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BRILL–NOETHER THEORY FOR GENERAL BRANCHED COVERINGS OF $\mathbf{P}^1$

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ABSTRACT. We study the Brill–Noether theory of special divisors of a general branched covering of the complex projective line with one total ramification point, while the other ramification points are ordinary.

1. The statements

Fix integers g, k with $2 \le k \le g$. Let $M_g(k)$ be the set of all smooth complex genus g curves, X, such that there exists $P \in X$ with $h^0(X, \mathbf{O}_X(kP)) \ge 2$; this algebraic set is usually denoted by $D_{k,k}$. By [A1], [A2], [L1], or [L2] $M_g(k)$ is an irreducible subvariety of M_g with $\dim(M_g(k)) = 2g - 3 + k$. Let $M_g[k]$ be the set of all pairs (X, P) with $X \in M_g(k)$, $P \in X$ and $h^0(X, \mathbf{O}_X(kP)) \ge 2$. We have $M_g(g) = M_g$, and a general $X \in M_g$ has $(g-1)g(g+1)/6$ Weierstrass points, all of them of weight 1, i.e., with $h^0(X, \mathbf{O}_X((g-1)P)) = 1$ and $h^1(X, \mathbf{O}_X((g+1)P)) = 0$. By [D2, Th. 4.9], if $2 < k < g$, for a general $X \in M_g(k)$ there is a unique $P \in X$ such that $(X, P) \in M_g[k]$.

Following [A1] and [A2], for integers $w > k \ge 2$ with $k + w$ even, let $\mathrm{WH}[k, w]$ be the set of all pairs (X, f) with X a smooth connected curve of genus $(w-k)/2$ and $f \colon X \to \mathbf{P}^1$ a branched covering of degree k with one total ramification point and $w - 1$ simple ramification points with different images in $\mathbf{P}^1$. For integers $w \ge 3k \ge 6$ with $k + w$ even, let $\mathrm{WH}(k, w)$ be the set of all smooth curves, X, of genus $(w - k)/2$, such that there is $f \colon X \to \mathbf{P}^1$ with $(X, f) \in \mathrm{WH}[k, w]$. By [A1, Th. 2.3] or [A2] or [D2, Lemma 3.2] $\mathrm{WH}(k, 2g+k)$ is connected and $M_g(k)$ is the closure of $\mathrm{WH}(k, 2g + k)$. In particular, for integers x, k, g with $2 \le x < k \le g$ we have $M_g(x) \subset M_g(k)$. For all integers g, r, and d set $\rho(g, r, d) := g - (r + 1)(g + r - d)$ (the so-called Brill–Noether number).

Our main results are as follows.

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THEOREM 1.1. *Fix integers* k, g, r, d *with* $3 \leq k < g$, $r > 0$, $d - g < r \leq \min\{k-2, [(g-3)/2]\}$ *and* $\rho(g,r,d) \geq 0$. *Let* X *be a general element of* $\mathrm{WH}(k, 2g+k)$ *and* $P \in X$ *the total ramification point of the associated degree* k *pencil* $X \to \mathbf{P}^1$. *Then there exists an irreducible component* Z *of* $W^r_d(X)$ *with* $\dim(Z) = \rho(g,r,d)$ *and such that for a general* $M \in Z$ *we have* $h^0(X, M(-kP)) = 0$.

THEOREM 1.2. *Fix integers* g, k, d *with* $3 < k < [(g+3)/2]$ *and* $2d-g-2 < 0$. *Let* X *be a general element of* $M_g(k)$ *and* L *the associated degree* k *pencil. Then for every* $M \in \mathrm{Pic}(X)$ *with* $\deg(M) = d$ *and* $h^0(X,M) \geq 2$ *we have* $h^0(X, M \otimes L^*) > 0$.

THEOREM 1.3. *Fix integers* g, k, d *with* $g \geq 5$, $k \geq [(g+3)/2]$ *and* $2d - g - 2 < 0$. *Let* X *be a general element of* $M_g(k)$. *Then for every* $M \in \mathrm{Pic}(X)$ *with* $\deg(M) = d$ *we have* $h^0(X,M) \leq 1$.

THEOREM 1.4. *Fix integers* g, k, d *with* $g \geq 5$, $k \geq [(g+3)/2]$ *and* $g + 2 \leq 2d \leq 2g$. *Let* X *be a general element of* $M_g(k)$. *Then we have* $\dim(W^1_d(X)) = \rho(g,1,d) = 2d - g - 2$.

We do not know if these results are true for the general member, X, of other subvarieties, T, of M_g. If T is contained in the locus of the k-gonal curves, where $k < [(g+3)/2]$, the proofs of [CM1] and [CM2] show that essentially we need only that X has a unique degree k pencil, L, that L satisfies the conditions of Remark 2.2 below (i.e., $h^0(X, L^{\otimes t}) = t+1$ if $t \leq [g/(k-1)]$ and $h^1(X, L^{\otimes t}) = 0$ if $t > [g/(k-1)]$), and that $\dim(T)$ is rather large.

2. The proofs

REMARK 2.1. By [D1, Th. 2], for a general $X \in M_g(k)$ there exists a Weierstrass point $P \in X$ with semigroup consisting only of multiples of k until after the greatest gap, i.e., such that for every integer $x \geq 0$, we have $h^0(X, \mathbf{O}_X(xP)) = \max\{1 + [x/k], x + 1 - g\}$. If $k \geq 1 + g/2$, this also follows from [EH]. In particular, $\mathbf{O}_X(kP)$ has no base point and $h^0(X, \mathbf{O}_X(kP)) = 2$. If $k \geq 1 + g/2$, the condition on the Weierstrass semigroup of P means that $h^1(X, \mathbf{O}_X((g+1)P)) = 0$. By its very definition, for any pair $(X, f) \in \mathrm{WH}[k, w]$ there exists a point $P \in X$ which is a total ramification point of f and hence satisfies $\mathbf{O}_X(kP) \cong f^*(\mathbf{O}_{\mathbf{P}^1}(1))$. Thus $h^0(X, \mathbf{O}_X(kP)) \geq 2$, and $\mathbf{O}_X(kP)$ is spanned by its global sections, and therefore P is a Weierstrass point of X. It is easy to check that for a general (X, f) the corresponding total ramification point P satisfies $h^0(X, \mathbf{O}_X(kP)) = 2$. By [Co], if $k \geq 3$, all other Weierstrass points of X are normal, i.e., their gap sequence is $(1, 2, 3, \ldots, g-2, g-1, g+1)$. This is obviously false if $k = 2$ (i.e. for hyperelliptic curves).

Remark 2.2. Fix a general $X \in M_g(k)$, $2 < k < g$. By [D2, Th. 4.9] there exists a unique $P \in X$ with $h^0(X, \mathbf{O}_X(kP)) \geq 2$. Set $L := \mathbf{O}_X(kP)$. By Remark 2.1 we have $h^0(X, L^{\otimes t}) = t+1$ if $0 \leq t \leq g/(k-1)$ and $h^0(X, L^{\otimes t}) = kt + 1 - g$ (i.e., $h^1(X, L^{\otimes t}) = 0$) if $t > g/(k-1)$.

The next result is implicit in the Arbarello stratification WH$[x, w]$, $g = (w-x)/2$, of M_g. It can probably be deduced from [Co], but we prefer to give a direct proof because we will use that proof quite often later on.

Proposition 2.3. *Fix integers g, k with $3 \leq k < [(g+3)/2]$. Let X be a general k-gonal curve of genus g. Then the first non-gap of all Weierstrass points of X is g.*

Proof. Assume that the result is not true. Thus for a general k-gonal curve X we can find $Q \in X$ and an integer t with $2 \leq t \leq g-1$ and $h^0(X, \mathbf{O}_X(tQ)) \geq 2$. We choose t minimal (for general X), so that $h^0(X, \mathbf{O}_X(tQ)) = 2$ and $\mathbf{O}_X(tQ)$ is spanned by its global sections. Denote by L the unique k-gonal pencil of X ([AC2, Th. 2.6]). By the generality of X, the curve X has no pencil of degree at most $k-1$ ([AC2, Th. 2.6]). Hence we have $h^0(X, L) = 2$, and L is spanned by its global sections. The set of all k-gonal curves of genus g has dimension $2g+2k-5$, while the set of all smooth curves of genus g which are multiple coverings of some curve of genus > 0 has dimension at most $2g-2$ ([L]). Thus by the generality of X the morphism $v\colon X \to \mathbf{P}^1$ induced by L does not factor through an intermediate curve. Hence the pair $(L, \mathbf{O}_X(tQ))$ induces a birational morphism $u\colon X \to \mathbf{P}^1 \times \mathbf{P}^1$ with $u(X)$ of type (k, t). The elementary theory of the deformation of branched coverings of smooth curves implies that for general X the pencil $v\colon X \to \mathbf{P}^1$ has exactly $2g + 2k - 2$ ramification points, all of them ordinary ramification points, and that no two of them are on the same fiber of v (see [L1] or apply [AC1, Scolium 5.6]).

We first assume that Q is not one of these ramification points; the other subcase will be discussed at the end of the proof. Our assumption implies that $u(Q)$ is a smooth point of a local branch of $u(X)$ at $u(Q)$. Since the second factor of u is just the map induced by $\mathbf{O}_X(tQ)$, there is a smooth branch of $u(X)$ at $u(Q)$ which contains a length t subscheme, Z, of the line, D_0, of type $(1, 0)$ of $\mathbf{P}^1 \times \mathbf{P}^1$ containing $u(Q)$ and with $Z_{\mathrm{red}} = \{u(Q)\}$. Since the intersection number of $u(X)$ with D_0 is t, this implies that $u(X)$ is unibranch at $u(Q)$. Hence $u(X)$ is smooth at $u(Q)$.

Vice versa, let $C \subset \mathbf{P}^1 \times \mathbf{P}^1$ be an integral curve of type (k, t) that contains Z and is smooth at Z_{red}. Assume that the normalization, Y, of C has genus g. Then Y has gonality at most k and $u(Q)$ corresponds to a Weierstrass point of Y with t in its gap sequence.

Fix $B \in \mathbf{P}^1 \times \mathbf{P}^1$. Choosing a basis of $H^0(X, L)$ and $H^0(X, \mathbf{O}_X(tQ))$ we rigidify all triples (X, L, Q) in such a way that for the associated morphism $u\colon X \to \mathbf{P}^1 \times \mathbf{P}^1$ we have $u(Q) = B$. Hence there is a quasi-finite covering, U, of $M_g(k)$ such that for every element of U (say, corresponding to a curve X with exceptional Weierstrass point Q and with birational morphism $u\colon X \to \mathbf{P}^1 \times \mathbf{P}^1$) we have $u(Q) = B$.

Set $A_0 := \mathbf{P}^1 \times \mathbf{P}^1$. Let A_1 be the blowing-up of A_0 at B. Denote by E the exceptional divisor of A_1 and let D_1 be the strict transform of D_0 in A_1. Hence $\mathrm{card}(D_1 \cap E) = 1$. Let A_2 be the blowing-up of A_1 at the point $D_1 \cap E$, and let D_2 be the strict transform of D_1 in A_2, F the strict transform of E in A_2, and E' the exceptional divisor of A_2. Hence $F + E'$ is the total transform of E and we have $(F + E') \cdot E' = 0$, $E'^2 = (F + E')^2 = -1$. We have $\mathrm{card}(D_2 \cap E') = 1$. Let A_3 be the blowing-up of A_2 at the point $D_2 \cap E'$, and let D_3 be the strict transform of D_2 in A_3. We continue this construction until we arrive at a surface A_t obtained from A_0 making t blowing-ups that contains a smooth rational curve, D_t, which is the strict transform of D_0 and has the following properties.

Let I and J be the total transforms in A_t of the generators of type $(1, 0)$ and of type $(0, 1)$, respectively, of $\mathrm{Pic}(A_0)$. Let F_i be the strict stransform in A_t of the exceptional divisor of the blowing-up $A_i \to A_{i-1}$ with the convention that F_t is the exceptional divisor of $A_t \to A_{t-1}$. Hence $\mathrm{Pic}(A_t) \cong \mathbf{Z}^{\otimes(t+2)}$ with generators I and J, and F_i, $1 \le i \le t$, which are all smooth, irreducible, and rational.

For $1 \le i \le t$ set $E_i := \sum_{i \le j \le t} F_j$. Hence E_i is the total transform of the exceptional divisor of the blowing-up $A_i \to A_{i-1}$. Thus $E_i^2 = -1$ for every i, $E_i \cdot I = E_i \cdot J = 0$ for every i, and $E_i \cdot F_j = 0$ if $i < j \le t+1$. Hence $E_i \cdot E_j = 0$ if $i < j$. The canonical line bundle K_{A_t} of A_t is $-2I - 2J + \sum_{1 \le j \le t} E_j$. By assumption, for every curve X associated to U the strict transform, D, of $u(X)$ in A_t is an element of $|kI + tJ - \sum_{1 \le j \le t} E_j|$ with geometric genus g. Hence $-D \cdot K_{A_t} = 2k + 2t - t = 2k + t$. The family, M, of all such strict transforms has dimension $2g + 2k - 5 + 4$. Indeed, the subgroup of $\mathrm{Aut}(\mathbf{P}^1 \times \mathbf{P}^1)$ fixing B has dimension 4; by a result proved in [AC2] and contained in [Co] (see [CKM, bottom of p. 147]) we have $\dim(M) \le -D \cdot K_{A_t} + g - 1 = 2k + t + g - 1 < 2g + 2k - 1$, which is a contradiction.

Now assume that Q is a ramification point of the pencil v. Since v has only ordinary ramification points, $u(X)$ has an ordinary cusp at $u(Q)$, the line D_0 is tangent to $u(X)$ at $u(Q)$ and contains no other point of $u(X)$. We repeat the previous construction. Let D be the strict transform of $u(X)$ in A_t and Y the strict transform of $u(X)$ in A_1. Since $u(X)$ has an ordinary cusp at $u(Q)$ with the tangent to D_0 as tangent cone of $u(X)$ at Q, $Y \in |kI + tJ - 2E_1|$. Iterating we obtain $D \in |kI + tJ - 2E_1 - \sum_{1 < j < t} E_j|$ and complete the proof as in the previous case. □

Proof of Theorem 1.1. Set $L := \mathbf{O}_X(tP)$ and $r := r(d, g, r)$. First assume $r = k - 2$ and $k < [(g + 3)/2]$. Let R be the degree k pencil on a general k-gonal curve C. By Remark 2.2 and the same assertion for C proved in [B] or

[CKM, Prop. 1.1], for any integer t we have $h^1(X, L^{\otimes t}) = h^1(C, R^{\otimes t})$. Hence we can apply verbatim the proof of [CM2, 2.3.1] and obtain the case $r = k-2$, $k < [(g+3)/2]$ of Theorem 1.1.

Now assume $r < k-2$ and $k < [(g+3)/2]$. Let Y be a general element of $M_g(r+2)$ and Q the associated total ramification point. Set $R := \mathbf{O}_Y(kQ)$. We have $h^0(Y, R) = 2$, and $(k-r-2)Q$ is the base divisor of R. By the deformation theory of coverings or the more general theory of admissible coverings introduced in [HM, §4] (applying, for instance, part (a) of [HM, Th. 5] and keeping track of the ramification of order $r+2$ at Q, or using [AC1, Scolium 5.1], or the method of [L1]), the pair (X, R) is the flat limit of a flat family of smooth k-gonal curves, i.e., we may regard (Y, Q) as the limit of a flat family of general elements, say (X_λ, Q_λ), of $M_g[k]$ in which the pencil $\mathbf{O}_{X\lambda}(kQ_\lambda)$ has as flat limit the line bundle R. Hence we can use the proof of [CM2, 2.3.2] to reduce this case to the case $k = r+2$ previously proved.

Now assume $k \geq [(g+3)/2]$ and set $k' := [(g+1)/2]$. By assumption we have $r \leq k'-2$. We apply the degeneration argument used in the second part to reduce the case (r, k) to the case (r, k') proved in the first two parts, and hence obtain the result. □

Proof of Theorems 1.2 and 1.3. Assume the result is false and take d minimal among all counterexamples and fix a corresponding line bundle M. We first assume $h^0(X, M) \geq 3$. Since $h^0(X, M(-A)) = h^0(X, M) - 1$ for a general $A \in X$, we could take $M(-A)$ instead of M, contradicting the minimality of d. Hence $h^0(X, M) = 2$. If M has a base point A, then similarly $M(-A)$ contradicts the minimality of d. Hence M is spanned by its global sections. Therefore the pair (L, M) induces a morphism $u\colon X \to \mathbf{P}^1 \times \mathbf{P}^1$. Since X is general in $M_g(k)$, it has a unique non-ordinary Weierstrass point ([Co]), and the proof in [Co] implies that X is not a multiple covering of a smooth curve of genus at least one. The condition $h^0(X, M \otimes L^*) = 0$ and the fact that X is not a multiple covering of a smooth curve of genus at least one imply that u is birational. We then let D be the line of $\mathbf{P}^1 \times \mathbf{P}^1$ with type $(0,1)$, and apply the proof of Proposition 2.3 with d instead of t and with the two families of lines of $\mathbf{P}^1 \times \mathbf{P}^1$ interchanged. The same argument gives Theorem 1.3. □

Proof of Theorem 1.4. The inequality $\dim(W^1_d(X)) \geq \rho(g, 1, d)$ is obvious by the existence theorem for special divisors. Hence it suffices to prove the inequality $\dim(W^1_d(X)) \leq \rho(g, 1, d)$. By [ACGH, VII ex. C], it is sufficient to consider the case $d = [(g+3)/2]$. Applying semicontinuity as in the proof of Proposition 2.3, we see that it is sufficient to prove the case $k = [(g+3)/2]$.

Fix any $M \in W^1_{[(g+3)/2]}(X)$. By Theorem 1.3 we have $h^0(X, M(-P)) \leq 1$ for every $P \in X$, i.e., M has no basepoint and $h^0(X, M) = 2$. Applying the proof of Theorem 1.1 (i.e., of Proposition 2.3) with respect to the invariants

$k = [(g+3)/2]$ and $d = [(g+3)/2]$, we obtain $\dim(W^1_{[(g+3)/2]}(X)) \le \rho(g, 1, [(g+3)/2])$ for general X. □

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