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COMPLEX SCALING AND DOMAINS WITH NON-COMPACT AUTOMORPHISM GROUP

KANG-TAE KIM AND STEVEN G. KRANTZ

Abstract. In this paper we study the boundary orbit accumulation points of smoothly bounded domains in $\mathbf{C}^2$ with non-compact automorphism group. We prove that a boundary orbit accumulation point cannot be exponentially flat. This confirms a version of a conjecture of Greene and Krantz. The proof uses a new result on the *a priori* convergence of convex scaling methods, which in particular implies the equivalence of two different scaling methods on convex domains.

0. Introduction

The notion of studying the automorphism group (i.e., the group of biholomorphic self-maps) of a domain in complex space is an old one. Certainly the primacy of this idea comes in part from Poincaré's study of the biholomorphic inequivalence of the ball in $\mathbb{C}^2$ and the bidisc in $\mathbb{C}^2$. The nub of his proof is to note that if the domains were equivalent then their automorphism groups would be isomorphic; then he provides a clever argument to show that in fact those groups *cannot* be isomorphic.

More generally, following from Poincaré's program, and from more recent work of Burns/Shnider/Wells [BSW] and Greene/Krantz [GRK2], it is known that two topologically equivalent domains in any given $\mathbb{C}^n$ are generically biholomorphically inequivalent. While it makes sense, in principle, to distinguish or to compare domains by using boundary differential invariants (i.e., the so-called Chern-Moser invariants), in practice it is not feasible to do so. Even if all the Chern-Moser invariants of two given domains "match up" in some sense, it does not follow that these domains are biholomorphically equivalent.

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Thus it is natural to seek other "invariants" by means of which we might compare and contrast various domains in space. The automorphism group is a natural invariant to use, for it has interesting algebraic, topological, and differential-geometric properties. While automorphism groups cannot always be computed, it is often possible to infer properties of the automorphism groups of domains (compactness vs. non-compactness, dimension, transitivity, size of isotropy subgroups, orbit structure, etc.) that allow them to be useful tools.

Of particular interest in recent years has been the study of those domains that have non-compact automorphism groups. Part of the motivation for this study is the following consideration: A domain with transitive automorphism group (especially a bounded, symmetric one, and particularly one with smooth boundary) is restricted to live on a very short list. Transitivity is just too rigid a condition to impose on the automorphism group of a domain in $\mathbb{C}^n$. Non-compactness of the automorphism group still captures some of the robustness of a "large" automorphism group, but is not nearly so restrictive.

It is a fundamental fact—following on work of Cartan, and amounting to a rather sophisticated normal families argument (see [NAR]) that if a bounded domain $\Omega \subseteq \mathbb{C}^n$ has non-compact automorphism group then there is a point $q \in \Omega$, a point $p \in \partial\Omega$, and automorphisms $\phi_j \in \mathrm{Aut}(\Omega)$ such that $\phi_j(q) \to p$. The converse is true as well. We call the boundary point p a "boundary orbit accumulation point" or "orbit accumulation point" for short. Recent studies (see [GRK1]–[GRK7], [BEP1]–[BEP4], [KIM1]–[KIM6]) demonstrate that the (Levi) geometry of the boundary accumulation point reveals important global information about the geometry of the domain and vice versa. It is therefore of interest to know what sorts of points in $\partial\Omega$ can be orbit accumulation points.

The primordial result in this subject is the theorem of Bun Wong [BW], later generalized by Rosay [ROS]. A version of the result is as follows:

> *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain. Let $p \in \partial\Omega$ be a point of strong pseudoconvexity (in particular, $\partial\Omega$ is C^2 and is strongly pseudoconvex near p). If p is an orbit accumulation point for the action of* $\mathrm{Aut}(\Omega)$ *on Ω then Ω is biholomorphic to the unit ball in $\mathbb{C}^n$.*

It was this theorem that first suggested to Greene and Krantz that the Levi geometry of a boundary orbit accumulation point can be a determining factor in the global geometry of the domain (see [GRK1]). A survey of their studies up to 1986 appears in [GRK2]. Later work, particularly [GRK3], [GRK4], led to specific conjectures about the geometric nature of orbit accumulation points. A discussion of more recent results appears in [IK].

Recent work of Cheung, Fu, Krantz, and Wong [CFKW] shows that a version of the Bun Wong/Rosay theorem is also true on complex manifolds.

In particular, the paper [CFKW] gives a characterization of domains in a complex manifold that have compact quotient.

One of the first results along the lines just suggested (see [GRK4]) is that a boundary orbit accumulation point must be pseudoconvex. The ideas in that same paper led to a conjecture, now the object of much attention, known as the Greene/Krantz conjecture:

> *Let $\Omega \subseteq \mathbb{C}^n$ have C^∞ boundary and non-compact automorphism group. If $p \in \partial\Omega$ is an orbit accumulation point then p is of finite type in the sense of Catlin, D'Angelo, and Kohn.*

The first substantial result to support the conjecture is due to Kim [KIM3]. It says that if $\Omega \subseteq \mathbb{C}^n$ is convex, if $p \in \partial\Omega$ is a boundary orbit accumulation point, and if there is an open neighborhood of p in which $\partial\Omega$ is Levi flat, then Ω must be a product domain. Such a point p must be of infinite type, but Kim's result shows that the domain Ω cannot have smooth boundary. So the result can be considered a special case of the Greene/Krantz conjecture.

The second result supporting the conjecture is due to Fu, Isaev, and Krantz [FIK]. It says that if Ω is a smoothly bounded Reinhardt domain with non-compact automorphism group then Ω has a defining function that is a polynomial. Certainly it follows that the domain in question must be of finite type. Thus the Greene/Krantz conjecture is confirmed (in a strong form) for Reinhardt domains.

The present paper is another step in the verification of the Greene/Krantz conjecture. We formulate the results here for convex domains in $\mathbb{C}^2$ (although a careful examination of the proof shows that the results are true for a limited class of domains in $\mathbb{C}^n$, and for some non-convex domains as well). Informally, the result is that if Ω is a convex domain in $\mathbb{C}^2$ and if a point $p \in \partial\Omega$ is infinitely flat in a certain calculable sense, then p cannot be an orbit accumulation point of the action of $\mathrm{Aut}(\Omega)$. While our "infinitely flat" condition is not literally equivalent to "infinite type" in the sense of Catlin, D'Angelo, and Kohn, it includes all known examples and many new ones as well; consequently—at least in dimension two—the theorem provides some theoretical framework for understanding points of infinite type and why the Greene/Krantz conjecture should be true.

One of the main difficulties in proving the Greene/Krantz conjecture has been coming to grips with C^∞ functions that vanish to infinite order. Work on non-compact automorphism groups, to now, has typically hypothesized that the orbit accumulation point in question be of finite type. Such an assumption has made it possible to exploit Taylor polynomial analysis. Analysis of points of infinite type has proved, so far, to be resistive. One of the main contributions of the present work is to demonstrate how to analyze a function that vanishes to infinite order and thereby to understand a point of a certain infinite type. In particular, the points of infinite type that we treat are obtained by composition of the function $\exp(-1/|z|^2)$ with the defining function for a finite type point.

An additional technique introduced here is a new, sharpened form of scaling (as introduced in [FRA], [PIN], [KIM1]–[KIM3]) together with some careful analysis of possible orbits and orders of contact of orbits. We anticipate developing these techniques further, and applying them to a broader class of domains, in later papers.

We present two significant new technical results in this paper. One of them (Section 4) is an *a priori* convergence result for Pinchuk scaling. The other (Section 5) is an equivalence statement for the scaling methods of Pinchuk and of Frankel. In addition to being crucial for the theorems developed here, these tools should prove, both conceptually and practically, useful in future work.

Section 1 of the paper introduces essential notation and definitions and enunciates the principal results. It also describes the proof strategy. Section 2 performs an analysis of the so-called "model domain". It is here that all the hard analysis takes place. Section 3 provides proofs of the two main theorems, and ties together all the preceding ideas. Section 4 considers convergence of the scaling methods; indeed it is proved there that the Pinchuk scaling method, suitably formulated, always converges on a convex domain. Section 5 proves an equivalence statement for the scaling method of Pinchuk and the scaling method of Frankel. It should be noted that the results of Sections 4 and 5 are used in earlier arguments, but they are given independent proofs in these later sections. Section 6 provides concluding remarks and proposes ideas for future investigation.

1. Basic notions and enunciation of principal results

1.1. The class $\mathcal{G}_p$. Throughout, we denote by $\mathcal{G}_p$ the collection of bounded domains Ω in $\mathbb{C}^2$ with $p \in \partial\Omega$ satisfying the property that (after a normalization of coordinates and an application of the implicit function theorem) $p = 0$ and there is a neighborhood U of p and a C^∞ function φ of a single real variable, vanishing at the origin, such that

$$U \cap \Omega = \{(z, w) \in U : \operatorname{Re} z > \varphi(|w|^2)\}.$$

Here we assume:

(A) $\varphi(x) = 0\ ,\ \forall x \leq 0.$
(B) $\varphi(x) > 0$ and $\varphi''(x) > 0$, $\forall x > 0.$
(C) For $x > 0$, the function $\psi(x) = -1/\log \varphi(x)$ (defined to be 0 for $x < 0$) extends to a function that is C^∞ smooth at 0, vanishing to a finite order m at that point.

1.2. Principal results. The main result of the present paper is as follows:

THEOREM 1.2.1. *Let $\Omega \subseteq \mathbb{C}^2$ be a bounded domain that belongs to the class $\mathcal{G}_p$. Then there cannot exist a point $q \in \Omega$ and automorphisms $\varphi_j \in \mathrm{Aut}(\Omega)$ such that $\varphi_j(q) \to p$ as $j \to \infty$.*

In addition to this theorem pertaining to the Greene-Krantz conjecture, we have obtained an *a priori* convergence of the Pinchuk scaling sequence when an automorphism orbit accumulates at a variety-free convex boundary point. We also show in this article that the scaling method of S. Frankel is equivalent to the Pinchuk scaling. However, we choose to keep the statement in the exposition of this paper.

1.3. Organization of the proofs. In order to make the entire exposition as clear as possible, we now give a rough sketch of the proof of Theorem 1.2.1.

First assume that $0 = (0,0) \in \partial\Omega$ and that $\Omega \in \mathcal{G}_0$. Seeking a contradiction we assume that

there exists $q \in \Omega$ and $f_j \in \mathrm{Aut}(\Omega)$ for $j = 1, 2, \ldots$ such that $\lim_{j\to\infty} f_j(q) = (0,0)$.

Then we prove (using an *a priori* convergence theorem for convex scaling methods; see Section 4 of this paper):

Ω is biholomorphic to either the bidisc or to a certain strongly pseudoconvex Kobayashi hyperbolic tube domain.

One may use boundary geometry and an application of the scaling methods developed by S. Pinchuk (for instance) to show that Ω cannot be biholomorphic to a strongly pseudoconvex Kobayashi hyperbolic tube domain; details are presented in Sections 3 and 4. Now we are left with the possibility that Ω is biholomorphic to the bidisc. In this case, Ω is in particular homogeneous. Since Ω has a strongly pseudoconvex boundary point, Rosay's generalization of Wong's theorem implies that Ω is biholomorphic to the ball. This leads to the desired contradiction, since the ball cannot be biholomorphic to the bidisc by the previously cited theorem of Poincaré.

A careful application of the scaling method followed by a new *a priori* convergence theorem for Pinchuk's scaling method plays a crucial role in our arguments.

2. The model domain case

2.0. Introductory remarks. In this section we will scale the given (model) domain along the hypothesized non-compact orbit. The argument divides into a great many cases and sub-cases. In each instance, we will see that the scaling does not yield any hyperbolic limit domain except the bidisc or a certain tube domain (as described in Section 1.3). We will then show that the model domain cannot be biholomorphic to either of those limit domains, whereas our *a priori* paradigm mandates that the limit domains *must* exist and be biholomorphic to the model domain. Proposition 2.7.1 gives a summary statement of the conclusions that our calculations yield.

2.1. The standard model. Let $\Omega_0 \in \mathbb{C}^2$ be defined by

$$(\dagger) \qquad \Omega_0 = \{(z,w) \in \mathbb{C}^2 \mid \operatorname{Re} z > \varphi(|w|^2)\}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a C^∞ function that satisfies the conditions (A), (B) and (C) of Section 1.1.

Throughout this article, this domain will serve as the standard model for the class $\mathcal{G}_p$ (despite its unboundedness, which is immaterial).

2.2. Notation. Let $f_j \in \operatorname{Aut}(\Omega_0)$, for $j = 1, 2, \ldots$, and $q \in \Omega_0$ satisfy that $f_j(q) \to (0,0)$ as $j \to \infty$. Write $f_j(q) = (a_j, b_j)$. We will usually drop the subscripts for the sake of convenience. For instance, we drop the subscript j and write $(a_j, b_j) = (a, b)$. Observe that we have $\operatorname{Re} a > \varphi(|b|^2)$.

2.3. Centering of the orbit. Choose $a^* \in \mathbb{C}$ such that $(a^*, b) \in \partial\Omega_0$, i.e.,

$$\operatorname{Re} a^* = \varphi(|b|^2).$$

For convenience, we will also choose a^* so that it satisfies the condition

$$\operatorname{Im} a^* = \operatorname{Im} a.$$

Then we perform a linear coordinate change $\psi(z,w) = (\zeta, \xi)$ such that

$$\begin{cases} \zeta &= (z - a^*) + c(w - b) \\ \xi &= w - b \end{cases}$$

for some $c \in \mathbb{C}$ to be chosen later.

We set the notation:

$$\begin{aligned} \rho(z,w) &= \operatorname{Re} z - \varphi(|w|^2), \\ \rho^* &= \rho \circ \psi^{-1}. \end{aligned}$$

The domain $\psi(\Omega_0)$ near $(0,0)$ is defined by the equation $\rho^*(\zeta,\xi) > 0$, that is

$$\rho^*(\zeta,\xi) \equiv \operatorname{Re} a^* + \operatorname{Re}(\zeta - c\xi) - \varphi(|\xi + b|^2) > 0.$$

We now require that

$$\left.\frac{\partial \rho^*}{\partial \xi}\right|_0 = 0, \quad \left.\frac{\partial \rho^*}{\partial \overline{\xi}}\right|_0 = 0, \quad \text{and} \quad \left.\frac{\partial \rho^*}{\partial (\operatorname{Im} \zeta)}\right|_0 = 0.$$

In our situation, the third condition is automatically satisfied. The first and the second conditions are equivalent to each other. Hence we only consider the first condition, which is equivalent to

$$c = -2\varphi'(|b|^2) \cdot \overline{b}.$$

Therefore the resulting defining equation for $\psi(\partial\Omega)$ near $(0,0)$ now becomes

$$\operatorname{Re}\zeta = \varphi(|\xi + b|^2) - \varphi(|b|^2) - 2\varphi'(|b|^2)\operatorname{Re}\bar{b}\xi.$$

2.4. Pinchuk's scaling and its convergence. We write

$$\mu = a - a^*$$

and choose $\lambda \in \mathbb{C}$ such that

$$(\mu, \lambda) \in \psi(\partial\Omega) \text{ and } |\lambda| = \min\{|\eta| : \eta \in \mathbb{C}, (\mu, \eta) \in \psi(\partial\Omega))\}.$$

Then we let $L(z,w) = (z/\mu, w/\lambda)$ for $(z,w) \in \mathbb{C}^2$.

In order to state the convergence theorem precisely, we revive the subscripts for the sequences for a moment. Let us write

$$\Phi_j \equiv L_j \circ \psi_j \circ f_j : \Omega_0 \to \mathbb{C}^2$$

for each $j = 1, 2, \ldots$. Now we present one of the key techniques.

Proposition 2.4.1. *Every subsequence of the $\{\Phi_j\}$ admits a subsequence that converges uniformly to a holomorphic embedding, say $\widehat{\Phi} : \Omega_0 \to \mathbb{C}^2$, on compact subsets of Ω_0 in such a way that $\widehat{\Phi}(\Omega_0)$ coincides with the local Hausdorff limit of the corresponding subsequence of $\Phi_j(\Omega_0)$.*

We give a more general statement and its proof in Section 5.

2.5. Analysis of the scaling of the model domain. Denote by $(a_j, b_j) = (a, b)$ the j^{th} automorphism orbit point for the scaling process. Since the defining inequality (†) above for the model domain is rotationally symmetric in the w-variable, we may assume that

$$b > 0 \tag{1}$$

throughout.

Given (a, b) with $b > 0$, we consider the "centered" defining inequality

$$\operatorname{Re}\zeta > \varphi(|\xi + b|^2) - \varphi(b^2) - 2b\varphi'(b^2)\operatorname{Re}\xi \tag{2}$$

and denote by Ω_j the domain defined by (2). Then we choose $\lambda \in \mathbb{C}$ satisfying

$$(\mu, \lambda) \in \partial\Omega_j \text{ and } |\lambda| = \min\{|\eta| \mid (\mu, \eta) \in \partial\Omega_j\}, \tag{3}$$

where

$$\mu = \operatorname{Re} a - \varphi(b^2). \tag{4}$$

Now we analyze λ. In order to do this, we write $\lambda = x + \sqrt{-1}\,y$ with $x, y \in \mathbb{R}$. Then λ realizes the minimum value of the function $x^2 + y^2$ under the constraint equation

$$\varphi((x+b)^2 + y^2) - \varphi(b^2) - 2b\varphi'(b^2)x = \mu.$$

This is a typical Lagrange multiplier problem; one immediately arrives at the following two cases:

Case 1. $\lambda \in \mathbb{R}$.

Case 2. $\lambda \notin \mathbb{R}$ and $|\lambda + b|^2 = b^2$.

We will take this branching into account when we apply the scaling method to (†).

2.5.1. The non-tangential orbit case. It is enough to consider the limit expression

$$\operatorname{Re}\zeta > \lim_{\lambda\to 0+} \frac{\varphi(|\lambda\xi|^2)}{\varphi(|\lambda|^2)}. \tag{2.5.1.1}$$

Since Conditions (A), (B) and (C) imply that

$$\lim_{r\to 0+} \frac{\varphi(rx)}{\varphi(r)} = \infty \text{ for every } x > 1, \tag{2.5.1.2}$$

we see that the expression (2.5.1.1) becomes

$$\operatorname{Re}\zeta > \begin{cases} \infty & \text{if } |\xi| > 1, \\ 0 & \text{if } |\xi| < 1. \end{cases} \tag{2.5.1.3}$$

Consequently, the scaled limit domain $\widehat{\Omega}$ defined by (2.5.1.3) is biholomorphic to the bidisc.

2.5.2. Several cases with tangential orbits. In this case, we are concerned with the convergence of the expression

$$\operatorname{Re}\zeta > \frac{\varphi(|\lambda\xi + b|^2) - \varphi(|b|^2) - 2b\varphi'(|b|^2)\operatorname{Re}\lambda\xi}{\varphi(|\lambda + b|^2) - \varphi(|b|^2) - 2b\varphi'(|b|^2)\operatorname{Re}\lambda}\,. \tag{2.5.2.1}$$

We shall analyze the possible limits of this expression with several possible choices for λ.

At this point, we remark that the *a priori* convergence theorem for Pinchuk's scaling sequence (Proposition 2.4.1) says that every subsequence of a scaling sequence must have a subsequence that converges and consequently yields a scaled limit domain which is biholomorphic to the original domain. Now, in the below, we shall extract subsequences whenever necessary and show that the scaled subsequential limits obtained thereof are either non-existent or the ones that can never be biholomorphic to the original domain. Notice that this immediately yields a contradiction to the hypothetical assumption that the non-compact automorphism orbit existed.

Division of our arguments into several cases and sub-cases below is also based upon extracting subsequences as such.

2.5.2.A. *The case $\lambda > 0$.* Recall the convexity of φ and the centering process. It follows that

$$\varphi((\lambda + b)^2) > \varphi(b^2) + 2b\varphi'(b^2)\lambda\,, \tag{2.5.2.A.1}$$

and each term in this expression is non-negative. (In fact, the convexity and the process of centering guarantees that the numerator of (2.5.2.1) is positive for all ξ near 0.) Thus, choosing a subsequence (from the given noncompact sequence of automorphisms), we may assume that

$$c_1 \equiv \lim \frac{\varphi(b^2)}{\varphi((\lambda+b)^2)} \text{ and } c_2 \equiv \lim \frac{2b\varphi'(b^2)\lambda}{\varphi((\lambda+b)^2)}. \tag{2.5.2.A.2}$$

Then (2.5.2.1) becomes
(2.5.2.A.3)

$$\operatorname{Re}\zeta > \frac{1}{1-(\sim c_1)-(\sim c_2)}\left\{\frac{\varphi(|\lambda\xi+b|^2)}{\varphi((\lambda+b)^2)} - (\sim c_1) - (\sim c_2)\operatorname{Re}\xi\right\}.$$

Therefore we focus upon the convergence of $\lim \dfrac{\varphi(|\lambda\xi+b|^2)}{\varphi((\lambda+b)^2)}$ in several different cases.

2.5.2.A.I. *b/λ has a bounded subsequence.* This case allows us to choose a subsequence (of the given automorphism orbit) so that

$$\beta \equiv \lim \frac{b}{\lambda}$$

Then the limit expression of (2.5.2.1) (or, equivalently, (2.5.2.A.3)) becomes

$$\operatorname{Re}\zeta > \begin{cases} \infty & \text{if } |\xi+\beta|^2 > |1+\beta|^2, \\ (-c_1 - c_2\operatorname{Re}\xi)/(1-c_1-c_2) & \text{if } |\xi+\beta|^2 < |1+\beta|^2, \end{cases} \tag{2.5.2.A.4}$$

Since the limit domain has to exist, be biholomorphic to the original domain (in this note the model defined by $\operatorname{Re} z > \varphi(|w|^2)$), and be supported by the hyperplane $\operatorname{Re}\zeta = 0$, we must have $c_1 = c_2 = 0$. In conclusion, the scaled limit domain $\widehat{\Omega}$ is defined by

$$\operatorname{Re}\zeta > \begin{cases} \infty & \text{if } |\xi+\beta|^2 > |1+\beta|^2, \\ 0 & \text{if } |\xi+\beta|^2 < |1+\beta|^2, \end{cases} \tag{2.5.2.A.5}$$

which is clearly biholomorphic to the bidisc.

2.5.2.A.II. *b/λ diverges to infinity.* Again, we look at the limit

$$\lim \frac{\varphi(|b+\lambda\xi|^2)}{\varphi(|b+\lambda|^2)}$$

with the condition that $\lambda/b \to 0$ as the automorphism orbit approaches the boundary.

In order to handle this situation, we consider $\psi(x) = -1\big/\log(\varphi(x))$ ($x \in \mathbb{R}$), where we have (by Conditions (A),(B) and (C) above)

$$\psi(x) = a_m x^m + \cdots \tag{2.5.2.A.6}$$

for some integer $m > 0$, and $a_m > 0$.

Then

$$\frac{\varphi(|b+\lambda\xi|^2)}{\varphi(|b+\lambda|^2)} = \exp \frac{\psi(|b+\lambda\xi|^2) - \psi(|b+\lambda|^2)}{\psi(|b+\lambda\xi|^2)\cdot\psi(|b+\lambda|^2)}. \tag{2.5.2.A.7}$$

Since

$$\psi(|b+\lambda\xi|^2) - \psi(|b+\lambda|^2) = ma_m b^{2m}\left\{2|\lambda/b|\operatorname{Re}(\xi-1) + o(|\lambda/b|)\right\}$$

we have that

$$\frac{\varphi(|b+\lambda\xi|^2)}{\varphi(|b+\lambda|^2)} \sim \exp\left\{\frac{2m|\lambda/b|\ \operatorname{Re}(\xi-1) + o(|\lambda/b|)}{b^{2m}a_m\left((\sim 1) + (\sim 0)\right)}\right\}.$$

Since $\lambda/b \to 0$, observe that the term $o(|\lambda/b|)$ in the last expression makes no contribution to the limit values. We will again consider several sub-cases.

2.5.2.A.IIa. $\lambda\big/b^{2m+1}$ *has a convergent subsequence.* Choosing a subsequence again, we may assume that

$$\lim \lambda\big/b^{2m+1} = c_4.$$

Then the limit domain $\widehat{\Omega}$ is expressed by the inequality

$$\operatorname{Re}\zeta > A_1\left(\exp\{A_2\left(\operatorname{Re}\xi - 1\right)\} - c_1 - c_2\operatorname{Re}\xi\right) \tag{2.5.2.A.8}$$

for some non-negative constants A_1 and A_2.

Since the limit domain must be biholomorphic to the model—which is Kobayashi hyperbolic—we must have $A_1 > 0$. We also see that A_2 cannot be zero. For, if $A_2 = 0$ (i.e., $c_4 = 0$), then to have the hypersurface $\operatorname{Re}\zeta = 0$ support the limit domain, the only choice for the limit domain comes from $c_2 = 0$ and that is the one defined by

$$\operatorname{Re}\zeta > A_1 c_1,$$

which cannot be Kobayashi hyperbolic.

Therefore the limit domain is given by the inequality (2.5.2.A.8) with $A_1 > 0$, and $A_2 > 0$. In this case, the limit domain is biholomorphic to the domain defined by the inequality

$$\operatorname{Re}\zeta > \exp(\operatorname{Re}\xi). \tag{2.5.2.A.9}$$

2.5.2.A.IIb. $\lambda\big/b^{2m+1}$ *diverges to infinity.* Then we obtain the limit domain immediately:

$$\operatorname{Re}\zeta > \begin{cases} \infty & \text{if } \operatorname{Re}(\xi-1) > 0, \\ -(c_1 + c_2\operatorname{Re}\xi)\big/(1-c_1-c_2) & \text{if } \operatorname{Re}(\xi-1) < 0. \end{cases}$$

In order for the limit domain to be supported by the hyperplane $\operatorname{Re}\zeta = 0$, we must have $c_1 = c_2 = 0$. But then this limit domain is given by

$$\operatorname{Re}\zeta > \begin{cases} \infty & \text{if } \operatorname{Re}(\xi - 1) > 0, \\ 0 & \text{if } \operatorname{Re}(\xi - 1) < 0. \end{cases}$$

This domain is biholomorphic to the bidisc.

2.5.2.B. *The case* $\lambda < 0$. We are led at first to compare the sequences $\varphi(|\lambda + b|^2)$, $\varphi(|b|^2)$, and $2b\lambda \cdot \varphi'(|b|^2)$, all of which tend to zero. Choosing a subsequence whenever necessary, we may consider the following three cases, which cover all the cases but are not mutually exclusive.

2.5.2.B.I. *Both* $\varphi(|b|^2)/\varphi(|\lambda+b|^2)$ *and* $2b\lambda\varphi'(|b|^2)/\varphi(|\lambda+b|^2)$ *admit bounded subsequences.* In this case, there exist non-negative constants C_1, C_2 such that

$$\lim \frac{\varphi(b^2)}{\varphi((\lambda+b)^2)} = C_1 \text{ and } \lim \frac{2b\lambda \cdot \varphi'(b^2)}{\varphi((\lambda+b)^2)} = -C_2.$$

Here we have to choose a subsequence if necessary.

Then the expression (2.5.2.1) is equivalent to

$$\operatorname{Re}\zeta > \frac{1}{1 - (\sim C_1) + (\sim C_2)} \cdot \left\{ \frac{\varphi(|\lambda\xi + b|^2)}{\varphi((\lambda+b)^2)} - (\sim C_1) + (\sim C_2) \operatorname{Re}\xi \right\}.$$

Therefore the analysis of this limit expression is similar to the case of $\lambda > 0$. The only possible scaled limit domains are either biholomorphic to the bidisc or the strongly pseudoconvex tube domain of Case 2.5.2.A.

2.5.2.B.II. *Both* $\varphi(|\lambda+b|^2)/\varphi(|b|^2)$ *and* $2b\lambda\varphi'(|b|^2)/\varphi(|b|^2)$ *admit bounded subsequences.* In this case, choosing a subsequence again if necessary, we may assume that

$$\lim \frac{\varphi((b+\lambda)^2)}{\varphi(b^2)} = A_1 \quad \text{and} \quad \lim \frac{2b\lambda \cdot \varphi'(b^2)}{\varphi(b^2)} = A_2$$

for some constants $A_1 \geq 0$ and $A_2 \leq 0$. From the conditions satisfied by φ, we deduce that

$$A_2 = 2b\lambda \cdot \frac{m}{a_m} \cdot \frac{1}{b^{2m+2}} \cdot (\sim 1) = (\sim 1) \cdot \frac{2m}{a_m} \cdot \frac{\lambda}{b^{2m+1}}.$$

Consequently,

$$\lim \frac{\lambda}{b^{2m+1}} = \frac{A_2 a_m}{2m}$$

as $\lambda, b \to 0$. In particular, we see that

$$\lim \frac{\lambda}{b} = 0.$$

Now we observe:

$$\begin{aligned}\frac{\varphi(|b+\lambda\xi|^2)}{\varphi(b^2)} \quad &\sim \quad \exp\left(\frac{m}{a_m}\cdot\frac{|\lambda|}{b^{2m+1}}\cdot\frac{\operatorname{Re}\xi-1+o(1)}{1+o(1)}\right)\\ &\to \quad c_1\exp(c_2\operatorname{Re}\xi)\end{aligned}$$

for some constants $c_1 > 0$ and $c_2 \geq 0$. As a consequence, we deduce that the expression (2.5.2.1) is equivalent to

$$\operatorname{Re}\zeta > \frac{1}{(\sim A_1)-1-(\sim A_2)}\cdot\left(c_1\exp(c_2\operatorname{Re}\xi)-1-A_2\operatorname{Re}\xi\right).$$

Since $c_2 \geq 0$ and $A_2 \leq 0$, and since the limit domain must be supported by the hyperplane defined by $\operatorname{Re}\zeta = 0$, we can see immediately that this case yields no limit domain that can be Kobayashi hyperbolic.

2.5.2.B.III. *Both* $\varphi(|\lambda+b|^2)/(2b\lambda\varphi'(|b|^2))$ *and* $\varphi(|b|^2)/(2b\lambda\varphi'(|b|^2))$ *admit bounded subsequences.* Choosing a subsequence we may again assume that

$$\lim\frac{\varphi((\lambda+b)^2)}{2b\lambda\cdot\varphi'(b^2)} = B_1 \text{ and } \lim\frac{\varphi(b^2)}{2b\lambda\cdot\varphi'(b^2)} = B_2$$

for some non-positive constants B_1, B_2.

Notice that the crucial term to handle in the scaling of this case is

$$\lim\frac{\varphi(|\lambda\xi+b|^2)}{2b\lambda\cdot\varphi'(b^2)}.$$

Since φ is a strictly increasing function on the positive real axis, we may choose $\mu = \mu(\lambda, b) > 0$ such that $\varphi(\mu^2) = -2b\lambda\cdot\varphi'(b^2)$. Thus the limit above can be re-written as

$$-\lim\frac{\varphi(|\lambda\xi+b|^2)}{\varphi(\mu^2)}.$$

2.5.2.B.IIIa. *The value* ∞ *or* $\gamma > 0$ *is a subsequential limit for* λ/b. The value for $-\lim\varphi(\lambda^2|\xi+b/\lambda|^2)/\varphi(\mu^2)$ depends upon the subsequential limits of λ^2/μ^2. If ∞ is a subsequential limit of λ/μ, then the value of the above limit becomes identically $-\infty$. This fact will imply that the scaled limit domain is not Kobayashi hyperbolic. If 0 is a subsequential limit of λ/μ, then the value of the above limit becomes identically 0. Again, the scaled limit domain in this case cannot be Kobayashi hyperbolic. Finally, if $\beta > 0$ is a subsequential limit of λ/μ, then we have

$$-\lim\frac{\varphi(\lambda^2|\xi+b/\lambda|^2)}{\varphi(\mu^2)} = \begin{cases}0 & \text{if } |\xi+\gamma^{-1}|^2 < \beta^{-2},\\ -\infty & \text{if } |\xi+\gamma^{-1}|^2 > \beta^{-2}.\end{cases}$$

In this case the scaled limit domain is given by

$$\operatorname{Re}\zeta > \lim\frac{1}{(\sim B_1)-(\sim B_2)+1}\cdot\left\{-\frac{\varphi(|\lambda\xi+b|^2)}{\varphi(\mu^2)}-(\sim B_2)-\operatorname{Re}\xi\right\},$$

so this domain cannot be supported by the hyperplane defined by $\operatorname{Re}\xi = 0$.

2.5.2.B.IIIb. *The sequence λ/b tends to* 0. Recall the conditions which the function φ satisfies. We first have that

$$\frac{\varphi(|b+\lambda\xi|^2)}{\varphi(\mu^2)} = \exp\left\{\frac{\psi(|b+\lambda\xi|^2)-\psi(\mu^2)}{\psi(\mu^2)\,\psi(|b+\lambda\xi|^2)}\right\}.$$

Moreover, $\psi(x)$ is smooth and vanishes to finite order $m > 0$ at the origin, i.e., its Taylor expansion at the origin is given by $\psi(x) = a_m x^m + R_m(x)$, where $R_m(x) = o(|x|^m)$.

Now

$$\begin{aligned}\psi(|b+\lambda\xi|^2) \quad - \quad & \psi(\mu^2) \\ = \quad & a_m(b^{2m}-\mu^{2m}) + a_m b^{2m}\left(\frac{\lambda}{b}\operatorname{Re}\xi + o((\lambda/b)\operatorname{Re}\xi)\right) \\ & +R_m(b^2|1+\lambda\xi/b|^2) - R_m(\mu^2).\end{aligned}$$

At this point, we need to compare $b^{2m}-\mu^{2m}$ and λb^{2m-1}. We do this by considering sub-cases again:

2.5.2.B.IIIb′. *The sequence $(b^{2m}-\mu^{2m})/(\lambda b^{2m-1})$ has a subsequence that converges.* Let the subsequential limit be L_1. Note that L_1 is a real number. Then it turns out that

$$\psi(|b+\lambda\xi|^2) - \psi(\mu^2) = a_m b^{2m-1}\lambda\left(\operatorname{Re}\xi + L_1 + o(1+|\xi|)\right).$$

Thus

$$\frac{\varphi(|\lambda\xi+b|^2)}{\varphi(\mu^2)} = \exp\left\{\frac{1}{a_m}\cdot\frac{b^{2m-1}\lambda}{b^{2m}\mu^{2m}}\cdot\frac{\operatorname{Re}\xi + L_1 + o(1+|\xi|)}{1+o(1+|\xi|)}\right\}.$$

If a subsequence of $\lambda/(b\mu^{2m})$ has a bounded limit, the corresponding scaled limit domain is biholomorphic either to a strongly pseudoconvex tube domain of Case 2.5.2.A or to the bidisc. If $\lambda/(b\mu^{2m}) \to -\infty$, then either the scaled limit has to be biholomorphic to the bidisc, or the scaled limit domain cannot be supported by the hyperplane $\operatorname{Re}\zeta = 0$.

2.5.2.B.IIIb″. *The sequence $(b^{2m}-\mu^{2m})/(\lambda b^{2m-1})$ diverges to $\pm\infty$.* Then we see that

$$\psi(|b+\lambda\xi|^2) - \psi(\mu^2) = a_m(b^{2m}-\mu^{2m})(1+o(1+|\xi|)).$$

Hence,

$$\frac{\varphi(|\lambda\xi+b|^2)}{\varphi(\mu^2)} = \exp\left\{\frac{1}{a_m}\cdot\frac{b^{2m-1}\lambda}{b^{2m}-\mu^{2m}}\cdot(1+o(1+|\xi|))\right\}.$$

If this last expression is convergent, then it converges to a constant that is independent of ξ. If this expression diverges, the scaled limit domain, at its best choices, cannot be a Kobayashi hyperbolic domain.

2.5.2.C. *The case* $\lambda \in \mathbb{C} \setminus \mathbb{R}$. In this case, we have $\varphi(|\lambda + b|^2) = \varphi(b^2)$. Therefore the scaling is dependent upon the analysis of the expression

$$\operatorname{Re} \zeta > \frac{1}{2\varphi'(b^2) b Re\lambda} \left\{ \varphi(|\lambda\xi + b|^2) - \varphi(b^2) - 2\varphi'(b^2) b \operatorname{Re} \lambda\xi \right\}.$$

Since the function φ is strictly increasing over the positive real axis, we also have

$$|\lambda + b|^2 = b^2.$$

Thus it follows that

$$\operatorname{Re} \lambda \leq 0 \text{ and } \left| \frac{\lambda}{b} + 1 \right|^2 = 1.$$

In particular, λ/b stays bounded. Choosing a subsequence, we may assume that

$$\lim \frac{\lambda}{b} = L.$$

Furthermore, notice that

$$\frac{\operatorname{Re} \lambda}{|\lambda|} = -\frac{1}{2} \cdot \frac{|\lambda|}{b} \longrightarrow -\frac{1}{2}|L|.$$

Now we consider subcases.

2.5.2.C.I. $L \neq 0$. In this case, we see that

$$\begin{aligned} \frac{\varphi(b^2)}{2b\varphi'(b^2) \operatorname{Re} \lambda} &\sim \frac{\varphi(b^2)}{-|L|^2 b^2 \varphi'(b^2)} \\ &\sim -ma_m |L|^{-2} b^{2m-2} \\ &\to \begin{cases} 0 & \text{if} \quad m > 1, \\ -ma_m |L|^{-2} & \text{if} \quad m = 1. \end{cases} \end{aligned}$$

Hence, choosing a subsequence, we see that the scaled limit is determined by the expression

$$\operatorname{Re} \zeta > \frac{\varphi(|\lambda\xi + b|^2)}{2b\varphi'(b^2) \operatorname{Re} \lambda} + c_1 + c_2 \operatorname{Re}(\bar{\gamma}\xi),$$

where c_1 and c_2 are constants with $c_1 \geq 0$ and $c_2 > 0$, and where γ is a complex number with modulus 1. Therefore the analysis of the scaled limit domain again focuses upon

$$\lim \frac{\varphi(|\lambda\xi + b|^2)}{2b\varphi'(b^2) \operatorname{Re} \lambda}.$$

Notice that in our case $|\lambda|/b \to |L| > 0$. Hence, the analysis of the above expression is similar to that of Sub-Case 2.5.2.B.IIIa. However, since $c_1 \geq 0$ and $c_2 > 0$, our scaled limit domain can never by supported by the hyperplane $\operatorname{Re} \xi = 0$. In order to avoid repeating elementary ideas that we have used before, we omit the details of this argument.

2.5.2.C.II. $L = 0$, *i.e.*, $\lambda/b \to 0$. Choose a subsequence again so that $\lambda/|\lambda| \to \gamma$ in $\mathbb{C}$. Now, notice that

$$\begin{aligned} \frac{2b\varphi'(b^2)\operatorname{Re}\lambda\xi}{2b\varphi'(b^2)\operatorname{Re}\lambda} &= \frac{|\lambda|}{\operatorname{Re}\lambda}\cdot\operatorname{Re}\frac{\bar\lambda}{|\lambda|}\xi \\ &\to \infty \end{aligned}$$

for all $\xi \in \mathbb{C}$ away from the line $\operatorname{Re}\bar\gamma\xi = 0$. Let us write

$$S(\xi,\lambda,b) = \frac{\varphi(b^2)}{2b\varphi'(b^2)\operatorname{Re}\lambda} + \frac{2b\varphi'(b^2)\operatorname{Re}\lambda\xi}{2b\varphi'(b^2)\operatorname{Re}\lambda}.$$

Then $\lim S(\xi,\lambda,b) = \infty$ for almost all $\xi \in \mathbb{C}$.

Now we consider the term

$$T(\xi,\lambda,b) \equiv \frac{\varphi(|\lambda\xi + b|^2)}{2b\varphi'(b^2)\operatorname{Re}\lambda}.$$

As in the case of Sub-Case 2.5.2.B.IIIb, we let $\mu = \mu(\lambda,b) > 0$ be chosen such that

$$\varphi(\mu^2) = -2b\varphi'(b^2)\operatorname{Re}\lambda.$$

Then we have

$$T(\xi,\lambda,b) = \exp\left\{\frac{a_m(b^{2m}-\mu^{2m}) + a_m b^{2m}\left(\operatorname{Re}(\lambda/b)\xi\right)}{a_m^2 b^{2m}\mu^{2m}}\cdot[1+o(1+|\xi|)]\right\}.$$

Recall the analysis of Sub-Case 2.5.2.B.IIIb. With some minor adjustments, we can deduce that for all ξ away from a thin subset of $\mathbb{C}$ we have the following four possibilities for the value of $\lim T(\xi,\lambda,b)$:

(a) It converges to a constant independent of ξ.

(b) It converges to a function of type $\exp(A\operatorname{Re}\gamma\xi + B)$ for some real constants A, B and a constant $\gamma \in \mathbb{C}$ with norm 1.

(c) It diverges and the speed of its divergence is not affected by $\operatorname{Re}\xi$. More precisely, for any $\xi_1, \xi_2 \in \mathbb{C}$, and for any $\epsilon > 0$, we have

$$\lim \frac{T(\xi_1,\lambda,b)}{(T(\xi_2,\lambda,b))^{1+\epsilon}}. = 0$$

(d) There is an open region $G \subset \mathbb{C}$ such that $\lim T(\xi,\lambda,b) = 0$ for every $\xi \in R$ and $\lim T(\xi,\lambda,b) = \infty$ for every $\xi \in \mathbb{C}\setminus\overline{R}$ except possibly for a thin set. Moreover, we have one more property. For points ξ_1, ξ_2 at which $\lim T$ diverges, we consider the case of $\operatorname{Re}\xi_1 < \operatorname{Re}x_2$. Then there is a constant $r = r(\xi_1,\xi_2) > 0$ such that

$$\lim \frac{T(\xi_2,\lambda,b)}{(T(\xi_1,\lambda,b))^{1+\delta}} = \infty$$

for every δ with $0 < \delta < r$.

In the cases (a), (b) and (c), it follows immediately that the scaled limit domain cannot exist as a Kobayashi hyperbolic domain. In the case (d), we reach at the same conclusion as well because the speeds of divergence for $T(\xi, \lambda, b)$ and $S(\xi, \lambda, b)$ vary quite differently according to the values of ξ.

In conclusion, the assumptions of this section do not yield any Kobayashi hyperbolic scaled limit domain other than the bidisc or the tube.

2.6. Holomorphic inequivalences. Now we establish the needed biholomorphic inequivalences.

Proposition 2.6.1. *The model domain Ω_0 is not biholomorphic to the bidisc.*

Proof. The proof is immediate. First of all, considering two complex independent supporting hyperplanes of Ω_0, a linear fractional transform maps Ω_0 onto a bounded domain, say G_0, in $\mathbb{C}^2$. Then G_0 has a C^2 smooth strongly pseudoconvex boundary point. If Ω_0 were biholomorphic to the bidisc, then Ω_0 (and hence G_0 also) would be homogeneous. Then Rosay's generalization of Wong's theorem would imply that G_0 is biholomorphic to the unit ball in $\mathbb{C}^2$. Consequently, we have to conclude that the ball and the bidisc are biholomorphic to each other, which contradicts the previously cited theorem of Poincaré. □

Proposition 2.6.2. *The model domain Ω_0 is not biholomorphic to the tube domain $R = \{(z, w) \in \mathbb{C}^2 : \operatorname{Re} z > \exp(\operatorname{Re} w)\}$.*

Proof. This proof is again by the scaling method of Pinchuk. Seeking a contradiction, we suppose that Ω_0 and R are biholomorphic to each other. We choose two supporting hyperplanes for R that are complex linearly independent, and use linear fractional transformations to realize R as a bounded domain, say $\widetilde{R}$, in $\mathbb{C}^2$. Notice that $\widetilde{R}$ is strongly pseudoconvex at every boundary point except possibly at one point, say $\widetilde{p}_\infty \in \partial\widetilde{R}$, which corresponds to the point at infinity for R.

Now consider the induced biholomorphic mapping $F : \Omega_0 \to \widetilde{R}$. Let

$$A = \Omega_0 \cap \{(z, 0) \in \mathbb{C}^2\} = \{(z, 0) \in \mathbb{C}^2 : \operatorname{Re} z > 0\} .$$

Then consider $F(A) \subset \widetilde{R}$. By Fatou's theorem, $F|_A$ has non-tangential boundary limits almost everywhere. If the boundary limits consisted of one single boundary point, then Privalov's theorem would imply that $F|_A$ is constant. Since F is a biholomorphic mapping, this cannot happen. Therefore we may choose $p \in \partial A$ and a sequence $p_j \in A$ approaching p radially such that $F(p_j)$ accumulates at a boundary point $\widetilde{p}$ of $\widetilde{R}$ with $\widetilde{p} \neq \widetilde{p}_\infty$. Thus $\widetilde{p}$ is a strongly pseudoconvex boundary point of $\widetilde{R}$. Now consider the scaling sequence $\Lambda_j \circ \psi_j$ (centering followed by scaling; see Sections 2.2–2.4 for notation) with respect to p_j in Ω_0, the scaling sequence $\widetilde{\Lambda}_j \circ \widetilde{\psi}_j$ (again, centering followed by scaling) with respect to $F(p_j)$ in $\widetilde{R}$ respectively, and the sequence of mappings

$$\widetilde{\Lambda}_j \circ \widetilde{\psi}_j \circ F \circ (\Lambda_j \circ \psi_j)^{-1} .$$

It is not hard to see that there is a subsequence that converges to a holomorphic mapping; this mapping in turn gives rise to a holomorphic mapping, say $\widehat{F}$, from a bidisc into the ball. This last is true because Ω_0 scales to the domain biholomorphic to a bidisc along such a point sequence and any convex domain scales to the domain biholomorphic to the ball along any point sequence that accumulates at a strongly pseudoconvex point. Consider the inverse of the above sequence. Choosing subsequences if necessary, it is easy to see that the mapping $\widehat{F}$ is in fact a biholomorphic mapping from the bidisc to the ball, which is absurd. Therefore the proof is complete. □

2.7. Compactness of Aut (Ω_0)**.** Summarizing the arguments presented so far in this section, we arrive at the following result:

PROPOSITION 2.7.1. *In the model domain* Ω_0*, there do not exist a point* $q \in \Omega_0$ *and a sequence* $\{f_j : j = 1, 2, \ldots\} \subset \mathrm{Aut}(\Omega_0)$ *satisfying* $\lim_{j\to\infty} f_j(q) = (0, 0)$.

In fact, one may even conclude that *the model domain* Ω_0 *admits no automorphism orbit accumulating at any boundary point.* This shows that $\mathrm{Aut}\,(\Omega_0)$ is compact with respect to the topology of uniform convergence on compact subsets.

3. Proof of the main theorem

We first need the following modification of convex scaling techniques.

PROPOSITION 3.0.2. *Suppose that* Ω *is a bounded domain in* $\mathbb{C}^2$ *with* $p \in \partial\Omega$ *and* $\Omega \in \mathcal{G}_p$*. Suppose also that there exists* $q \in \Omega$ *and* $f_j \in \mathrm{Aut}(\Omega)$ *(*$j = 1, 2, \ldots$*) satisfying* $\lim\limits_{j\to\infty} f_j(q) = p$*. Then every subsequence of Pinchuk's scaling sequence* Φ_j *defined in the preceding section admits a subsequence, say* Φ_{j_k}*, that converges to a holomorphic embedding* $\widehat{\Phi} : \Omega \to \mathbb{C}^2$ *uniformly on compact subsets of* Ω*. Furthermore,* $\widehat{\Phi}(\Omega)$ *is the limit domain of the sequence* $\Phi_{j_k}(\Omega)$ *in the sense of local Hausdorff set-convergence.*

The detailed proof of this proposition is given in Section 4, in a more general setting.

Now we can finish the proof of Theorem 1.2.1. If $\Omega \in \mathcal{G}_p$ admits an automorphism orbit $f_j(q)$ as in the hypothesis of the proposition above with $f_j(q) \to p$ as $j \to \infty$, then our scaling method applies to Ω. So we are left with the conclusion that Ω is biholomorphic to either the bidisc or the tube domain $\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z > \exp(2 \operatorname{Re} w)\}$. As a result, the same arguments as in the preceding section on "inequivalences" imply that neither conclusion is possible. Such a contradiction originates from the assumption that Ω admits a non-compact automorphism orbit accumulating at p. Therefore the assertion of Theorem 1.2.1 follows. □

4. Convergence of the scaling

We prove the convergence of Pinchuk's scaling on convex domains. We first treat the globally convex case, and then generalize it to the locally convex orbit accumulation point case. Since the proof that we present in complex dimension two actually applies to two-dimensional slices in any dimension, the argument applies to convex domains in all dimensions. In fact, we explain how the proof works in all dimensions in a later part of this section in which we show that the two scaling methods of Pinchuk and Frankel are equivalent.

4.1. Globally convex case. Let $\Omega \subset \mathbb{C}^2$ be a bounded, convex domain admitting

$$0 = (0,0) \in \partial\Omega, \quad q \in \Omega, \quad \text{and} \quad f_j \in \operatorname{Aut}(\Omega) \qquad (j = 1, 2, \ldots)$$

which satisfy $\lim_{j\to\infty} f_j(q) = 0$.

4.1.1. Normalization at the orbit accumulation point. Let us denote by (z,w) the standard coordinates of $\mathbb{C}^2$, and let $z = x+iy, w = u+iv$, where x, y, u, v are real variables.

Changing the coordinates of $\mathbb{C}^2$ at 0 linearly, if necessary, we may assume that Ω near 0 is represented by the inequality

$$v > \psi(x, y, u)$$

where ψ is a non-negative convex function.

4.1.2. Centering the orbit. For each $j = 1, 2, \ldots$, denote by $f_j(q) = (q_{j1}, q_{j2}) \in \mathbb{C}^2$. Then consider the $\mathbb{C}$-affine change of coordinates $\Lambda_j : \mathbb{C}^2 \to \mathbb{C}^2 : (z,w) \mapsto (\zeta, \xi)$ defined by

$$\begin{aligned} \zeta &= z - q_{j1}, \\ \xi &= e^{i\theta_j}(w - q^*_{j2}) - c_j(z - q_{j1}). \end{aligned}$$

We may select the values $\theta_j \in \mathbb{R}$, $c_j \in \mathbb{C}$ and $q^*_{j2} \in \mathbb{C}$ so that we have

- $(q_{j1}, q^*_{j2}) \in \partial\Omega$,
- $\Lambda_j(\Omega)$ is supported by the hyperplane $\operatorname{Im} \xi = 0$,
- $\Lambda_j(\Omega)$ is contained in the half space $\operatorname{Im} \xi > 0$, and
- $\Lambda_j(f_j(q)) = (0, i\epsilon_j)$ for some $\epsilon_j > 0$.

4.1.3. Scaling with the centered orbit. Let

$$\lambda_j = \min\{|z| : z \in \mathbb{C}, (z, \sqrt{-1}\,\epsilon_j) \in \partial(\Lambda_j(\Omega))\}$$

and define the linear map $S_j : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$S_j(z, w) = \left(\frac{z}{\lambda_j}, \frac{w}{\epsilon_j}\right)$$

for each $j = 1, 2, \ldots$. The Pinchuk scaling sequence is

$$\Phi_j \equiv S_j \circ \Lambda_j \circ f_j : \Omega \to \mathbb{C}^2 \quad (j = 1, 2, \ldots).$$

4.1.4. Convergence. We derive the convergence in three steps.

Step 1: Horizontal discs. Choose a point $(\zeta_j, \sqrt{-1}\epsilon_j) \in \partial(\Lambda_j(\Omega))$ such that $|\zeta_j| = \lambda_j$, for each j. Then the set

$$E_j = \{(z, \sqrt{-1}\epsilon_j) \in \mathbb{C}^2 : |z| \leq \lambda_j\}$$

is contained in the domain $\Lambda_j(\Omega)$ and touches the boundary of $\Lambda_j(\Omega)$ at $(\zeta_j, \sqrt{-1}\epsilon_j)$. Therefore the set

$$E = \{(z, \sqrt{-1}) \in \mathbb{C}^2 : |z| < 1\}$$

is contained in $S_j \circ \Lambda_j(\Omega) = \Phi_j(\Omega)$ for every j. (Notice that $E = S_j(E_j)$ for every j.) Moreover, there exists $z_j \in \mathbb{C}$ with $|z_j| = 1$ such that $(z_j, \sqrt{-1}) \in \partial E \cap \partial(\Lambda_j(\Omega))$ for every j.

Step 2: Vertical rhombi. Now consider the planar domain

$$X_j = \{(0, w) \in \mathbb{C}^2 : w \in \mathbb{C}\} \cap \Lambda_j(\Omega).$$

For $\alpha > 0$, consider the rhombus

$$R_j \equiv \{(0, u + \sqrt{-1}v) \in \mathbb{C}^2 : |u| + \alpha|v - \epsilon_j| < \alpha\epsilon_j\}.$$

Since Ω is convex, and since $f_j(q) \to 0$ as $j \to \infty$, we see that there exists a positive real value for α independent of j satisfying $R_j \in \lambda_j(\Omega)$ for every j. Thus the rhombus $R \equiv \{(0, u + \sqrt{-1}v) \in \mathbb{C}^2 : |u| + \alpha|v - 1| < \alpha\}$ is contained in $S_j \circ \Lambda_j(\Omega) = \Phi_j(\Omega)$, since $S_j(R_j) = R$ for every j.

Step 3: Two supporting hyperplanes. Let us consider the convex hull Q of E and R above. It is an open subset of $\mathbb{C}^2$ containing $(0, \sqrt{-1})$. Moreover, $(z_j, i) \in \partial Q \cap \Lambda_j(\Omega)$ for every j. Choosing a subsequence (from f_j) whenever necessary, we have the following:

- The sequence $\Phi_j(\Omega)$ converges to some convex domain $\widehat{\Omega}$ in the topology of local Hausdorff set convergence. (Notice that $\widehat{\Omega}$ exists because of the Blaschke selection theorem on convex bodies, taking subsequences whenever necessary.)
- The point sequence z_j converges to $z_\infty \in \mathbb{C}$ with $|z_\infty| = 1$.
- The supporting planes Π_j of $\Phi_j(\Omega)$ at $(z_j, \sqrt{-1})$ converge to the hyperplane Π, which supports $\widehat{\Omega}$ at $(z_\infty, \sqrt{-1})$.

Recall that the origin $0 = (0,0) \in \mathbb{C}^2$ is always a boundary point of $\Phi_j(\Omega)$ for every j, and the hyperplane

$$\Xi \equiv \{(0,w) : \operatorname{Im} w = 0\}$$

is a supporting hyperplane of $\Phi_j(\Omega)$ for every j.

Since Ξ and Π_j also support Q at 0 and at $(z_j, \sqrt{-1})$, respectively, for every j, the hyperplanes Ξ and Π must support Q at 0 and at $(z_\infty, \sqrt{-1})$, respectively. Consequently, their normal lines are linearly independent over $\mathbb{C}$. This shows that the limit domain $\widehat{\Omega}$ of the Pinchuk scaling sequence is contained in the region bounded by two hyperplanes whose normal lines are $\mathbb{C}$-linearly independent. This implies:

Proposition 4.1.1. *The sequence of maps $\Phi_j : \Omega \to \mathbb{C}^2$ is pre-compact, in the sense that every one of its subsequences has itself a subsequence that converges uniformly on compact subsets.*

We have yet to establish that the subsequential limits are holomorphic embeddings of Ω into $\mathbb{C}^2$.

4.1.5. Holomorphic re-embedding. Pinchuk's original method will now play a decisive role. Let us consider a compact exhaustion K_ℓ of $\widehat{\Omega}$. Then consider the sequence of mappings

$$\Phi_j{}^{-1} : K_\ell \to \Omega$$

for $j = 1, 2, \ldots$. Since Ω is bounded, the sequence is clearly a pre-compact normal family. We also have $\Phi_j(q) = (0, \sqrt{-1})$ for every j, and $(0, i)$ is an interior point of $\widehat{\Omega}$. (See Section 4.1.4 above.) Therefore we must have that $\|(d\Phi_j(q))^{-1}\|$ is a bounded sequence, by the Cauchy estimates. Since Proposition 4.1.1 above has shown that $\|d\Phi_j(q)\|$ is a bounded sequence, Hurwitz's theorem together with local uniform convergence yields:

Proposition 4.1.2. *Every subsequence of the Pinchuk scaling sequence $\Phi_j : \Omega \to \mathbb{C}^2$ contains a subsequence that converges uniformly on compact subsets to a holomorphic mapping, say $\widehat{\Phi}$, of Ω into $\mathbb{C}^2$ whose holomorphic Jacobian determinant is nowhere vanishing in Ω.*

We further demonstrate the following result:

Proposition 4.1.3. *The map $\widehat{\Phi} : \Omega \to \mathbb{C}^2$ in the preceding proposition is globally injective.*

Proof. Seeking a contradiction, we assume the contrary that the map $\widehat{\Phi} : \Omega \to \mathbb{C}^2$ is not injective. Then there exist two distinct points $x, y \in \Omega$ such that

$$\widehat{\Phi}(x) = \widehat{\Phi}(y) = \xi$$

Choose $r > 0$ so that we may have

- $\overline{B(x;2r)} \cup \overline{B(y;2r)} \subset\subset \Omega$,
- $\overline{B(x;2r)} \cap \overline{B(y;2r)} = \emptyset$ (the empty set),
- The restricted maps $\widehat{\Phi}|_{B(x;2r)}$ and $\widehat{\Phi}|_{B(y;2r)}$ are injective, respectively.

Note that $\widehat{\Phi}$ is an open mapping, since it is locally invertible. Thus we choose $s > 0$ to satisfy

$$B(\xi; 2s) \subset \widehat{\Phi}(B(x;r)) \cap \widehat{\Phi}(B(y;r)).$$

Then we have two open sets, $G_1 \equiv \widehat{\Phi}|_{B(x;2r)}{}^{-1}(B(\xi;s))$ in $B(x;2r)$, and $G_2 \equiv \widehat{\Phi}|_{B(y;2r)}{}^{-1}(B(\xi;s))$ in $B(y;2r)$.

Since Φ_j is globally injective for every j, we must have

$$\Phi_j(G_1) \cap \Phi_j(G_2) = \phi \text{ (the empty set) } \forall j = 1, 2, \ldots$$

while

$$\Phi_j(G_1) \cup \Phi_j(G_2) \subset B(\xi; 2s)$$

for all sufficiently large values of j. Altogether, we have

$$\begin{aligned} 0 &\leq \operatorname{dis}(\Phi_j(x), B(\xi;2s) \setminus \Phi_j(G_1)) \\ &\leq \operatorname{dis}(\Phi_j(x), \Phi_j(y)) \\ &\to 0 \text{ as } j \to \infty. \end{aligned}$$

Therefore we have (because of uniform convergence of Φ_j to $\widehat{\Phi}$ on compact subsets of Ω)

$$\begin{aligned} 0 &= \lim_{j\to\infty} \operatorname{dis}(\Phi_j(x), B(\xi;2s) \setminus \Phi_j(G_1)) \\ &= \operatorname{dis}(\widehat{\Phi}(x), B(\xi;2s) \setminus (\widehat{\Phi}(G_1))) \\ &= \operatorname{dis}(\xi, B(\xi;2s) \setminus B(\xi;s)) \\ &= s \\ &> 0\,; \end{aligned}$$

this is a contradiction. □

Altogether, we arrive at the main result of this subsubsection.

THEOREM 4.1.1. *Every subsequence of the Pinchuk scaling sequence $\Phi_j : \Omega \to \mathbb{C}^2$ contains a subsequence that converges uniformly on compact subsets to a holomorphic embedding of Ω into $\mathbb{C}^2$.*

4.1.6. The local Hausdorff limit domain. As above, we denote by $\widehat{\Phi} : \Omega \to \mathbb{C}^2$ the holomorphic re-embedding of Ω into $\mathbb{C}^2$ obtained as a subsequential limit of the Pinchuk scaling sequence. The aim of this subsubsection is to analyze the set $\widehat{\Phi}(\Omega)$.

First we consider the local Hausdorff set-limit of the sequence $\Phi_j(\Omega)$. To be precise, we take the local Hausdorff (subsequential) set limit of the sequence of the closures of $\Phi_j(\Omega)$ first. It is a convex set, since each $\Phi_j(\Omega)$ is convex. We then denote its interior by $\widehat{\Omega}$. Then, as in Pinchuk's original scaling method, we may choose a subsequence from Φ_j (and denote the subsequence by the same notation) so that we have:

- for every compact subset K of Ω the sequence $\Phi_j|_K : K \to \mathbb{C}^2$ converges uniformly;
- $\Phi_j(K) \subset \widehat{\Omega}$ for j large;
- for every compact subset $\widetilde{K}$ of $\widehat{\Omega}$, $\Phi_j^{-1}(\widetilde{K}) \subset \Omega)$ for j large;
- the sequence $\Phi_j^{-1}|_{\widetilde{K}} : \widetilde{K} \to \Omega$ converges uniformly.

Letting K and $\widetilde{K}$ grow to exhaust Ω and $\widehat{\Omega}$, respectively, we see that the limit maps $\widehat{\Phi} \equiv \lim \Phi_j$ and $\Psi \equiv \lim \Phi_j^{-1}$ define the holomorphic mappings $\widehat{\Phi} : \Omega \to \overline{\widehat{\Omega}}$ and $\Psi : \widehat{\Omega} \to \overline{\Omega}$.

Since $\Psi \circ \widehat{\Phi}(q) = q$ and $d(\Psi \circ \widehat{\Phi})(q) = I$ and $\widehat{\Phi} \circ \Psi(0, i) = (0, i)$ and $d(\widehat{\Phi} \circ \Psi)(0, i) = I$, a generalization (see [WU]) of Schwarz's lemma implies that $\widehat{\Phi}(\Omega) \subset \widehat{\Omega}$ and furthermore that $\widehat{\Phi} : \Omega \to \widehat{\Omega}$ is a biholomorphism. In conclusion, we have:

Proposition 4.1.4. *$\widehat{\Phi}(\Omega) = \widehat{\Omega}$.*

Remark 4.1.1. The main result of this paper is a proof of the Greene-Krantz conjecture for a broad class of convex domains in $\mathbb{C}^2$. While many of the techniques apply to higher dimensions, and to non-convex domains, there are a number of technical difficulties to overcome if we are to extend our results to a genuinely more general setting. However, we also remark that the contents of the present section can easily be generalized to all dimensions. One simply need to consider several vertical rhombi (see the proof of Theorem 5.1.1 for a detailed description) following the concept called the *line type*. For the concept of line types see [MCS] and references therein, for instance.

4.2. Generalization to the locally convex case. Let Ω be a bounded domain in $\mathbb{C}^2$. Assume that Ω admits a boundary point $p \in \partial\Omega$ with the following properties:

(i) $\partial\Omega$ is variety-free at p.
(ii) p admits an open neighborhood $U \subset \mathbb{C}^2$ and a one-to-one holomorphic mapping $g : U \to \mathbb{C}^2$ such that $g(U \cap \Omega)$ is convex.
(iii) There exists $f_j \in \mathrm{Aut}(\Omega)$ $(j = 1, 2, \ldots)$ and a point $q \in \Omega$ such that $\lim_{j\to\infty} f_j(q) = p$.

Now we build Pinchuk's scaling sequence for $g(U \cap \Omega)$ for the sequence $g \circ f_j(q)$ accumulating at $g(p) \in g(U \cap \partial\Omega)$; we follow the exposition of Sections 4.1.1–4.1.3.

The domain to scale is now $g(U \cap \Omega)$ and the reference sequence is $g \circ f_j(q)$. Thus we instead write

$$g \circ f_j(q) = (q_{j1}, q_{j2})$$

and then repeat the construction of the scaling sequence as in subsections 4.1.1–4.1.3 with $g(U \cap \partial\Omega)$ replacing $\partial\Omega$ in this case. We shall use the same notation, but in such a way that

- Λ_j denotes the centering which transforms $g(U \cap \partial\Omega)$ to a hypersurface which passes through the origin and is contained in $\{(z, w) \in \mathbb{C}^2 : \operatorname{Im} w > 0\}$,
- $\Lambda_j \circ g \circ f_j(q) = (0, \sqrt{-1}\, \epsilon_j)$ for $\epsilon_j > 0$,
- $\lambda_j = \min\{|z| : (z, i\epsilon_j) \in \Lambda_j \circ g \circ (U \cap \partial\Omega)\}$, and
- $S_j(z, w) = (z/\lambda_j, w/\epsilon_j)$.

Since $\partial\Omega$ is variety-free at p, we immediately have:

LEMMA 4.2.1. *$\lambda_j \to 0$ as $j \to \infty$.*

We also obtain:

LEMMA 4.2.2. *For every compact subset K of Ω, there exists a constant $J_K > 0$ such that $f_j(K) \subset U \cap \Omega$ for all $j > J_K$.*

For a detailed proof, see [KIK].

Now choose a compact exhaustion K_ℓ ($\ell = 1, 2, \ldots$) of Ω. Lemma 4.2.2 implies that for each K_ℓ, we may choose $j_\ell > 0$ such that the composition

$$\Phi_{j_\ell} \equiv S_{j_\ell} \circ \Lambda_{j_\ell} \circ g \circ f_{j_\ell}|_{K_\ell}$$

is a well-defined holomorphic embedding of K_ℓ into $\mathbb{C}^2$. Changing the indices and choosing a subsequence, we may say that

$$\Phi_j \equiv S_j \circ \Lambda_j \circ g \circ f_j|_{K_j} : K_j \to \mathbb{C}^2$$

defines a holomorphic embedding of K_j into $\mathbb{C}^2$ for each j. We again call this Φ_j the *Pinchuk scaling sequence*.

Then the arguments in Sections 4.1.4–4.1.6 apply to the current case, starting from the vertical rhombus and horizontal disc arguments which yield that $S_j \circ \Lambda_j \circ g(U \cap \Omega)$ is supported by two complex independent hyperplanes that converge to two independent hyperplane as before. Altogether, with the notation and terminology above, we arrive at the following result:

THEOREM 4.2.1. *Let $\Omega \subset \mathbb{C}^2$ be a bounded domain that is locally convex and variety-free at $p \in \partial\Omega$. Assume further that there exists an automorphism orbit $\{f_j(q)\}$ in Ω accumulating at p. Then every subsequence of the Pinchuk scaling sequence Φ_j constructed above contains a subsequence that converges uniformly on compact subsets to a holomorphic embedding $\widehat{\Phi}$, say, of Ω into $\mathbb{C}^2$. Moreover, $\widehat{\Phi}(\Omega)$ coincides with the unbounded convex domain which is* *the limit domain* $\lim S_j \circ \Lambda_j \circ g(\Omega \cap U)$ *in the sense of local Hausdorff set-convergence.*

Notice that the last statement of this theorem uses Lemma 4.2.1.

5. Equivalence of the two scaling methods

The main content of Section 4 concerns the scaling method developed by S. Pinchuk. There is, of course, another scaling method that was developed by S. Frankel in the late 1980s (see [FRA]). Frankel's scaling method consists of considering the mappings

$$\omega_j(z) \equiv (df_j(q))^{-1}(f_j(z) - f_j(q))$$

for a *convex Kobayashi hyperbolic* domain $\Omega \subset \mathbb{C}^n$ which admits a non-compact automorphism orbit $f_j(q)$ accumulating at a boundary point of Ω. The main results can be summarized as:

THEOREM (Frankel [FRA]). *Every subsequence of the sequence* $\omega_j : \Omega \to \mathbb{C}^n$ *contains a sequence that converges uniformly on compact subsets of* Ω *that converges to a 1-1 holomorphic mapping, say* $\widehat{\omega}$*, of* Ω *into* $\mathbb{C}^n$*. Moreover,* $\widehat{\omega}(\Omega)$ *is the limit domain (convex unbounded) of the sequence* $(df_j(q))^{-1}(\Omega - f_j(q))$ *in the sense of local Hausdorff set-convergence.*

First notice that the convergence of Pinchuk's scaling on convex domains extends immediately to all dimensions: *after the centering with* λ_j *of the orbit* $f_j(q)$, one first considers the vertical rhombus, and then lets $V_{j,1}$ be its orthogonal complement with respect to the standard Hermitian inner product of $\mathbb{C}^n$. Then $V_{j,1}$ is a complex $n-1$ dimensional subspace of $\mathbb{C}^n$. Then consider

$$\lambda_{j1} = \min\{|\Lambda_j \circ f_j(q) - \zeta| : \zeta \in \Lambda_j(\partial\Omega) \cap V_{j,1}\} .$$

Let $\zeta_{j,1} \in \partial\Omega \cap V_{j,1}$ be a point that realizes the value of $\lambda_{j,1}$. Then consider $V_{j,2}$, which is the orthogonal complement of the vector $\Lambda_j \circ f_j(q) - \zeta_{j,1}$ in $V_{j,1}$. Then choose $\lambda_{j,2}$ and the point $\zeta_{j,2} \in \Lambda_j(\partial\Omega) \cap V_{j,2}$ similarly. Inductively, we are left with the orthonormal vectors

$$\begin{aligned} v_{j,\ell} &= \left[\Lambda_j \circ f_j(q) - \zeta_{j,1}\right] / \left[|\Lambda_j \circ f_j(q) - \zeta_{j,1}|\right] \ , \ j = 1, \ldots, n-1, \\ v_{j,n} &= \left[\Lambda_j \circ f_j(q)\right] / \left[|\Lambda_j \circ f_j(q)|\right], \end{aligned}$$

and the linear mapping $S_j : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$S_j(v_{j,\ell}) = \frac{v_{j,\ell}}{\lambda_{j,\ell}}, \text{ for } \ell = 1, \ldots, n\, ,$$

where $\lambda_{j,n} = |\Lambda_j \circ f_j(q)|$. Then it is easy to see that the set $S_j \circ \Lambda_j(\Omega)$ is contained in the convex set supported by the supporting hyperplanes of the convex hull of the union of vertical rhombus and the "horizontal" discs in each $v_{j,\ell}$ directions. Thus all these supporting hyperplanes are linearly independent over $\mathbb{C}$, and they converge to complex linearly independent hyperplanes.

This natural higher dimensional generalization of our considerations in Section 4.1.4 immediately establishes the convergence of Pinchuk's scaling on convex domains of all dimensions.

Now the convergence of Pinchuk's scaling sequence $S_j \circ \Lambda_j \circ f_j : \Omega \to \mathbb{C}^n$ implies in particular that the sequence $S_j \circ d\Lambda_j \circ df_j(q)$ is a bounded sequence with its determinant bounded away from zero. Therefore we may summarize the situation as follows:

Theorem 5.1.1. *The sequence of $n \times n$ matrices*

$$df_j(q) = A_j \circ S_j \circ d\Lambda_j, \text{ for each } j = 1, 2, \ldots,$$

satisfies

(a) $\|A_j\| \leq C$ *for all* $j = 1, 2, \ldots,$ *and*
(b) $\det A_j \geq c$

for some positive constants c, C that are independent of j.

These arguments show that the end results of the scaling methods of Frankel and of Pinchuk are equivalent up to a non-singular linear change of coordinates on their limit domains. Furthermore, this observation gives a quantitative understanding (for bounded convex domains) for the longstanding intuition that the automorphism sequence that gives an orbit accumulating at the boundary must in fact conform to the boundary Levi geometry at the orbit accumulation point.

6. Concluding remarks

We have proved a version of the Green-Krantz conjecture for convex domains in complex dimension two. The methods that we introduce here suggest that progress in higher dimensions is possible, and we intend to explore that avenue in future papers. It is also of interest to remove the hypothesis of convexity.

Finally, our discussion of the convergence of, and the equivalence of, the standard scaling methods is both a powerful tool in the present work and will prove to be useful in future investigations.

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Kang-Tae Kim, Department of Mathematics, Pohang University of Science and Technology, Pohang 790-784, The Republic of Korea (South)

E-mail address: `kimkt@postech.ac.kr`

Steven G. Krantz, Department of Mathematics, Washington University in St. Louis, St. Louis, MO 63130, USA

E-mail address: `sk@math.wustl.edu`