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A SPECTRAL MULTIPLIER THEOREM FOR H^1 SPACES ASSOCIATED WITH SCHRÖDINGER OPERATORS WITH POTENTIALS SATISFYING A REVERSE HÖLDER INEQUALITY

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ABSTRACT. Let $\{T_t\}_{t>0}$ be the semigroup of linear operators generated by a Schrödinger operator $-A = \Delta - V$ on $\mathbb{R}^d$, where V is a nonnegative nonzero potential satisfying a reverse Hölder inequality, and let $\int_0^\infty \lambda \, dE_A(\lambda)$ be the spectral resolution of A. We say that a function f is an element of H^1_A if the maximal function $\mathcal{M}f(x) = \sup_{t>0} |T_t f(x)|$ belongs to L^1. We prove that if a function F satisfies a Mihlin condition with exponent $\alpha > d/2$ then the operator $F(A) = \int_0^\infty F(\lambda)\, dE_A(\lambda)$ is bounded on H^1_A.

1. Introduction

Let $A = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^d$, $d \geq 3$, where V is a nonnegative nonzero potential that satisfies the reverse Hölder inequality with exponent $q \geq d/2$; that is, there exists a constant C_0 such that for every ball $B(x,r)$

$$\left(\frac{1}{|B(x,r)|} \int_{B(x,r)} V(y)^q \, dy \right)^{1/q} \leq \frac{C_0}{|B(x,r)|} \int_{B(x,r)} V(y)\, dy. \tag{1.1}$$

Let $\{T_t\}_{t>0}$ be the semigroup of linear operators generated by $-A$, and let $T_t(x,y)$ denote the integral kernels of these operators. Since $V(x) \geq 0$ and $V \in L^q_{\mathrm{loc}}(\mathbb{R}^d)$,

$$0 \leq T_t(x,y) \leq (4\pi t)^{-d/2} \exp\left(-|x-y|^2/(4t)\right). \tag{1.2}$$

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We say that a function f is in the space H^1_A if the maximal function

$$\mathcal{M}f(x) = \sup_{t>0} |T_t f(x)| = \sup_{t>0} \left| \int_{\mathbb{R}^d} T_t(x,y) f(y)\, dy \right|$$

belongs to $L^1(\mathbb{R}^d)$, and we set

$$\|f\|_{H^1_A} = \|\mathcal{M}f\|_{L^1(\mathbb{R}^d)}. \tag{1.3}$$

Let $\int_0^\infty \lambda\, dE_A(\lambda)$ be the spectral resolution for the operator A. For a bounded function F on $\mathbb{R}_+$ we define the operator $F(A)$ by setting

$$F(A)f = \int_0^\infty F(\lambda)\, dE_A(\lambda) f.$$

For $s \geq 0$ let $C(s)$ denote the space of functions F on $\mathbb{R}$ for which

$$\|F\|_{C(s)} = \begin{cases} \sum_{k=0}^{s} \sup_{\lambda\in\mathbb{R}} \left|F^{(k)}(\lambda)\right| & \text{if } s \in \mathbb{Z}, \\ \left\|F^{([s])}\right\|_{\mathrm{Lip}\,(s-[s])} + \sum_{k=0}^{[s]} \sup_{\lambda\in\mathbb{R}} \left|F^{(k)}(\lambda)\right| & \text{if } s \notin \mathbb{Z}, \end{cases}$$

is finite.

Our goal is to prove the following theorem.

THEOREM 1.4. *Let F be a bounded continuous function on $\mathbb{R}_+$. If for some $\alpha > d/2$ and some nonzero function $\psi \in C_c^\infty(0,\infty)$ there exists a constant $C > 0$ such that*

$$\|\psi(\cdot)F(t\cdot)\|_{C(\alpha)} \leq C \quad \textit{for every } t > 0, \tag{1.5}$$

then $F(A)$ is a bounded operator on H^1_A.

We recall that the classical Hörmander multiplier theorem [13] applied to the Laplace operator $-\Delta$ on $\mathbb{R}^d$ implies that if for some $\alpha > d/2$ a bounded continuous function F defined on $\mathbb{R}^+$ satisfies

$$\sup_{t>0} \|\psi(\cdot)F(t\cdot)\|_{H(\alpha)} < \infty,$$

where $\|\cdot\|_{H(\alpha)}$ is the Sobolev norm, then the operator $F(-\Delta)$ is of weak type $(1,1)$ and bounded on $L^p(\mathbb{R}^d)$, $1 < p < \infty$. Moreover, $F(-\Delta)$ is bounded on the classical Hardy space $H^1(\mathbb{R}^d)$.

The spaces H^1_A we consider in the present paper are substantially larger than the classical Hardy spaces. It was proved in [7] that every element f of H^1_A can be decomposed into a sum of special atoms, defined as follows. Let $m(x,V)$ be given by

$$m(x,V)^{-1} = \sup\left\{ r > 0 : \frac{1}{r^{d-2}} \int_{B(x,r)} V(y)\, dy \leq 1 \right\}. \tag{1.6}$$

Then $\mathbb{R}^d = \bigcup_{n\in\mathbb{Z}} \mathcal{B}_n$, where

$$\mathcal{B}_n = \left\{ x : 2^{n/2} \leq m(x,V) < 2^{(n+1)/2} \right\}.$$

We say that a function a is an H^1_A atom associated to a ball $B(y_0, r)$ of center $y_0 \in \mathbb{R}^d$ and radius $r > 0$, if

$$\operatorname{supp} a \subset B(y_0, r), \tag{1.7}$$

$$\|a\|_{L^\infty} \le |B(y_0, r)|^{-1}, \tag{1.8}$$

$$\text{if } y_0 \in \mathcal{B}_n, \text{ then } r \le 2^{1-n/2}, \tag{1.9}$$

$$\text{if } y_0 \in \mathcal{B}_n \text{ and } r \le 2^{-1-n/2}, \text{ then } \int a(x)\, dx = 0. \tag{1.10}$$

The atomic norm in the space H^1_A is defined by

$$\|f\|_{H^1_A\text{atom}} = \inf\left\{\sum |c_j|\right\}, \tag{1.11}$$

where the infimum is taken over all decompositions $f = \sum_j c_j a_j$, where the a_j are H^1_A atoms and the c_j are scalars. It was proved in [7] that the norm $\| \ \|^p_{H^1_A\text{atom}}$ is equivalent to the norm $\| \ \|_{H^1_A}$; that is, there exists a constant $C > 0$ such that

$$\frac{1}{C}\|f\|_{H^1_A} \le \|f\|_{H^1_A\text{atom}} \le C\|f\|_{H^1_A}. \tag{1.12}$$

Our H^1_A atoms are scaled $(1, \infty)$ atoms from the local Hardy space $\mathbf{h}^1$, where the scale and localization is adapted to the behavior of the potential V. The atoms are supported on balls, satisfy the size condition (1.8), but for some of them the mean zero condition is not needed. Therefore, in order to show the boundedness of the operator $F(A)$ on such atoms, we derive appropriate estimates for kernels associated with the multiplier F (see Theorem 3.8). In the case where V is a nonnegative polynomial, similar results were obtained in [5] by using nilpotent Lie group methods. We recall that every nonnegative polynomial V satisfies (1.1) for all q, $1 < q < \infty$.

The function $m(x, V)$ was introduced by Shen [16]. In the next section we state some properties of the function which we will use in this paper.

The problem of finding sufficient conditions on a function F that guarantee the boundedness of the operator $F(L)$ on $L^p(\mathfrak{M})$, where L is a positive self-adjoint operator (on $L^2(\mathfrak{M})$), has been investigated by many authors (cf. [1], [2], [3], [10], [12], [14], [15], [17], [18]). E. Stein [17] showed that if $-L$ is the infinitesimal generator of a symmetric diffusion semigroup and F is of Laplace transform type, then $F(L)$ is bounded on L^p, $1 < p < \infty$. In the case where L is a sublaplacian on a stratified nilpotent Lie group, spectral multiplier theorems where proved, for example, by A. Hulanicki and E. Stein (see [8, p. 208]), M. Christ [1], G. Mauceri and S. Meda [14], D. Müller and E. Stein [15], and W. Hebisch [11]. We also refer the reader to the papers [10] and [18], where multipliers on L^p spaces in the case when L is a Schrödinger operator were considered.

2. Auxiliary lemmas

For $t > 0$ and $V \geq 0$ satisfying (1.1) we define the Schrödinger operator $A^{\{t\}}$ by setting

$$A^{\{t\}} = -\Delta + V^{\{t\}}, \quad V^{\{t\}}(x) = tV\left(t^{1/2}x\right). \tag{2.1}$$

Obviously, for every $t > 0$,

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V^{\{t\}}(y)^q\, dy\right)^{1/q} \leq \frac{C_0}{|B(x,r)|}\int_{B(x,r)} V^{\{t\}}(y)\, dy, \tag{2.2}$$

with a constant C_0 independent of $t > 0$. Moreover,

$$m\left(t^{-1/2}x, V^{\{t\}}\right) = t^{1/2}m(x,V). \tag{2.3}$$

The following lemma is a simple consequence of (2.3) and a result of Shen [16, Lemma 1.4].

Lemma 2.4. *There exist constants $C,\ k_0 > 0$ such that for every $t > 0$*

$$C^{-1}m\left(x,V^{\{t\}}\right) \leq m\left(y,V^{\{t\}}\right) \leq Cm\left(x,V^{\{t\}}\right) \qquad \text{if } |x-y| \leq \frac{4}{m\left(x,V^{\{t\}}\right)}, \tag{2.5}$$

$$m\left(y,V^{\{t\}}\right) \leq C\left(1+|x-y|m\left(x,V^{\{t\}}\right)\right)^{k_0} m\left(x,V^{\{t\}}\right), \tag{2.6}$$

$$m\left(y,V^{\{t\}}\right) \geq \frac{m\left(x,V^{\{t\}}\right)}{C\left(1+|x-y|m(x,V^{\{t\}})\right)^{k_0/(k_0+1)}}. \tag{2.7}$$

We shall denote by $\Gamma^{\{t\}}(x,y,\tau)$ the fundamental solution of the operator $-\Delta + V^{\{t\}} + i\tau$, $\tau \in \mathbb{R}$.

Applying (2.2) and two results of Shen ([16, Theorem 2.7] and [16, p. 535]), we obtain the following result.

Proposition 2.8. *For every $k > 0$ there exists a constant C_k such that for every $t > 0$*

$$\left|\Gamma^{\{t\}}(x,y,\tau)\right| \leq C_k(1+|\tau|^{1/2}|x-y|)^{-k}\left(1+m\left(x,V^{\{t\}}\right)|x-y|\right)^{-k}|x-y|^{2-d}. \tag{2.9}$$

Moreover, there exist constants $C, \delta > 0$ such that for every $t > 0$

$$\left|\Gamma^{\{t\}}(x,y+h,\tau) - \Gamma^{\{t\}}(x,y,\tau)\right| \leq \frac{C|h|^\delta}{\left(1+|\tau|^{1/2}|x-y|\right)^3|x-y|^{d-2+\delta}} \tag{2.10}$$

for $|h| \leq |x-y|/4$.

THEOREM 2.11. *For every $l \in \mathbb{N}$ there exists a constant C_l such that for every $t > 0$*

$$\left| m\left(x, V^{\{t\}}\right)^{2l} \left(-\Delta + V^{\{t\}}\right)^{-l} f(x) \right| \leq C_l \mathbf{M}^l f(x), \tag{2.12}$$

where $\mathbf{M}$ is the classical Hardy–Littlewood maximal operator.

Proof. The proof is by induction on l (cf. [19]). Let

$$\Gamma^{\{t\}}(x, y) = \Gamma^{\{t\}}(x, y, 0).$$

Fix $x \in \mathbb{R}^d$, and set $r = m(x, V^{\{t\}})^{-1}$. Applying (2.9) with $k = 3$, we obtain

$$\begin{aligned}
\left| \left(-\Delta + V^{\{t\}}\right)^{-1} f(x) \right| &= \left| \int \Gamma^{\{t\}}(x, y) f(y)\, dy \right| \\
&\leq C \int_{B(x,r)} |x - y|^{2-d} |f(y)|\, dy \\
&\quad + C \int_{B(x,r)^c} |x - y|^{2-d-3} m\left(x, V^{\{t\}}\right)^{-3} |f(y)|\, dy \\
&\leq C m\left(x, V^{\{t\}}\right)^{-2} \mathbf{M} f(x),
\end{aligned}$$

which proves (2.12) for $l = 1$.

Assume now that $m\left(y, V^{\{t\}}\right)^{2l} \left(-\Delta + V^{\{t\}}\right)^{-l} f(y) \leq C_l \mathbf{M}^l f(y)$. We have

$$\begin{aligned}
\left| \left(-\Delta + V^{\{t\}}\right)^{-l-1} f(x) \right| &= \left| \int \Gamma^{\{t\}}(x, y) \left(-\Delta + V^{\{t\}}\right)^{-l} f(y)\, dy \right| \\
&\leq \int C_l \left| \Gamma^{\{t\}}(x, y) \right| m\left(y, V^{\{t\}}\right)^{-2l} \mathbf{M}^l f(y)\, dy \\
&= \int_{B(x,r)} + \int_{B(x,r)^c} = I_1 + I_2.
\end{aligned}$$

Using (2.5) and (2.9), we get

$$\begin{aligned}
I_1 &\leq C \int_{B(x,r)} |x - y|^{2-d} m\left(x, V^{\{t\}}\right)^{-2l} \mathbf{M}^l f(y)\, dy \\
&\leq C m\left(x, V^{\{t\}}\right)^{-2l-2} \mathbf{M}^{l+1} f(x).
\end{aligned}$$

Applying (2.7) and (2.9) (with $k = 3 + 2lk_0/(k_0 + 1)$), we obtain

$$\begin{aligned}
I_2 &\leq C \int_{B(x,r)^c} |x - y|^{2-d-k} m\left(x, V^{\{t\}}\right)^{-k} m\left(y, V^{\{t\}}\right)^{-2l} \mathbf{M}^l f(y)\, dy \\
&\leq C \int_{B(x,r)^c} |x - y|^{2-d-k+2lk_0/(k_0+1)} m\left(x, V^{\{t\}}\right)^{-k-2l+2lk_0/(k_0+1)} \mathbf{M}^l f(y)\, dy \\
&\leq C m\left(x, V^{\{t\}}\right)^{-2l-2} \mathbf{M}^{l+1} f(x).
\end{aligned}$$

□

3. Estimates of kernels

Let $\int_0^\infty \lambda \, dE_{A^{\{t\}}}(\lambda)$ be the spectral resolution for $A^{\{t\}}$. Obviously, if for a bounded continuous function G on $\mathbb{R}_+$ the operator

$$G\left(A^{\{t\}}\right) = \int_0^\infty G(\lambda)\, dE_{A^{\{t\}}}(\lambda)$$

has an integral kernel $G(A^{\{t\}})(x,y)$, that is, if

$$G\left(A^{\{t\}}\right) f(x) = \int_{\mathbb{R}^d} G\left(A^{\{t\}}\right)(x,y) f(y)\, dy,$$

then the integral kernel $G(tA)(x,y)$ of the operator $G(tA)$ is given by

$$G(tA)(x,y) = t^{-d/2} G\left(A^{\{t\}}\right)\left(t^{-1/2}x, t^{-1/2}y\right). \tag{3.1}$$

We denote by $T_s^{\{t\}}(x,y)$ the integral kernels of the semigroup $\{T_s^{\{t\}}\}_{s>0}$ generated by the Schrödinger operator $-A^{\{t\}}$.

For an integral kernel $K(x,y)$ and $b>0$ we define

$$\|K(x,y)\|_{\omega(b)} = \sup_{x\in\mathbb{R}^d} \int |K(x,y)| \left(1+|x-y|\right)^b \, dy + \sup_{y\in\mathbb{R}^d} \int |K(x,y)| \left(1+|x-y|\right)^b \, dx.$$

The following theorem is a consequence of (1.2) and results of W. Hebisch [9, Theorem 2.10].

THEOREM 3.2. *Given $b, s>0$ with $s > b + d/2$ there exists a constant $C = C(b,s,d)$ such that for every function $\xi \in C(s)$ with $\operatorname{supp}\xi \subset (1/4, 4)$ and for every $t>0$ we have*

$$\left\| \xi\left(A^{\{t\}}\right)(x,y) \right\|_{\omega(b)} \le C\|\xi\|_{C(s)}. \tag{3.3}$$

COROLLARY 3.4. *For every $M>0$ there exist constants $C, s>0$ such that for every $\xi \in C(s)$ with $\operatorname{supp}\xi \subset (1/4,4)$ and for every $t>0$ we have*

$$\left| \xi\left(A^{\{t\}}\right)(x,y) \right| \le C\|\xi\|_{C(s)} \left(1+|x-y|\right)^{-M}. \tag{3.5}$$

Proof. Set $\eta(\lambda) = e^{\lambda}\xi(\lambda)$. Obviously $\operatorname{supp}\eta \subset (1/4,4)$ and $\|\eta\|_{C(s)} \le C_s\|\xi\|_{C(s)}$. Since $\xi(A^{\{t\}}) = T_1^{\{t\}} \circ \eta(A^{\{t\}})$, we have

$$\begin{aligned}(1+|x-y|)^M &\left|\xi\left(A^{\{t\}}\right)(x,y)\right| \\ &\le \left| \int T_1^{\{t\}}(x,z)\left(1+|x-z|\right)^M \eta\left(A^{\{t\}}\right)(z,y)\left(1+|z-y|\right)^M dz \right|.\end{aligned}$$

Applying Theorem 3.2 with $b = M$ and using (1.2) we obtain that there exist constants s and C' such that

$$(1+|x-y|)^M \left|\xi\left(A^{\{t\}}\right)(x,y)\right| \le C'\|\eta\|_{C(s)} \le C'C_s\|\xi\|_{C(s)}. \qquad \square$$

We fix a real valued function $\psi \in C_c^\infty(1/2,\, 2)$ such that

$$\sum_{\mu\in\mathbb{Z}} \psi^2\left(2^{-\mu}\lambda\right) = 1 \quad \text{for } \lambda > 0. \tag{3.6}$$

Let F be a continuous function on $\mathbb{R}_+$ that satisfies (1.5). We set

$$\begin{aligned} Q_{\mu,t}(\lambda) &= F(\lambda)e^{-t\lambda}\psi^2\left(2^{-\mu}\lambda\right), \\ \tilde{Q}_{\mu,t}(\lambda) &= Q_{\mu,t}\left(2^\mu\lambda\right) = F\left(2^\mu\lambda\right)e^{-t2^\mu\lambda}\psi^2(\lambda). \end{aligned}$$

It follows from (3.1) that the integral kernels $Q_{\mu,t}(x,y)$ of the operators $Q_{\mu,t}(A)$ satisfy

$$Q_{\mu,t}(x,y) = 2^{\mu d/2}\tilde{Q}_{\mu,t}\left(A^{\{2^{-\mu}\}}\right)\left(2^{\mu/2}x, 2^{\mu/2}y\right). \tag{3.7}$$

Our goal in this section is to prove the following two theorems.

THEOREM 3.8. *Assume that F satisfies* (1.5). *Then there exists a constant $\delta > 0$ and a family of kernels $K_\mu(x,y) \ge 0$, $\mu \in \mathbb{Z}$, with $\|K_\mu(x,y)\|_{\omega(0)} \le 1$, such that for every $M > 0$ there exists a constant C_M such that*

$$\begin{aligned} |Q_{\mu,t}(x,y)| \le C_M K_\mu(x,y)\left(1+2^{\mu/2}|x-y|\right)^{-\delta} \times \\ \times\left(1+2^{-\mu/2}m(x,V)\right)^{-M}\left(1+2^{-\mu/2}m(y,V)\right)^{-M}. \end{aligned} \tag{3.9}$$

THEOREM 3.10. *There exist constants $C, \varepsilon > 0$ such that for every $\mu \in \mathbb{Z}$*

$$\int_{\mathbb{R}^d} \sup_{t>0} \left|Q_{\mu,t}(x,y) - Q_{\mu,t}(x,y_0)\right| dx \le C2^{\varepsilon\mu/2}\left|y-y_0\right|^\varepsilon. \tag{3.11}$$

Proof of Theorem 3.8. We set, for $t > 0$ and $\mu \in \mathbb{Z}$,

$$\begin{aligned} \varphi_{\mu,t}(\lambda) &= e^{-t\lambda}\psi\left(2^{-\mu}\lambda\right), & \theta_\mu(\lambda) &= \psi\left(2^{-\mu}\lambda\right)F(\lambda), \\ \tilde{\varphi}_{\mu,t}(\lambda) &= \varphi_{\mu,t}\left(2^\mu\lambda\right) = e^{-t2^\mu\lambda}\psi(\lambda), & \tilde{\theta}_\mu(\lambda) &= \theta_\mu\left(2^\mu\lambda\right) = \psi(\lambda)F\left(2^\mu\lambda\right). \end{aligned}$$

Since for every $k > 0$ there exists a constant C_k independent of t and μ such that $\|\tilde{\varphi}_{\mu,t}\|_{C^k(1/4,\,4)} \le C_k$, Corollary 3.4 asserts that for every $M > 0$ there is a constant C_M such that for every $\mu \in \mathbb{Z}$ and $t > 0$ we have

$$\left|\tilde{\varphi}_{\mu,t}\left(A^{\{2^{-\mu}\}}\right)(x,y)\right| \le C_M\left(1+|x-y|\right)^{-M}. \tag{3.12}$$

Moreover, by Theorem 3.2 there exist constants $C, \delta > 0$ such that

$$\left\|\tilde{\theta}_\mu\left(A^{\{2^{-\mu}\}}\right)(x,y)\right\|_{\omega(\delta)} \le C. \tag{3.13}$$

Since

$$\tilde{Q}_{\mu,t}\left(A^{\{2^{-\mu}\}}\right)(x,y) = \int \tilde{\varphi}_{\mu,t}\left(A^{\{2^{-\mu}\}}\right)(x,z)\tilde{\theta}_{\mu}\left(A^{\{2^{-\mu}\}}\right)(z,y)\,dz,$$

it follows that there is a constant $C > 0$ independent of μ such that

$$\left\| \left\{ \sup_{t>0} \left| \tilde{Q}_{\mu,t}\left(A^{\{2^{-\mu}\}}\right)(x,y) \right| \right\} \right\|_{\omega(\delta)} \leq C. \tag{3.14}$$

Let $\zeta \in C_c^\infty(1/4, 4)$ be such that $\zeta(\lambda) = 1$ for $\lambda \in [1/2, 2]$. Fix l (large) and set $\xi(\lambda) = \lambda^l \zeta(\lambda)$. Obviously, by Corollary 3.4,

$$\left| \xi\left(A^{\{2^{-\mu}\}}\right)(x,y) \right| \leq C_l \left(1 + |x-y|\right)^{-d-1}, \tag{3.15}$$

with C_l independent of μ. Moreover,

$$\zeta\left(A^{\{2^{-\mu}\}}\right)(x,y) = \left(\left(A^{\{2^{-\mu}\}}\right)^{-l} \xi\left(A^{\{2^{-\mu}\}}\right) \right)(x,y). \tag{3.16}$$

For fixed $y \in \mathbb{R}^d$ we put $h(x) = \xi(A^{\{2^{-\mu}\}})(x,y)$. Therefore, by (3.15), (3.16), and Theorem 2.11, we obtain

$$\left| m\left(x, V^{\{2^{-\mu}\}}\right)^{2l} \zeta\left(A^{\{2^{-\mu}\}}\right)(x,y) \right| \leq C_l \mathbf{M}^l h(x) \leq C_l', \tag{3.17}$$

where C_l' does not depend on μ, x, and y.

On the other hand, for every $N > 0$ there exists a constant C_N such that

$$\left| \zeta\left(A^{\{2^{-\mu}\}}\right)(x,y) \right| \leq C_N \left(1 + |x-y|\right)^{-2N},$$

which combined with (3.17) gives

$$\left| \zeta\left(A^{\{2^{-\mu}\}}\right)(x,y) \right| \leq C_{l,N} \left(1 + m\left(x, V^{\{2^{-\mu}\}}\right)\right)^{-l} \left(1 + |x-y|\right)^{-N}. \tag{3.18}$$

Obviously,

$$\tilde{Q}_{\mu,t}\left(A^{\{2^{-\mu}\}}\right) = \zeta\left(A^{\{2^{-\mu}\}}\right) \tilde{Q}_{\mu,t}\left(A^{\{2^{-\mu}\}}\right).$$

Therefore, applying (3.18) we obtain

$$\begin{aligned} &(1 + |x-y|)^\delta \left| \tilde{Q}_{\mu,t}\left(A^{\{2^{-\mu}\}}\right)(x,y) \right| \\ &\leq C_l \left(1 + m\left(x, V^{\{2^{-\mu}\}}\right)\right)^{-l} \int (1 + |x-z|)^{-N} \left\{ \sup_{t>0} \left| \tilde{Q}_{\mu,t}\left(A^{\{2^{-\mu}\}}\right)(z,y) \right| \right\} \times \\ &\qquad \times (1 + |x-z|)^\delta (1 + |z-y|)^\delta \, dz. \end{aligned} \tag{3.19}$$

Setting

$$\begin{aligned} &\tilde{K}_\mu'(x,y) \\ &\quad = \int (1 + |x-z|)^{-N+\delta} \left\{ \sup_{t>0} \left| \tilde{Q}_{\mu,t}\left(A^{\{2^{-\mu}\}}\right)(z,y) \right| \right\} (1 + |z-y|)^\delta \, dz \end{aligned}$$

and using (3.14), we get

$$\left\|\tilde{K}'_\mu(x,y)\right\|_{\omega(0)} \le C, \tag{3.20}$$

with C independent of μ. Thus

$$\begin{aligned}\left|\tilde{Q}_{\mu,t}\left(A^{\{2^{-\mu}\}}\right)(x,y)\right| \\ \le C_l \tilde{K}'_\mu(x,y)\left(1+m\left(x,V^{\{2^{-\mu}\}}\right)\right)^{-l}(1+|x-y|)^{-\delta}.\end{aligned} \tag{3.21}$$

Since

$$\tilde{Q}^*_{\mu,t}\left(A^{\{2^{-\mu}\}}\right)(x,y)=\overline{\tilde{Q}_{\mu,t}\left(A^{\{2^{-\mu}\}}\right)(y,x)}$$

is the integral kernel of the operator $\overline{\tilde{Q}_{\mu,t}}\left(A^{\{2^{-\mu}\}}\right)$, we obtain

$$\begin{aligned}\left|\tilde{Q}_{\mu,t}\left(A^{\{2^{-\mu}\}}\right)(x,y)\right| \\ \le C_l \tilde{K}''_\mu(x,y)\left(1+m\left(y,V^{\{2^{-\mu}\}}\right)\right)^{-l}(1+|x-y|)^{-\delta}.\end{aligned} \tag{3.22}$$

Now (3.9) follows from (3.7), (2.3), (3.21), and (3.22). □

Lemma 3.23. *There exist constants $C, \varepsilon > 0$ independent of μ such that*

$$\int \left|T_1^{\{2^{-\mu}\}}(x,y+h)-T_1^{\{2^{-\mu}\}}(x,y)\right| dx \le C|h|^\varepsilon. \tag{3.24}$$

Proof. Since the kernels $T_t^{\{2^{-\mu}\}}(x,y)$ satisfy (1.2), it suffices to prove (3.24) for $|h|<1$. By a functional calculus,

$$T_1^{\{2^{-\mu}\}}(x,y)=c\int_{-\infty}^{\infty} e^{i\tau}\Gamma^{\{2^{-\mu}\}}(x,y,\tau)\,d\tau.$$

Therefore, applying (2.10), we get

$$\left|T_1^{\{2^{-\mu}\}}(x,y+h)-T_1^{\{2^{-\mu}\}}(x,y)\right| \le \frac{C|h|^\delta}{|x-y|^{d+\delta}} \quad \text{for } |h| \le \frac{|x-y|}{4}. \tag{3.25}$$

Using (1.2) and (3.25), we have

$$\begin{aligned}\int \left|T_1^{\{2^{-\mu}\}}(x,y+h)-T_1^{\{2^{-\mu}\}}(x,y)\right| dx \\ \le \int_{|x-y|\le 4|h|^{1/2}} + \int_{|x-y|>4|h|^{1/2}} \\ \le C|h|^{d/2} + C\int_{|x-y|>4|h|^{1/2}} \frac{|h|^\delta}{|x-y|^{d+\delta}}\,dx \le C|h|^\varepsilon. \quad \square\end{aligned}$$

Proof of Theorem 3.10. Set

$$\tilde{R}_{\mu,t}(\lambda) = \tilde{Q}_{\mu,t}(\lambda)e^{\lambda} = F\left(2^{\mu}\lambda\right)e^{-t2^{\mu}\lambda}e^{\lambda}\psi^2(\lambda).$$

Applying the same arguments as in the proof of Theorem 3.8, we obtain that (3.21) holds for $\tilde{R}_{\mu,t}$ instead of $\tilde{Q}_{\mu,t}$; that is, there exists a constant $C > 0$ and kernels $\tilde{K}''_{\mu}(x,y)$ such that

$$\left\|\tilde{K}''_{\mu}(x,y)\right\|_{\omega(0)} \le C, \tag{3.26}$$

$$\sup_{t>0}\left|\tilde{R}_{\mu,t}\left(A^{\{2^{-\mu}\}}\right)(x,y)\right| \le \tilde{K}''_{\mu}(x,y). \tag{3.27}$$

Using (3.26), (3,27), Lemma 3.23 and the fact that

$$\tilde{Q}_{\mu,t}\left(A^{\{2^{-\mu}\}}\right)(x,y) = \int \tilde{R}_{\mu,t}\left(A^{\{2^{-\mu}\}}\right)(x,z)T_1^{\{2^{-\mu}\}}(z,y)\,dz,$$

we have

$$\begin{aligned}
&\int \sup_{t>0}\left|\tilde{Q}_{\mu,t}\left(A^{\{2^{-\mu}\}}\right)(x,y) - \tilde{Q}_{\mu,t}\left(A^{\{2^{-\mu}\}}\right)(x,y_0)\right| dx\\
&\le \iint\left(\sup_{t>0}\left|\tilde{R}_{\mu,t}\left(A^{\{2^{-\mu}\}}\right)(x,z)\right|\right)\left|T_1^{\{2^{-\mu}\}}(z,y) - T_1^{\{2^{-\mu}\}}(z,y_0)\right| dz\,dx\\
&\le C\left|y-y_0\right|^{\varepsilon}.
\end{aligned}$$

Finally (3.11) follows from (3.7). □

4. Proof of Theorem 1.4

Let F be a continuous function on $\mathbb{R}_+$ that satisfies (1.5). By (1.12) it suffices to prove that there exists a constant $C > 0$ such that for every H^1_A atom a

$$\left\|\mathcal{M}F(A)a\right\|_{L^1} \le C. \tag{4.1}$$

Let a be an H^1_A atom associated to a ball $B(y_0, r)$. Obviously the operators $F(A)$ and $\mathcal{M}$ are bounded on $L^2(\mathbb{R}^d)$. Therefore, by (1.8),

$$\left\|\mathcal{M}F(A)a\right\|_{L^1(B(y_0,4r))} \le C. \tag{4.2}$$

Let $\psi(\lambda)$, $Q_{\mu,t}(\lambda)$ be as in Section 3. Using (3.6) we conclude

$$\left|\mathcal{M}F(A)a(x)\right| = \sup_{t>0}\left|T_tF(A)a(x)\right| \le \sum_{\mu\in\mathbb{Z}}\sup_{t>0}\left|Q_{\mu,t}(A)a(x)\right|. \tag{4.3}$$

By Theorem 3.8 we have, for $\mu \geq -2\log_2 r$,

$$\begin{aligned}\int_{B(y_0,4r)^c} \left(\sup_{t>0} |Q_{\mu,t}(A)a(x)|\right) dx \\ &\leq C \int_{B(y_0,4r)^c} \int_{B(y_0,r)} K_\mu(x,y) \left(2^{\mu/2} r\right)^{-\delta} |a(y)|\, dy\, dx \\ &\leq C \left(2^{\mu/2} r\right)^{-\delta}.\end{aligned}$$

Therefore

$$\sum_{\mu \geq -2\log_2 r} \int_{B(y_0,4r)^c} \left(\sup_{t>0} |Q_{\mu,t}(A)a(x)|\right) dx \leq C. \tag{4.4}$$

By virtue of (4.2), (4.3), and (4.4) it remains to show that there exists a constant $C > 0$ such that

$$\sum_{\mu < -2\log_2 r} \int_{B(y_0,4r)^c} \left(\sup_{t>0} |Q_{\mu,t}(A)a(x)|\right) dx \leq C.$$

Let n be the integer such that $y_0 \in \mathcal{B}_n$. We consider two cases.

Case 1: $r < 2^{-1-n/2}$. In this case a satisfies (1.10). Using (1.10) and Theorem 3.10, we get

$$\begin{aligned}\sum_{\mu < -2\log_2 r} \int_{B(y_0,4r)^c} \left(\sup_{t>0} |Q_{\mu,t}(A)a(x)|\right) dx \\ &\leq \sum_{\mu < -2\log_2 r} \int_{B(y_0,4r)^c} \left(\sup_{t>0} \left| \int_{B(y_0,r)} (Q_{\mu,t}(A)(x,y) \right.\right. \\ &\qquad \left.\left. - Q_{\mu,t}(x,y_0))a(y)\, dy \right|\right) dx \\ &\leq C \sum_{\mu < -2\log_2 r} \int_{B(y_0,r)} 2^{\varepsilon\mu/2} |y - y_0|^\varepsilon |a(y)|\, dy \\ &\leq C \sum_{\mu < -2\log_2 r} \int_{B(y_0,r)} 2^{\varepsilon\mu/2} r^{-d} |y - y_0|^\varepsilon\, dy \leq C.\end{aligned}$$

Case 2: $2^{-1-n/2} \le r \le 2^{1-n/2}$. Applying Theorem 3.8, we get

$$\sum_{\mu<-2\log_2 r} \int_{B(y_0,4r)^c} \left(\sup_{t>0} |Q_{\mu,t}(A)a(x)|\right) dx \le C_M \sum_{\mu<-2\log_2 r} \int_{B(y_0,4r)^c} \int_{B(y_0,r)} K_\mu(x,y) \left(1+2^{-\mu/2}m(y,V)\right)^{-M} \times \times |a(y)|\,dy\,dx.$$

Since $m(y,V) \sim 2^{n/2}$ for $y \in B(y_0,r)$, we obtain

$$\sum_{\mu<-2\log_2 r} \int_{B(y_0,4r)^c} \left(\sup_{t>0} |Q_{\mu,t}(A)a(x)|\right) dx \le C \sum_{\mu<n+2} 2^{-M(n-\mu)/2} \le C.$$

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