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COMPLETE MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE AND ALMOST BEST SOBOLEV CONSTANT

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ABSTRACT. We prove that for any given integer $n \geq 2$ and $q \in [1, n)$ there exists a constant $\epsilon = \epsilon(n, q) > 0$ such that any n-dimensional complete Riemannian manifold with nonnegative Ricci curvature, in which the Sobolev inequality

$$\left(\int_M |f|^{\frac{nq}{n-q}}\, dv\right)^{\frac{n-q}{nq}} \leq (K(n,q)+\epsilon)\left(\int_M |\nabla f|^q\, dv\right)^{1/q}, \ \forall f \in C_0^\infty(M)$$

holds with $K(n, q)$ the optimal constant of this inequality in the n-dimensional Euclidean space R^n, is diffeomorphic to R^n.

1. Introduction

Let M be an n-dimensional smooth complete Riemannian manifold. Given $q \in [1, n)$, we set $q^* = \frac{nq}{n-q}$. Let $C_0^\infty(M)$ be the space of smooth functions with compact support in M. Denote by dv and ∇ the Riemannian volume element and the gradient operator of M, respectively. Let $K(n, q)$ be the best constant for the Euclidean Sobolev inequality, that is,

$$K(n,q)^{-1} = \inf_{u \in C_0^\infty(R^n) - \{0\}} \frac{\left(\int_{R^n} |\nabla u|^q\, dv\right)^{1/q}}{\left(\int_{R^n} u^{q^*}\right)^{1/q^*}}. \tag{1.1}$$

It is well known that (cf. [Au1], [Au2], [H1], [Ta])

$$K(n,1) = n^{-1}\omega_n^{-1/n}, \tag{1.2}$$

and for $q > 1$

$$K(n,q) = \frac{1}{n}\left(\frac{n(q-1)}{n-q}\right)^{(q-1)/q}\left(\frac{\Gamma(n+1)}{n\omega_n\Gamma\left(\frac{n}{q}\right)\Gamma\left(n+1-\frac{n}{q}\right)}\right)^{1/n}, \tag{1.3}$$

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where ω_n is the volume of the Euclidean unit ball in R^n and Γ is the Euler function. Moreover, for $q > 1$ the infimum in (1.1) is attained by the function $\left(\lambda + |x|^{\frac{q}{q-1}}\right)^{1-\frac{n}{q}}$, $\lambda > 0$, where $|x|$ is the Euclidean length of the vector x in R^n.

Ledoux [L] proved that a complete n-dimensional Riemannian manifold with nonnegative Ricci curvature, in which one of the Sobolev inequalities

$$\left(\int_M |f|^{q^*}\, dv\right)^{1/q^*} \leq K(n,q)\left(\int_M |\nabla f|^q\, dv\right)^{1/q}, \quad \forall f \in C_0^\infty(M),$$

is satisfied, is isometric to R^n.

In this paper, we are interested in the topology of manifolds M with nonnegative Ricci curvature in which one of the Sobolev inequalities

$$\left(\int_M |f|^{q^*}\, dv\right)^{1/q^*} \leq C\left(\int_M |\nabla f|^q\, dv\right)^{1/q}, \quad \forall f \in C_0^\infty(M),$$

holds for some constant C close to $K(n,q)$. Our main result can be stated as follows.

Theorem 1. *Given an integer $n \geq 2$ and $q \in [1,n)$, there exists a constant $\epsilon = \epsilon(n,q) > 0$ such that any n-dimensional complete Riemannian manifold with nonnegative Ricci curvature, in which the Sobolev inequality*

$$\left(\int_M |f|^{q^*}\, dv\right)^{1/q^*} \leq (K(n,q)+\epsilon)\left(\int_M |\nabla f|^q\, dv\right)^{1/q}, \quad \forall f \in C_0^\infty(M),$$

is satisfied, is diffeomorphic to R^n.

This expresses the stability of the theorem of Ledoux mentioned above.

A recent result of Cheeger and Colding [CC] states that, given an integer $n \geq 2$, there exists a constant $\delta(n) > 0$ such that any n-dimensional complete Riemannian manifold with nonnegative Ricci curvature and $\mathrm{Vol}[B(x,r)] \geq (1-\delta(n))V_0(r)$ for some $p \in M$ and all $r > 0$ is diffeomorphic to R^n. Here $B(x,r)$ is the geodesic ball of radius r with center x and $V_0(r) = \omega_n r^n$ the volume of the Euclidean ball of radius r in R^n. Theorem 1 is a consequence of this theorem and the following result.

Theorem 2. *Let $n \geq 2$ be an integer and let $q \in [1,n)$ and $C \geq K(n,q)$ be given. Let M be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature and satisfying*

$$\left(\int_M |f|^{q^*}\, dv\right)^{1/q^*} \leq C\left(\int_M |\nabla f|^q\, dv\right)^{1/q}, \quad \forall f \in C_0^\infty(M). \tag{1.4}$$

Then for any $x_0 \in M$ and any $r > 0$ we have

$$\mathrm{Vol}[B(x_0,r)] \geq \left(C^{-1}K(n,q)\right)^n V_0(r). \tag{1.5}$$

If we take $C = K(n,q)$ in Theorem 2, then (1.5) together with Bishop's comparison theorem [BC] implies that M is isometric to R^n. Thus Theorem 2 is also a generalization of Ledoux' theorem.

For related results on the topology of manifolds with nonnegative Ricci curvature we refer to [AG], [CX], [C], [SS], [S1], [S2] and [X].

2. Proof of Theorem 2

The case $q = 1$ is well-known. For the sake of completeness, we include a proof here. In the case $q = 1$ the Sobolev inequality is equivalent to the isoperimetric inequality (cf. [SY])

$$|\Omega|^{(n-1)/n} \le C|\partial\Omega|, \tag{2.1}$$

where $\partial\Omega$ is the boundary of a bounded open set Ω in M, and $|\Omega|$ and $|\partial\Omega|$ denote the volume of Ω and the area of $\partial\Omega$, respectively. Let x_0 be a fixed point of M. Since

$$\frac{d}{dr}|B(x_0,r)| = |\partial B(x_0,r)|,$$

we have from (2.1) that

$$|B(x_0,r)|^{(n-1)/n} \le C\frac{d}{dr}|B(x_0,r)|, \quad r > 0.$$

Integrating the above inequality yields

$$|B(x_0,r)| \ge (nC)^{-n} r^n = \left(C^{-1}K(n,1)\right)^n V_0(r).$$

This proves the case $q = 1$ of Theorem 2.

Now assume that (1.3) is satisfied for some $q \in (1,n)$. Let $\beta > 0$ and set $f = \beta^{-1}\, d(\cdot,x_0)$, where $d(\cdot,x_0)$ is the distance function from x_0 on M. We set, for any $\lambda > 0$,

$$F(\lambda) = \frac{1}{n-1}\int_M \frac{dv}{\left(\lambda + f^{\frac{q}{q-1}}\right)^{n-1}}.$$

By Fubini's theorem we have, for $\lambda > 0$,

$$\begin{aligned} F(\lambda) &= \frac{q}{q-1}\int_0^{+\infty} |\{x : d(x_0,x) < \beta s\}| \frac{s^{\frac{1}{q-1}}\, ds}{(\lambda + s^{\frac{q}{q-1}})^n} \\ &= \frac{q}{q-1}\int_0^{+\infty} |B(x_0,\beta s)| \frac{s^{\frac{1}{q-1}}\, ds}{(\lambda + s^{\frac{q}{q-1}})^n}. \end{aligned} \tag{2.2}$$

Bishop's comparison theorem [BC] tells us that $\mathrm{Vol}[B(x_0,s)] \le \omega_n s^n$ for all $s > 0$. Thus $0 \le F(\lambda) < \infty$ and F is differentiable.

The extremal functions in the Sobolev inequality with $C = K(n,q)$ in R^n are the functions $\left(\lambda + |x|^{\frac{q}{q-1}}\right)^{1-\frac{n}{q}}$, $\lambda > 0$. We will use similar functions on M

in the Sobolev inequality (1.4) to obtain a differential inequality that allows comparison to the extremal Euclidean case.

By a simple approximation procedure, we can apply (1.4) to the function $(\lambda + f^{\frac{q}{q-1}})^{1-\frac{n}{q}}$ to get

$$\left(\int_M \frac{dv}{\left(\lambda + f^{\frac{q}{q-1}}\right)^{n-1}}\right)^{1/q^*} \le C\left(\frac{n-q}{q-1}\right)\left(\int_M \frac{f^{\frac{q}{q-1}}\, dv}{\left(\lambda + f^{\frac{q}{q-1}}\right)^{n}}\right)^{1/q} \tag{2.3}$$

for every $\lambda > 0$. Set

$$d = \left(C\left(\frac{n-q}{q-1}\right)\right)^{-q}. \tag{2.4}$$

Then (2.3) implies

$$d\left(\int_M \frac{dv}{\left(\lambda + f^{\frac{q}{q-1}}\right)^{n-1}}\right)^{q/q^*} \le \int_M \frac{dv}{\left(\lambda + f^{\frac{q}{q-1}}\right)^{n-1}} - \int_M \frac{dv}{\left(\lambda + f^{\frac{q}{q-1}}\right)^{n}},$$

and hence

$$d\left(-F'(\lambda)\right)^{1-\frac{q}{n}} - \lambda F'(\lambda) \le (n-1)F(\lambda) \tag{2.5}$$

for every $\lambda > 0$.

The idea now is to compare the solutions of (2.5) to the solutions H of the differential equation

$$d\left(-H'(\lambda)\right)^{1-\frac{q}{n}} - \lambda H'(\lambda) = (n-1)H(\lambda). \tag{2.6}$$

It is easy to check that a particular solution of (2.6) is given by

$$H_1(\lambda) = \frac{B}{\lambda^{\frac{n}{q}-1}}, \tag{2.7}$$

where

$$\begin{aligned} B &= \frac{q}{n-q}\left(\frac{d(n-q)}{n(q-1)}\right)^{n/q} \\ &= \left(C^{-1}K(n,q)\right)^n \cdot \frac{q}{n-q}\left(\frac{\left(K(n,q)\left(\frac{n-q}{q-1}\right)\right)^{-q}(n-q)}{n(q-1)}\right)^{n/q} \\ &= \left(C^{-1}K(n,q)\right)^n \cdot \frac{1}{n-1}\int_{R^n} \frac{dx}{\left(1 + |x|^{\frac{q}{q-1}}\right)^{n-1}}. \end{aligned} \tag{2.8}$$

We set, for any $\lambda > 0$,

$$H_0(\lambda) = \frac{1}{n-1} \int_{R^n} \frac{dx}{\left(\lambda + |x|^{\frac{q}{q-1}}\right)^{n-1}} = \frac{q}{q-1} \int_0^{+\infty} \omega_n s^n \frac{s^{\frac{1}{q-1}}}{(\lambda + s^{\frac{q}{q-1}})^n} \, ds. \tag{2.9}$$

Then

$$\begin{aligned} H_1(\lambda) &= \left(C^{-1}K(n,q)\right)^n \cdot \frac{1}{\lambda^{\frac{n}{q}-1}} \cdot \frac{1}{n-1} \int_{R^n} \frac{dx}{\left(1 + |x|^{\frac{q}{q-1}}\right)^{n-1}} \\ &= \left(C^{-1}K(n,q)\right)^n \frac{1}{n-1} \int_{R^n} \frac{dx}{\left(\lambda + |x|^{\frac{q}{q-1}}\right)^{n-1}} \\ &= \left(C^{-1}K(n,q)\right)^n \cdot H_0(\lambda). \end{aligned} \tag{2.10}$$

Since $F(\lambda)$ and $H_1(\lambda)$ satisfy (2.5) and (2.6), respectively, one can use the arguments in [L] to show that if $F(\lambda_0) < H_1(\lambda_0)$ for some $\lambda_0 > 0$, then $F(\lambda) < H_1(\lambda)$ for every $\lambda \le \lambda_0$. Also, as proved in [L], the local geometry gives

$$\liminf_{\lambda \to 0} \frac{F(\lambda)}{H_0(\lambda)} \ge \beta^n > 1,$$

and so

$$\liminf_{\lambda \to 0} \frac{F(\lambda)}{H_1(\lambda)} = \left(\frac{C}{K(n,q)}\right)^n \liminf_{\lambda \to 0} \frac{F(\lambda)}{H_0(\lambda)} > 1.$$

Thus we have

$$F(\lambda) \ge H_1(\lambda)$$

for every $\lambda > 0$, that is,

$$\int_0^\infty \left(|B(x_0, \beta s)| - (C^{-1}K(n,q))^n V_0(s)\right) \frac{s^{\frac{1}{q-1}} \, ds}{\left(\lambda + s^{\frac{q}{q-1}}\right)^n} \ge 0, \quad \lambda > 0.$$

Letting $\beta \to 1$, we get

$$\int_0^\infty \left(|B(x_0, s)| - (C^{-1}K(n,q))^n V_0(s)\right) \frac{s^{\frac{1}{q-1}} \, ds}{\left(\lambda + s^{\frac{q}{q-1}}\right)^n} \ge 0, \quad \lambda > 0. \tag{2.11}$$

By the Bishop-Gromov comparison theorem (cf. [BC], [Ch], [GLP]) the function $|B(x_0,s)|/V_0(s)$ is decreasing. Set $L = (C^{-1}K(n,q))^n$ and let

$$L_0 = \lim_{s \to +\infty} \frac{|B(x_0, s)|}{V_0(s)}.$$

In order to prove (1.5), it suffices to show that $L_0 \geq L$. Suppose on the contrary that $L_0 = L - \epsilon_0$ for some $\epsilon_0 > 0$. Then there exists an $N > 0$ such that

$$\frac{|B(x_0,s)|}{V_0(s)} \leq L - \frac{\epsilon_0}{2},\ \forall s \geq N. \tag{2.12}$$

Substituting (2.12) into (2.11) and observing (2.7)–(2.10), we obtain that, for every $\lambda > 0$,

$$\begin{aligned}
0 &\leq \int_0^\infty \left(\frac{|B(x_0,s)|}{V_0(s)} - L\right) \frac{\omega_n s^{n+\frac{1}{q-1}}\,ds}{\left(\lambda + s^{\frac{q}{q-1}}\right)^n} \\
&\leq \int_0^N \frac{|B(x_0,s)|}{V_0(s)} \cdot \frac{\omega_n s^{n+\frac{1}{q-1}}\,ds}{\left(\lambda + s^{\frac{q}{q-1}}\right)^n} + \int_N^{+\infty} \left(L - \frac{\epsilon_0}{2}\right) \cdot \frac{\omega_n s^{n+\frac{1}{q-1}}\,ds}{\left(\lambda + s^{\frac{q}{q-1}}\right)^n} \\
&\qquad\qquad - L \int_0^{+\infty} \frac{\omega_n s^{n+\frac{1}{q-1}}\,ds}{\left(\lambda + s^{\frac{q}{q-1}}\right)^n} \\
&= \int_0^N \frac{|B(x_0,s)|}{V_0(s)} \cdot \frac{\omega_n s^{n+\frac{1}{q-1}}\,ds}{\left(\lambda + s^{\frac{q}{q-1}}\right)^n} - \int_0^N \left(L - \frac{\epsilon_0}{2}\right) \frac{\omega_n s^{n+\frac{1}{q-1}}\,ds}{\left(\lambda + s^{\frac{q}{q-1}}\right)^n} \\
&\qquad\qquad - \frac{\epsilon_0}{2} \int_0^{+\infty} \frac{\omega_n s^{n+\frac{1}{q-1}}\,ds}{\left(\lambda + s^{\frac{q}{q-1}}\right)^n} \\
&\leq \left(1 - L + \frac{\epsilon_0}{2}\right) \int_0^N \frac{\omega_n s^{n+\frac{1}{q-1}}\,ds}{\left(\lambda + s^{\frac{q}{q-1}}\right)^n} - \frac{\epsilon_0}{2} \cdot \frac{(q-1)}{q} \left(\frac{C}{K(n,q)}\right)^n \cdot B \cdot \lambda^{1-\frac{n}{q}} \\
&\leq \left(1 - L + \frac{\epsilon_0}{2}\right) \omega_n \lambda^{-n} \int_0^N s^{n+\frac{1}{q-1}}\,ds - \frac{\epsilon_0}{2} \cdot \frac{(q-1)}{q} \left(\frac{C}{K(n,q)}\right)^n \cdot B \cdot \lambda^{1-\frac{n}{q}} \\
&= \left(1 - L + \frac{\epsilon_0}{2}\right) \omega_n \lambda^{-n} \cdot \frac{1}{\left(n+1+\frac{1}{q-1}\right)} N^{n+1+\frac{1}{q-1}} \\
&\qquad\qquad - \frac{\epsilon_0}{2} \cdot \frac{(q-1)}{q} \left(\frac{C}{K(n,q)}\right)^n \cdot B \cdot \lambda^{1-\frac{n}{q}}.
\end{aligned}$$

This implies that

$$0 \leq \left(1 - L + \frac{\epsilon_0}{2}\right) \cdot \frac{\omega_n N^{n+1+\frac{1}{q-1}}}{\left(n+1+\frac{1}{q-1}\right)} \cdot \lambda^{\frac{n}{q}-n-1} - \frac{\epsilon_0}{2} \cdot \frac{(q-1)}{q} \left(\frac{C}{K(n,q)}\right)^n \cdot B$$

for every $\lambda > 0$. Letting $\lambda \to +\infty$ yields a contradiction. Thus $L \geq L_0$.

This completes the proof of Theorem 2. $\square$

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