z^+ depending only on z such that, for any $z \in \mathbb{C}_-$ or $z \in \mathbb{C}_+$, and any function $f \in \mathcal{H}_0$, the following inequalities hold:*

$$|\Phi_{\nu_-}(f;z)| \le C_z^- \|f\|_-, \qquad z \in \mathbb{C}_-, \tag{3.12a}$$

and

$$|\Phi_{\nu_+}(f;z)| \le C_z^+ \|f\|_-, \qquad z \in \mathbb{C}_+. \tag{3.12b}$$

In order that the set $V(\mathcal{K})$ contain more than one element it is sufficient that the inequalities (3.12a) *and* (3.12b) *hold for some $z_- \in \mathbb{C}_-$ and $z_+ \in \mathbb{C}_+$, respectively.*

Proof. Suppose, for instance, that $n_- \neq 0$. Then for any $z \in \mathbb{C}_-$ there exists ν_- $(m+1 \le \nu_- \le m+n)$ such that $u_{\nu_-}(t;\bar{z}) \in \mathfrak{H}_+$ and the corresponding vector $\omega_{\nu_-} = \hat{\mathbf{K}}^{-1}u_{\nu_-}$ belongs to the space $\mathfrak{H}_-$. By (2.5), $\Phi_{\nu_-}(f;z)$ can be written in the form

$$\Phi_{\nu_-}(f;z) = (f, \omega_{\nu_-})_-,$$

which means that $\Phi_{\nu_-}(\cdot;z)$ is a continuous linear functional on the space $\mathfrak{H}_-$. Thus (3.12a) holds. Similarly we see that if $n_+ \neq 0$ then (3.12b) holds.

To obtain the other direction, we reverse these arguments. □

Corollary 1. *In order that the set $V(\mathcal{K})$ contain only one element it is necessary that, for any $z \in \mathbb{C}_-$ (or $z \in \mathbb{C}_+$),*

$$\sum_{j=1}^\infty \frac{1}{\rho_j} |\Phi_\nu(h_j;z)|^2 = \infty, \tag{3.13}$$

where $\nu = m+1, \dots, m+n$ *if* $z \in \mathbb{C}_-$ *(or* $\nu = 1, \dots, m$*, if* $z \in \mathbb{C}_+$*),* $\{h_j\}_{j=1}^{\infty}$ *is the complete system of eigenvectors of the operator* $\mathbf{K}$ *in the space* $\mathcal{H}_0$*, and* ρ_j *are the corresponding eigenvalues. Conversely, if these conditions hold for some* $z \in \mathbb{C}_-$ *(or some* $z \in \mathbb{C}_+$*), then* $V(\mathcal{K})$ *contain only one element.*

4. Description of the set $\mathbf{V}(\mathcal{K})$

1. We will later consider the case when the integral representation (2.1) is not unique. Here we concentrate on the case $m = n$ which is the most important case for applications. In other words, we assume that T_1, T_2, and Γ are square matrix functions of order m, and that the defect index of the operator A_- is (m, m). By the results of the previous section this implies that the vectors $u_\nu(t; z)$, $\nu = m+1, \dots, 2m$ for $z \in \mathbb{C}_+$ and $\nu = 1, \dots, m$ for $z \in \mathbb{C}_-$, belong to the space $\mathfrak{H}_+$. (We will consider the general case in another paper.)

By Krein's theorem [Kr3] there is a one-to-one correspondence between the set $V(\mathcal{K})$ and the set of spectral functions $F(\lambda)$ of the operator A_-, given by the formula

$$(R(z)f, f)_- = \int_{-\infty}^{\infty} \frac{\Phi(f; z)^* d\Sigma(\lambda) \Phi(f; z)}{\lambda - z}, \tag{4.1}$$

where $R(z)$ is the resolvent of the operator A_- corresponding to the spectral function $F(\lambda)$. Thus, we need to describe the resolvents of the operator A_-.

Since we assume that the defect numbers of the operator A_- are equal, A_- admits selfadjoint extensions in the space $\mathfrak{H}_-$. Let $\hat{A}_-$ be such an extension, and let $\hat{R}(z) = (\hat{A}_- - zI)^{-1}$ ($\Im z \neq 0$) be the resolvent.

The set of generalized resolvents $R(z)$ of the operator A_- for z in the upper half-plane $\mathbb{C}_+$ ($\Im z > 0$) is described by the Krein-Saakjan formula [Saa]

$$\begin{aligned} R(z) = \hat{R}(z) - \hat{U}_{iz}(I_{\mathfrak{N}} - w_+(z)) \times \\ \times [2iI_{\mathfrak{N}} + (z-i)P_{\mathfrak{N}}\hat{U}_{-iz}(I_{\mathfrak{N}} - w_+(z))]^{-1} P_{\mathfrak{N}} \hat{U}_{-iz}. \end{aligned} \tag{4.2}$$

Here $\mathfrak{N} = \mathfrak{N}_i = [(A_- + iI)\mathfrak{D}(A_-)]^\perp$ is the defect subspace of the operator A_-; $P_{\mathfrak{N}}$ is the orthogonal projector onto this subspace; $\hat{U}_{\zeta z}$ is given by

$$\hat{U}_{\zeta z} = (\hat{A} - \zeta I)(\hat{A} - zI)^{-1} = I + (z - \zeta)\hat{R}(z);$$

$w_+(z)$ is a holomorphic in the upper half-plane satisfying $|w_+(z)| \leq 1$, $z \in \mathbb{C}_+$, i.e., $w_+(z)$ belongs to the unit ball of the space H^∞ in the upper half-plane. A short proof of this formula was given and partially published by the author [Bek3].

2. Let us introduce some notation. We will often write a vector function f with $2m$ components in the form

$$f = \begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix},$$

where $f^{(1)}$ and $f^{(2)}$ are vector functions with m components. For any vector function f with m components and any complex number z, we define

$$F_{\pm}(f;z) = \int_0^{\infty} \exp(\pm izt) f(t) w(t) dt, \tag{4.3}$$

provided that the integral makes sense.

Let Ω_+ be a $2m \times m$ matrix whose columns form a basis for the defect subspace $\mathfrak{N}_i$ in the space $\mathfrak{H}_-$. By (3.11) the matrix Ω_+ can be chosen such that

$$\hat{K}\Omega_+ = \begin{pmatrix} 0 \\ \exp(-t) I_m \end{pmatrix}.$$

Analogously, we define a matrix Ω_-, whose columns form a basis of the defect subspace $\mathfrak{N}_{-i}$, and which is of the form

$$\hat{K}\Omega_- = \begin{pmatrix} \exp(-t) I_m \\ 0 \end{pmatrix}.$$

Using an orthogonal basis $\{\rho_j^{-1/2} h_j(t)\}$ of the space $\mathfrak{H}_-$ and the above notations, we can write

$$\Omega_+ = \begin{pmatrix} \Omega_+^{(1)} \\ \Omega_+^{(2)} \end{pmatrix},$$

where

$$\Omega_+^{(k)} = \sum_j \frac{1}{\rho_j} h_j^{(k)} (F_+(h_j^{(2)}; i))^*, \qquad k = 1, 2. \tag{4.4}$$

In the same way, we have

$$\Omega_- = \begin{pmatrix} \Omega_-^{(1)} \\ \Omega_-^{(2)} \end{pmatrix},$$

where

$$\Omega_-^{(k)} = \sum_j \frac{1}{\rho_j} h_j^{(k)} (F_-(h_j^{(1)}; -i))^*, \qquad k = 1, 2. \tag{4.5}$$

3. We suppose that for $\mu = 1, 2, \ldots, m$ the μ-th columns of the matrices Ω_+ and Ω_- have equal norms in the space $\mathfrak{H}_-$. (If this assumption is not satisfied, an additional constant factor is needed in the formulas below.)

We use the selfadjoint extension $\hat{A}$ of the operator A defined as follows. The domain $\mathfrak{D}(\hat{A})$ consists of those vectors $\hat{f}$ which can be represented in the form $\hat{f} = f + (\Omega_+ + \Omega_-)X$, where $f \in \mathfrak{D}(A)$ and X is a column vector with m fixed components. The operator $\hat{A}$ is given by

$$\hat{A}\hat{f} = Af + i(\Omega_+ - \Omega_-).$$

Now, for $f \in \mathfrak{H}_0$ the resolvent $\hat{R}(z) = (\hat{A} - Iz)^{-1}$ is given by

$$\begin{aligned}(4.6)\quad \hat{R}(z)f(t) = &-\frac{iJ}{w(t)} \exp(-iJzt) \int_0^t \exp(iJz\tau) f(\tau) w(\tau) dt \\ &- \frac{iJ}{w(t)} \exp(-iJzt) \int_0^t \exp(iJz\tau) \times \\ &\times [(z-i)\Omega_+ + (z+i)\Omega_-]\, w(\tau) d\tau X \\ &+ (\Omega_+ + \Omega_-) X.\end{aligned}$$

The vector X is defined as

$$(4.7a)\qquad X = -[(z-i)F_+(\Omega_+^{(1)}; z) + (z+i)F_+(\Omega_-^{(1)}; z)]^{-1} F_+(f^{(1)}; z)$$

for $z \in \mathbb{C}_+$, and

$$(4.7b)\qquad X = -[(z-i)F_-(\Omega_+^{(2)}; z) + (z+i)F_-(\Omega_-^{(2)}; z)]^{-1} F_-(f^{(2)}; z)$$

for $z \in \mathbb{C}_-$, where the matrix $F_+(\Omega_+^{(1)}; z)$ is defined by term by term integration of (4.4), that is,

$$(4.8)\qquad F_+(\Omega_+^{(1)}; z) = \sum_j \frac{1}{\rho_j} F_+(h_j^{(1)}; z)(F_+(h_j^{(2)}; i))^*,$$

and the other matrices $F_\pm(\Omega_\pm^{(k)}; z)$ are defined similarly. From the assumption about defect numbers and Corollary 1 it follows that the infinite series converge for the corresponding non-real values of z. Moreover, since for any j the vector function $h_j(t)w(t)$ belongs to the space $L^2(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$, the functions $F_+(h_j^{(k)}(t); z)$, $k = 1, 2$, are analytic in the upper half-plane $\mathbb{C}_+$ and belong to $H^2(\mathbb{C}_+) \cap (W_+)$, where $H^2(\mathbb{C}_+)$ is the Hardy space in the upper half-plane and (W_+) is the Wiener class in the upper half-plane. Similarly, the functions $F_-(h_j^{(k)}(t); z)$, $k = 1, 2$, are analytic in the lower half-plane $\mathbb{C}_-$ and belong to $H^2(\mathbb{C}_-) \cap (W_-)$. (As usual, we identify a function from $H^p(\mathbb{C}_\pm)$ with its boundary function on the real axis.)

4. Let $\Sigma(\lambda) = (\Sigma_{lj}(\lambda)_{l,j=1,2})$ be a matrix function from the set $V(\mathcal{K})$ corresponding to the resolvent $R(z)$, and let $\hat{\Sigma}(\lambda) = (\hat{\Sigma}_{lj}(\lambda)_{l,j=1,2})$ be the matrix function from the set $V(\mathcal{K})$ corresponding to the "standard" resolvent $\hat{R}(z)$. By Bochner's theorem the blocks $\Sigma_{11}(\lambda)$ and $\Sigma_{22}(\lambda)$ are defined uniquely. Moreover, it follows from (2.1) that the blocks $\Sigma_{21}(\lambda)$ and $\hat{\Sigma}_{21}(\lambda)$ satisfy

$$\int_{-\infty}^{\infty} \exp(i\lambda t) d[\Sigma_{21}(\lambda) - \hat{\Sigma}_{21}(\lambda)] = 0, \qquad \lambda \geq 0.$$

Therefore, by F. and M. Riesz' Theorem, we have

$$d\Sigma_{21}(\lambda) - d\hat{\Sigma}_{21}(\lambda) = H(\lambda) d\lambda,$$

with an $m \times m$ matrix function $H(\lambda)$ belonging to the Hardy space $H^1(\mathbb{C}_+)$ in the upper half-plane $\mathbb{C}_+$. Thus we have

$$d\Sigma(\lambda) - d\hat{\Sigma}(\lambda) = \begin{pmatrix} 0 & H^*(\lambda) \\ H(\lambda) & 0 \end{pmatrix} d\lambda. \tag{4.9}$$

THEOREM 3. *There is a one-to-one correspondence between the set of $m \times m$ matrix functions $H(\lambda) \in H^1(\mathbb{C}_+)$ satisfying equation (4.9) and the set of $m \times m$ matrix functions $w_+(z)$ on the unit ball of the space $H^\infty(\mathbb{C}_+)$, given by*

(4.10)
$$H(\lambda) = -\frac{1}{\pi}\left[(\lambda+i)F_+(\Omega_+^{(2)*};\lambda) + (\lambda-i)F_+(\Omega_-^{(2)*};\lambda)\right]^{-1} F_-(\Omega_+^{(2)};-i)\times$$
$$\times\left[I - w_+(\lambda)\right]\left[(\lambda-i)F_+(\Omega_+^{(1)};\lambda)w_+(\lambda) + (\lambda+i)F_+(\Omega_-^{(1)};\lambda)\right]^{-1}.$$

Here $w_+(\lambda)$ are the boundary values of the function $w_+(z)$ on the real axis $\mathbb{R}$.

Proof. The result follows by a straightforward calculation using (4.2). □

5. The generalized Nehari problem

In this section we consider the following problem:

Let $\varphi(\lambda)$ be a given $m \times m$ matrix function from $L^1 \cap L^\infty$ on the real axis $\mathbb{R}$. Is it possible to approximate this function by an $m \times m$ matrix function $h(\lambda) \in H^1 \cap H^\infty$, such that the pointwise difference satisfies

$$|\varphi(\lambda) - h(\lambda)| \le \psi(\lambda) \tag{5.1}$$

almost everywhere, where $\psi(\lambda) \ge 0$ is another given non-negative matrix function from $L^1 \cap L^\infty$?

This problem is called the *generalized Nehari problem.*

Condition (5.1) is equivalent to the following condition: For almost all $\lambda \in \mathbb{R}$ the $2m \times 2m$ matrix

$$S(\lambda) = \begin{bmatrix} \psi(\lambda) & [\varphi(\lambda) - h(\lambda)]^* \\ \varphi(\lambda) - h(\lambda) & \psi(\lambda) \end{bmatrix} \tag{5.2}$$

is positive definite.

Consider the generalized Toeplitz kernel defined by (2.1) with $d\Sigma(\lambda) = S(\lambda)d\lambda$, i.e.,

$$\mathcal{K}(t,s) = \int_{-\infty}^{\infty} \exp(-iJ\lambda t) S(\lambda) \exp(iJ\lambda s) d\lambda.$$

Since $S(\lambda) \ge 0$, this kernel is positive definite. Since the elements of the matrix $S(\lambda)$ are bounded functions, it is easy to see that the kernel $\mathcal{K}(t,s)$ generates a bounded operator in the Hilbert space $L^2_{2m}(\mathbb{R})$. If t, s are restricted to the positive semi-axis $\mathbb{R}_+$, then the kernel $\mathcal{K}$ generates a bounded operator in the space $L^2_{2m}(\mathbb{R}_+)$.

Conversely, suppose that a positive definite GTK $\mathcal{K}(t,s)$ (see (1.1)) generates a bounded operator in the space $L^2_{m+n}(\mathbb{R}_+)$. From this assumption, Parseval's identity, and (2.1) it follows that for any function $f \in L^2_{m+n}(\mathbb{R}_+)$ and any matrix function $\Sigma(\lambda) \in V(\mathcal{K})$ one has

$$\int_{-\infty}^{\infty} \tilde{f}^*(\lambda) d\Sigma(\lambda) \tilde{f}(\lambda) \leq K \int_{-\infty}^{\infty} \|\tilde{f}(\lambda)\|^2 d\lambda, \tag{5.3}$$

where K is a positive constant and

$$\tilde{f}(\lambda) = \int_0^{\infty} \exp{(iJt)} f(t) dt.$$

Lemma 2. *Suppose the kernel $\mathcal{K}$ generates a bounded integral operator in the Hilbert space $L^2_{m+n}(\mathbb{R}_+)$ with norm not exceeding K. Then any matrix function $\Sigma(\lambda) \in V(\mathcal{K})$ is absolutely continuous with respect to the Lebesgue measure $d\Sigma(\lambda) = S(\lambda)d\lambda$, and its "density" $S(\lambda)$ is a nonnegative matrix function for almost all $\lambda \in \mathbb{R}$ and satisfies $\|S\|_\infty \leq K$.*

Remark. Since the matrix function $\Sigma(\lambda)$ is bounded, it is obvious that $S(\lambda) \in L^1$.

Proof. Since $\Sigma(\lambda)$ is a non-decreasing matrix function, its non-diagonal elements are absolutely continuous with respect to the diagonal elements, which are non-decreasing scalar functions. Therefore, it is enough to prove that any diagonal element of the matrix function $\Sigma(\lambda)$ is absolutely continuous with respect to the Lebesgue measure.

Noting that the Fourier transform of a function from $L^2(\mathbb{R}_+)$ is a function in the Hardy space $H^2(\mathbb{C}_+)$, and using Parseval's identity, we see that the lemma is equivalent to the following statement:

Let $\sigma(\lambda)$ be a non-decreasing and bounded function on the real axis, and suppose there exists a positive constant K such that, for any function $h(\lambda) \in H^2(\mathbb{C}_+)$, the following inequality holds:

$$\int_{-\infty}^{\infty} |h(\lambda)|^2 d\sigma(\lambda) \leq K \int_{-\infty}^{\infty} |h(\lambda)|^2 d\lambda. \tag{$*$}$$

Then $d\sigma(\lambda) = s(\lambda)d\lambda$, where the "density" $s(\lambda)$ belongs to $L^1 \cap L^\infty$ and satisfies $\|s\|_\infty \leq K$.

Since any function $h(\lambda)$ from H^2 does not vanish on a set of positive Lebesgue measure, this statement is not trivial.

For any function $h(\lambda) \in H^2$ and for any real t the function $\exp{(it\lambda)}h(\lambda)$ also belongs to the space H^2. Applying inequality $(*)$ to a function of the form $(\sum c_k \exp(it_k\lambda))h(\lambda)$, where c_k are arbitrary complex numbers, t_k are real numbers, and the sum is finite, we obtain

$$\sum c_k \bar{c}_l \int_{-\infty}^{\infty} e^{i(t_k-t_l)\lambda} |h(\lambda)|^2 (K d\lambda - d\sigma(\lambda)) \geq 0.$$

Hence the function

$$\gamma(t) = \int_{-\infty}^{\infty} e^{it\lambda} |h(\lambda)|^2 (K d\lambda - d\sigma(\lambda)), \qquad t \in \mathbb{R},$$

is a continuous positive definite function. By Bochner's theorem, there exists a non-decreasing and bounded function $\rho(\lambda)$ on the real axis such that

$$\gamma(t) = \int_{-\infty}^{\infty} e^{it\lambda} d\rho(\lambda).$$

By the uniqueness theorem for Fourier transforms it follows that

$$K|h(\lambda)|^2 d\lambda = |h(\lambda)|^2 d\sigma(\lambda) + d\rho(\lambda).$$

Since the measures $|h(\lambda)|^2 d\sigma(\lambda)$ and $d\rho(\lambda)$ are both positive, the above statement follows, and the proof of the lemma is complete. □

From the lemma we obtain the following corollary.

COROLLARY 2. *Let $\mathcal{K}(t,s)$ be an $(m+n) \times (m+n)$ positive definite generalized Toeplitz kernel which generates a bounded operator in the space $L^2_{m+n}(\mathbb{R}_+)$. Then any extension of this kernel which preserves its structure and the positive definiteness generates a bounded operator in the space $L^2_{m+n}(\mathbb{R})$ with the same norm.*

Let

$$S(\lambda) = \begin{bmatrix} \psi_1(\lambda) & \varphi^*(\lambda) \\ \varphi^*(\lambda) & \psi_2(\lambda) \end{bmatrix} \geq 0$$

be a "density" matrix. Then $\psi_1(\lambda)$ is the symbol of the Toeplitz operator $\mathbf{T}_{\psi_1}$, which is the integral operator acting on the space $L^2_m(\mathbb{R}_+)$ by the formula

$$(\mathbf{T}_{\psi_1} f)(t) = \int_0^{\infty} T_1(t-s) f(s) ds. \tag{5.4}$$

Similarly, $\tilde{\psi}_2(\lambda) = \psi_2(-\lambda)$ is the symbol of the Toeplitz operator $\mathbf{T}_{\tilde{\psi}_2}$, which acts on the space $L^2_n(\mathbb{R}_+)$ and has kernel function T_2. The function $\varphi(\lambda)$ is one of the possible symbols of the Hankel operator $\mathbf{\Gamma}_\phi$, which maps $L^2_m(\mathbb{R}_+)$ to $L^2_n(\mathbb{R}_+)$ by the formula

$$(\mathbf{\Gamma}_\phi f)(t) = \int_0^{\infty} \Gamma(t+s) f(s) ds. \tag{5.5}$$

The condition for the positive definiteness of the kernel $\mathcal{K}$ can be formulated as follows:

$$\|\Gamma_\phi\|^2 \leq \|T_{\psi_1}\| \|T_{\psi_2}\|. \tag{5.6}$$

This leads to the following theorem.

THEOREM 4.

(1) *The generalized Nehari problem is solvable if and only if the norm of the Hankel operator* Γ_ϕ *with symbol* ϕ *is not greater than the norm of the Toeplitz operator* T_ψ *with symbol* ψ*, i.e., if* $\|\Gamma_\phi\| \leq \|T_\psi\|$.

(2) *The problem has a unique solution if and only if the corresponding GTK* $\mathcal{K}(t, s)$ *possesses a unique extension to the real axis* $\mathbb{R}$.

(3) *If the solution of the generalized Nehari problem is not unique, then a description of the solution set is given by* (4.10).

REMARK. After submitting this article, the author learned that W. Helton [Hel] solved the generalized Nehari problem for the case of rational functions $\phi(\lambda)$ and $\psi(\lambda)$.

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