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OPERATORS ON GENERALIZED POWER SERIES

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ABSTRACT. Given a ring C and a totally (resp. partially) ordered set of "monomials" $\mathfrak{M}$, Hahn (resp. Higman) defined the set of power series $C[[\mathfrak{M}]]$ with well-ordered (resp. Noetherian or well-quasi-ordered) support in $\mathfrak{M}$. This set $C[[\mathfrak{M}]]$ can usually be given a lot of additional structure: if C is a field and $\mathfrak{M}$ a totally ordered group, then Hahn proved that $C[[\mathfrak{M}]]$ is a field. More recently, we have constructed fields of "transseries" of the form $C[[\mathfrak{M}]]$ on which we defined natural derivations and compositions.

In this paper we develop an operator theory for generalized power series of the above form. We first study linear and multilinear operators. We next isolate a big class of so-called Noetherian operators $\Phi : C[[\mathfrak{M}]] \to C[[\mathfrak{N}]]$, which include (when defined) summation, multiplication, differentiation, composition, etc. Our main result is the proof of an implicit function theorem for Noetherian operators. This theorem may be used to explicitly solve very general types of functional equations in generalized power series.

1. Introduction

In [Hah07], Hahn introduced an abstract framework for algebraic computations on power series with generalized exponents like

$$\begin{aligned} f &= 1 + z^{\log 2} + z^{\log 3} + z^{\log 4} + \cdots ; \\ g &= 1 + z + z^2 + z^e + z^3 + z^{1+e} + z^4 + z^{2+e} + z^5 + z^{2e} + z^{3+e} + \cdots ; \\ h &= 1 + z^{1/2} + z^{3/4} + z^{7/8} + \cdots + z + z^{3/2} + z^{7/4} + \cdots + z^2 + \cdots + \cdots . \end{aligned}$$

One of his main results states that, given a field C and a totally ordered monomial group $\mathfrak{M}$, the set $C[[\mathfrak{M}]]$ of series $f : C \to \mathfrak{M}$ with well-ordered support in $\mathfrak{M}$ carries a natural field structure. This result was generalized by Higman [Hig52] to the case of partially ordered monomial monoids $\mathfrak{M}$.

More recently, Dahn and Göring [DG86] and Écalle [É92] constructed so-called fields of "transseries", which are fields of generalized power series $C[[\mathfrak{M}]]$

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in the sense of Hahn, with additional structure, such as exponentiation, differentiation, integration, composition, etc. Examples of transseries are

$$\begin{aligned} \varphi &= x + \log x + \log\log x + \log\log\log x + \cdots ; \\ \psi &= e^{e^x + e^{x/2} + e^{x/3} + \cdots} + e^{e^{x/2} + e^{x/3} + e^{x/4} + \cdots} + e^{e^{x/3} + e^{x/4} + e^{x/5} + \cdots} + \cdots ; \\ \xi &= \Gamma(x) = \sqrt{2\pi} e^{x \log x - x - \frac{1}{2} \log x} + \cdots \end{aligned}$$

In [vdH97], we have shown how to differentiate, integrate and compose such transseries, and how to solve algebraic differential equations (whenever possible).

In this paper, we will be concerned with the development of an abstract operator theory for generalized power series, in the setting of partially ordered monomial sets introduced by Higman. We start by recalling some basic results about Noetherian orderings (also called well-quasi-orderings) in Section 2. In Higman's setting, generalized power series have Noetherian support. For this reason, we shall actually call them Noetherian series.

In Section 3, we recall the definition of Noetherian series and develop the theory of strongly linear and strongly multilinear operators. More precisely, it is possible to define a notion of infinite summation on algebras $C[[\mathfrak{M}]]$ of Noetherian power series. One may think of this as something analogous to normal summable families in analysis. Strongly linear mappings will then be linear mappings which also preserve infinite summation.

The remainder of this article focuses on the resolution of certain functional equations. Translated into the terminology of operators, this comes down to the isolation of nice classes of operators on which some kind of implicit function theorem holds (actually, we will rather prove "parameterized fixed point theorems"). As a basic example, one would like to solve implicit equations like

$$f = g + f' f'' \tag{1.1}$$

in fields of transseries, where g is a sufficiently small parameter (say $g = o(e^{-x})$) and f the unknown.

In Section 4, we start by developing a theory of continuous and contracting functions for Noetherian series and we will prove the existence of a solution $f = \Psi(g)$ to equations like (1.1) using the technique of fixed points. Actually, we will prove an implicit function theorem which is very similar to fixed point theorems from [PC90] and [PCR93], although our proof is more constructive.

A more natural and even more explicit way of getting solutions to (1.1) would be to replace the left hand side by the right hand side in a recursive manner, while expanding all sums. This would lead to a formal solution of the form

$$\begin{aligned} f &= g + f'f'' \\ &= g + g'g'' + (f'f'')'g'' + g'(f'f'')'' + (f'f'')'(f'f'')'' \\ &= g + g'g'' + (g'g'')'g'' + g'(g'g'')'' + \cdots . \end{aligned}$$

The main difficulty then resides in proving that the obtained formal expansion is indeed summable in our generalized sense. In Sections 5 and 6, we will prove that this is indeed the case for a suitable class of "Noetherian operators".

2. Noetherian orderings

Throughout this paper, orderings are understood to be partial, except when we explicitly state them to be total. Actually, almost all ordered sets considered in this paper are *monomial sets*, and we denote them by fraktur letters $\mathfrak{M}, \mathfrak{N}, \ldots$. We denote by $\succcurlyeq$ (or by $\succcurlyeq_{\mathfrak{M}}, \succcurlyeq_{\mathfrak{N}}, \ldots$) the orderings on such monomial sets. Usually, $\mathfrak{M}$ is even a *monomial monoid* or *group*, on which the multiplication is assumed to be compatible with the ordering, i.e.,

$$\mathfrak{m} \preccurlyeq \mathfrak{n} \;\Leftrightarrow\; \mathfrak{m}\mathfrak{v} \preccurlyeq \mathfrak{n}\mathfrak{v} \;\Leftrightarrow\; \mathfrak{v}\mathfrak{m} \preccurlyeq \mathfrak{v}\mathfrak{n},$$

for all $\mathfrak{m}, \mathfrak{n}, \mathfrak{v} \in \mathfrak{M}$.

EXAMPLE 2.1.

(1) $\mathfrak{M} = \{x^\alpha e^{\beta x} \mid \alpha, \beta \in \mathbb{R}\}$ with $x^\alpha e^{\beta x} \succcurlyeq 1 \Leftrightarrow (\beta > 0 \vee (\beta = 0 \wedge \alpha > 0))$ is a totally ordered monomial group.

(2) If $\mathfrak{M}$ and $\mathfrak{N}$ are monomial sets, then their disjoint union $\mathfrak{M} \amalg \mathfrak{N}$ is naturally ordered, by taking the orderings on $\mathfrak{M}$ and $\mathfrak{N}$ on each part of the disjoint union, and by taking $\mathfrak{M}$ and $\mathfrak{N}$ mutually incomparable in $\mathfrak{M} \amalg \mathfrak{N}$.

(3) If $\mathfrak{M}$ and $\mathfrak{N}$ are monomial sets, then the Cartesian product $\mathfrak{M} \times \mathfrak{N}$ is naturally ordered by $(\mathfrak{m}, \mathfrak{n}) \succcurlyeq_{\mathfrak{M}\times\mathfrak{N}} (\mathfrak{m}', \mathfrak{n}') \Leftrightarrow \mathfrak{m} \succcurlyeq_{\mathfrak{M}} \mathfrak{m}' \wedge \mathfrak{n} \succcurlyeq_{\mathfrak{N}} \mathfrak{n}'$.

(4) Let $\mathfrak{M}^\star$ be the set of *non-commutative words* over a monomial set $\mathfrak{M}$ (and where one may think of the elements of $\mathfrak{M}$ as infinitesimals). Such words are denoted by sequences $\mathfrak{m}_1 \cdots \mathfrak{m}_m$, with $\mathfrak{m}_1, \ldots, \mathfrak{m}_m \in \mathfrak{M}$. The empty word is denoted by ε. The set $\mathfrak{M}^\star$ is "naturally" ordered by $\mathfrak{m}_1 \cdots \mathfrak{m}_m \succcurlyeq_{\mathfrak{M}^\star} \mathfrak{n}_1 \cdots \mathfrak{n}_n$, if and only if there exists a strictly increasing mapping $\varphi : \{1, \ldots, m\} \to \{1, \ldots, n\}$, such that $\mathfrak{m}_i \succcurlyeq_{\mathfrak{M}} \mathfrak{n}_{\varphi(i)}$ for all i.

Let $\mathfrak{M}$ be a monomial set. A *chain* in $\mathfrak{M}$ is a subset of $\mathfrak{M}$ which is totally ordered for the induced ordering. An *antichain* is a subset of $\mathfrak{M}$ of pairwise incomparable elements. The ordering on $\mathfrak{M}$ is said to be *well-founded*, if there are no infinite sequences $\mathfrak{m}_1 \prec \mathfrak{m}_2 \prec \cdots$ of elements in $\mathfrak{M}$. A *Noetherian* ordering is a well-founded ordering without infinite antichains.

REMARK 2.2. In the literature, an ordered set $(E, \leqslant)$ is usually said to be well-founded, if there are no infinite sequences $x_1 > x_2 > \cdots$ of elements in E. This definition is compatible with ours, if one interprets a monomial set $\mathfrak{M}$ to be ordered by the opposite ordering $\succcurlyeq$ of $\preccurlyeq$ (as we did).

Let $\mathfrak{M}$ be a monomial set. A *final segment* is a subset $\mathfrak{F}$ of $\mathfrak{M}$, such that $\mathfrak{m} \in \mathfrak{F} \wedge \mathfrak{m} \succcurlyeq \mathfrak{n} \Rightarrow \mathfrak{n} \in \mathfrak{F}$, for all $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$. Given an arbitrary subset $\mathfrak{S}$ of $\mathfrak{M}$, we denote by $(\mathfrak{S}) = \{\mathfrak{n} \in \mathfrak{M} \mid \exists \mathfrak{m} \in \mathfrak{S}, \mathfrak{m} \succcurlyeq \mathfrak{n}\}$ the final segment *generated* by $\mathfrak{S}$. Dually, an *initial segment* is a subset $\mathfrak{I}$ of $\mathfrak{M}$, such that $\mathfrak{n} \in \mathfrak{I} \wedge \mathfrak{m} \succcurlyeq \mathfrak{n} \Rightarrow \mathfrak{m} \in \mathfrak{I}$, for all $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$. The following characterizations of Noetherian orderings are classical [Mil85], [Pou85].

PROPOSITION 2.3. *Let $\mathfrak{M}$ be a monomial set. Then the following are equivalent:*

(a) *The ordering $\succcurlyeq$ on $\mathfrak{M}$ is Noetherian.*
(b) *Any final segment of $\mathfrak{M}$ is finitely generated.*
(c) *The ascending chain condition w.r.t. inclusion holds for final segments of $\mathfrak{M}$.*
(d) *Each sequence $\mathfrak{m}_1, \mathfrak{m}_2, \ldots \in \mathfrak{M}$ admits a subsequence $\mathfrak{m}_{i_1} \succcurlyeq \mathfrak{m}_{i_2} \succcurlyeq \cdots$.*
(e) *Any extension of the ordering on $\mathfrak{M}$ to a total ordering on $\mathfrak{M}$ yields a well-ordering.* □

The most elementary examples of Noetherian orderings are well-orderings, and orderings on finite sets. Proposition 2.3 allows us to construct more complicated Noetherian orderings from simpler ones:

PROPOSITION 2.4. *Assume that $\mathfrak{M}$ and $\mathfrak{N}$ are Noetherian monomial sets. Then:*

(a) *Any subset of $\mathfrak{M}$ with the induced ordering is Noetherian.*
(b) *Let $\mathfrak{M} \to \mathfrak{V}$ be an increasing mapping into a monomial set $\mathfrak{V}$. Then $\operatorname{Im} \varphi$ is Noetherian.*
(c) *Any extension of the ordering $\succcurlyeq$ on $\mathfrak{M}$ is Noetherian.*
(d) *$\mathfrak{M} \amalg \mathfrak{N}$ is Noetherian.*
(e) *$\mathfrak{M} \times \mathfrak{N}$ is Noetherian.* □

The following theorem is due to Higman [Hig52]. We will recall a proof due to Nash-Williams [NW63], because a similar proof technique will be used in Section 6.1.

THEOREM 2.5. *Let $\mathfrak{M}$ be a Noetherian monomial set. Then $\mathfrak{M}^\star$ is Noetherian.*

Proof. We say that $\mathfrak{n}_1, \mathfrak{n}_2, \ldots$ is a *bad sequence* in $\mathfrak{M}^\star$, if there do not exist $i < j$ with $\mathfrak{n}_i \succcurlyeq_{\mathfrak{M}^\star} \mathfrak{n}_j$. An ordering is Noetherian if and only if there are no bad sequences. Now assume for contradiction that $\mathfrak{n}_1, \mathfrak{n}_2, \ldots$ is a bad sequence in $\mathfrak{M}^\star$. Without loss of generality, we may assume that each $\mathfrak{n}_i$ is chosen in $\mathfrak{M}^\star \setminus (\mathfrak{n}_1, \ldots, \mathfrak{n}_{i-1})$ such that it has minimal length as a word. We say that $\mathfrak{n}_1, \mathfrak{n}_2, \ldots$ is a *minimal bad sequence*.

Now for all i, we must have $\mathfrak{n}_i \neq \varepsilon$, so we can factor $\mathfrak{n}_i = \mathfrak{m}_i \mathfrak{v}_i$, where $\mathfrak{m}_i$ is the first letter of $\mathfrak{n}_i$. By Proposition 2.3(d), we can extract a sequence $\mathfrak{m}_{i_1} \succcurlyeq_{\mathfrak{M}} \mathfrak{m}_{i_2} \succcurlyeq_{\mathfrak{M}} \cdots$ from $\mathfrak{m}_1, \mathfrak{m}_2, \ldots$. Now consider the sequence

$$\mathfrak{n}_1, \ldots, \mathfrak{n}_{i_1 - 1}, \mathfrak{v}_{i_1}, \mathfrak{v}_{i_2}, \ldots.$$

By the minimality of $\mathfrak{n}_1, \mathfrak{n}_2, \ldots$, this sequence is good. Hence, there exist $j < i_1$ and k with $\mathfrak{n}_j \succcurlyeq_{\mathfrak{M}^\star} \mathfrak{v}_{i_k}$, or $j < k$ with $\mathfrak{v}_{i_j} \succcurlyeq_{\mathfrak{M}^\star} \mathfrak{v}_{i_k}$. But then, $\mathfrak{n}_j \succcurlyeq_{\mathfrak{M}^\star} \mathfrak{v}_{i_k} \succcurlyeq_{\mathfrak{M}^\star} \mathfrak{m}_{i_k} \mathfrak{v}_{i_k} = \mathfrak{n}_{i_k}$ resp. $\mathfrak{n}_{i_j} = \mathfrak{m}_{i_j} \mathfrak{v}_{i_j} \succcurlyeq_{\mathfrak{M}^\star} \mathfrak{m}_{i_k} \mathfrak{v}_{i_k} = \mathfrak{n}_{i_k}$. This contradicts the badness of $\mathfrak{n}_1, \mathfrak{n}_2, \ldots$. □

3. Noetherian series

3.1. Noetherian series and infinite summation. Let C be a commutative additive group of *coefficients* and $\mathfrak{M}$ a set of monomials. The *support* of a mapping $f : \mathfrak{M} \to C$ is defined by

$$\operatorname{supp} f = \{\mathfrak{m} \in \mathfrak{M} \mid f(\mathfrak{m}) \neq 0\}.$$

If $\operatorname{supp} f$ is Noetherian for the induced ordering, then we call f a *generalized power series* or a *Noetherian series*. We denote the set of all Noetherian series with coefficients in C and monomials in $\mathfrak{M}$ by $C[[\mathfrak{M}]]$. We also write $f_{\mathfrak{m}} = f(\mathfrak{m})$ for the *coefficient* of $\mathfrak{m} \in \mathfrak{M}$ in such a series and $\sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ for f. Each $f_{\mathfrak{m}} \mathfrak{m}$ with $\mathfrak{m} \in \operatorname{supp} f$ is called a *term* occurring in f.

Given two Noetherian series $f, g \in \mathfrak{M}$, we define their sum by

$$f + g = \sum_{\mathfrak{m} \in \operatorname{supp} f \cup \operatorname{supp} g} (f_{\mathfrak{m}} + g_{\mathfrak{m}}) \mathfrak{m}.$$

This gives $C[[\mathfrak{M}]]$ the structure of a commutative group. More generally, consider a family $(f_i)_{i \in I}$ of series in $C[[\mathfrak{M}]]$. We say that $(f_i)_{i \in I}$ is a *Noetherian family*, if $\bigcup_{i \in I} \operatorname{supp} f_i$ is Noetherian and for each $\mathfrak{m} \in \mathfrak{M}$ there exist only a finite number of $i \in I$ such that $\mathfrak{m} \in \operatorname{supp} f_i$. In that case, we define its sum by

$$\sum_{i \in I} f_i = \sum_{\mathfrak{m} \in \mathfrak{M}} \left(\sum_{i \in I} f_{i, \mathfrak{m}} \right) \mathfrak{m}. \tag{3.1}$$

This sum is again a Noetherian series. In particular, given a series $f \in C[[\mathfrak{M}]]$, the family $(f_{\mathfrak{m}} \mathfrak{m})_{\mathfrak{m} \in \operatorname{supp} f}$ is Noetherian and we have $f = \sum_{\mathfrak{m} \in \operatorname{supp} f} f_{\mathfrak{m}} \mathfrak{m}$ in the sense of (3.1).

It is useful to see $C[[\mathfrak{M}]]$ as a *strong commutative group*, i.e., a commutative group with an additional "infinite summation structure" on it. In our case, this structure is reflected through the infinite summation of Noetherian families; it satisfies the following fundamental properties:

PROPOSITION 3.1.

(a) *Any zero family* $(0)_{i\in I}$ *is Noetherian, and* $\sum_i 0 = 0$.
(b) *For any* $f_1 \in [[\mathfrak{M}]]$, *the family* $(f_i)_{i\in\{1\}}$ *is Noetherian, and* $\sum_{i\in\{1\}} f_i = f_1$.
(c) *If* $(f_i)_{i\in I} \in C[[\mathfrak{M}]]^I$ *and* $(f_i)_{i\in J} \in C[[\mathfrak{M}]]^J$ *are Noetherian and* $I \cap J = \varnothing$, *then* $(f_i)_{i\in I\amalg J}$ *is Noetherian and* $\sum_{i\in I\amalg J} f_i = \sum_{i\in I} f_i + \sum_{i\in J} f_i$.
(d) *If* $(f_i)_{i\in I} \in C[[\mathfrak{M}]]^I$ *is a Noetherian family, then for any bijective mapping* $\varphi : J \to I$, *the family* $(f_{\varphi(j)})_{j\in J}$ *is Noetherian, and* $\sum_{j\in J} f_{\varphi(j)} = \sum_{i\in I} f_i$.
(e) *If* $(f_i)_{i\in I} \in C[[\mathfrak{M}]]^I$ *is a Noetherian family and* $I = \coprod_{j\in J} I_j$ *a decomposition of* I *into pairwise disjoint subsets, then* $(f_i)_{i\in I_j}$ *is a Noetherian family for each* $j \in J$, $(\sum_{i\in I_j} f_i)_{j\in J}$ *is a Noetherian family, and* $\sum_{j\in J}\sum_{i\in I_j} f_i = \sum_{i\in I} f_i$.

Proof. All properties are straightforward to prove. For illustration, we will prove (e). Let $(f_i)_{i\in I} \in C[[\mathfrak{M}]]^I$ be a Noetherian family and let $I = \coprod_{j\in J} I_j$ be a partition of I. For each $\mathfrak{m} \in \mathfrak{M}$ and $j \in J$, let $I_{;\mathfrak{m}} = \{i \in I \mid f_{i,\mathfrak{m}} \neq 0\}$ and $I_{j;\mathfrak{m}} = I_j \cap I_{;\mathfrak{m}}$, so that

$$I_{;\mathfrak{m}} = \coprod_{j\in J} I_{j;\mathfrak{m}}. \tag{3.2}$$

Now $(f_i)_{i\in I_j}$ is a Noetherian family for all $j \in J$, since

$$\bigcup_{i\in I_j} \operatorname{supp} f_i \subseteq \bigcup_{i\in I} \operatorname{supp} f_i$$

and $I_{j;\mathfrak{m}} \subseteq I_{;\mathfrak{m}}$ is finite for all $\mathfrak{m} \in \mathfrak{M}$. Furthermore,

$$\bigcup_{j\in J} \operatorname{supp} \sum_{i\in I_j} f_i \subseteq \bigcup_{j\in J}\bigcup_{i\in I_j} \operatorname{supp} f_i = \bigcup_{i\in I} \operatorname{supp} f_i$$

and for all $\mathfrak{m} \in \mathfrak{M}$, the set

$$\{j \in J \mid \left(\sum_{i\in I_j} f_j\right)_{\mathfrak{m}} \neq 0\} \subseteq \{j \in J \mid I_{j;\mathfrak{m}} \neq \varnothing\}$$

is finite, because of (3.2). Hence, the family $\left(\sum_{i\in I_j} f_i\right)_{j\in J}$ is Noetherian and for all $\mathfrak{m} \in \mathfrak{M}$, we have

$$\left(\sum_{j\in J}\sum_{i\in I_j} f_i\right)_{\mathfrak{m}} = \sum_{j\in J}\sum_{i\in I_{j;\mathfrak{m}}} f_{i,\mathfrak{m}} = \sum_{i\in I_{;\mathfrak{m}}} f_{i,\mathfrak{m}} = \left(\sum_{i\in I} f_i\right)_{\mathfrak{m}}.$$

This proves (e). □

REMARK 3.2. Given two monomial sets $\mathfrak{M}$ and $\mathfrak{N}$, it is often convenient to identify $C[[\mathfrak{M}]] \times C[[\mathfrak{N}]] = C[[\mathfrak{M}]] \oplus C[[\mathfrak{N}]]$ with $C[[\mathfrak{M} \amalg \mathfrak{N}]]$ via the natural isomorphism

$$\begin{array}{rcl} C[[\mathfrak{M} \amalg \mathfrak{N}]] & \to & C[[\mathfrak{M}]] \times C[[\mathfrak{N}]] \\ f & \mapsto & (\sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}, \sum_{\mathfrak{n} \in \mathfrak{N}} f_{\mathfrak{n}} \mathfrak{n}). \end{array}$$

In particular, multivariate operators

$$\Phi : C[[\mathfrak{M}_1]] \times \cdots \times C[[\mathfrak{M}_m]] \to C[[\mathfrak{N}_1]] \times \cdots \times C[[\mathfrak{N}_n]]$$

may actually be regarded as a univariate operators

$$\Phi : C[[\mathfrak{M}_1 \amalg \cdots \amalg \mathfrak{M}_m]] \to C[[\mathfrak{N}_1 \amalg \cdots \amalg \mathfrak{N}_m]].$$

Similarly, given a monomial set $\mathfrak{M}$, the Noetherian families $(f_i)_{i \in I} \in C[[\mathfrak{M}]]^I$ may be identified with series in $C[[I \times \mathfrak{M}]]$, where $I \times \mathfrak{M}$ is strictly ordered by $(i, \mathfrak{m}) \prec (j, \mathfrak{n}) \Leftrightarrow \mathfrak{m} \prec \mathfrak{n}$. We may thus view an operator $\Phi : C[[I \times \mathfrak{M}]] \to C[[\mathfrak{N}]]$ as an operator "in infinitely many variables", which assigns to each Noetherian family $(f_i)_{i \in I} \in C[[\mathfrak{M}]]^I$ a series in $C[[\mathfrak{N}]]$.

3.2. Algebras of Noetherian series. Assume now that C is a (not necessarily commutative) ring, and $\mathfrak{M}$ a (not necessarily commutative) monomial monoid. Then we may naturally see C and $\mathfrak{M}$ as subsets of $C[[\mathfrak{M}]]$ via $c \mapsto c \cdot 1$ resp. $\mathfrak{m} \mapsto 1 \cdot \mathfrak{m}$. Given f and g in $C[[\mathfrak{M}]]$, we define their product by

$$fg = \sum_{(\mathfrak{m}, \mathfrak{n}) \in \operatorname{supp} f \times \operatorname{supp} g} f_{\mathfrak{m}} g_{\mathfrak{n}} \mathfrak{m}\mathfrak{n}.$$

The right hand side is well defined by Propositions 2.4(e) and 2.4(b). Higman [Hig52] first observed that $C[[\mathfrak{M}]]$ is a ring for this product. Actually, it is even a *strong ring*, because the product is compatible with the infinite summation structure on $C[[\mathfrak{M}]]$ in the following way:

PROPOSITION 3.3. *For all Noetherian families $(f_i)_{i \in I} \in C[[\mathfrak{M}]]^I$ and $(g_j)_{j \in J} \in C[[\mathfrak{M}]]^J$, the family $(f_i g_j)_{(i,j) \in I \times J}$ is also Noetherian, and*

$$\sum_{(i,j) \in I \times J} f_i g_j = \left(\sum_{i \in I} f_i \right) \left(\sum_{j \in J} g_j \right).$$

Proof. First of all,

$$\begin{aligned}\bigcup_{(i,j)\in I\times J} \operatorname{supp} f_i g_j &\subseteq \bigcup_{(i,j)\in I\times J} (\operatorname{supp} f_i)(\operatorname{supp} g_j) \\ &= \left(\bigcup_{i\in I} \operatorname{supp} f_i\right)\left(\bigcup_{j\in J} \operatorname{supp} g_j\right)\end{aligned}$$

is Noetherian. Given $\mathfrak{m} \in \mathfrak{M}$, the set of couples

$$(\mathfrak{v}, \mathfrak{w}) \in \left(\bigcup_{i\in I} \operatorname{supp} f_i\right) \times \left(\bigcup_{j\in J} \operatorname{supp} g_j\right)$$

with $\mathfrak{v}\mathfrak{w} = \mathfrak{m}$ forms a finite anti-chain; let $(\mathfrak{v}_1, \mathfrak{w}_1), \ldots, (\mathfrak{v}_n, \mathfrak{w}_n)$ denote those couples. Then

$$\{(i,j) \in I \times J \mid (f_i g_j)_{\mathfrak{m}} \neq 0\} \subseteq \bigcup_{k=1}^{n} \{(i,j) \in I \times J \mid f_{i,\mathfrak{v}_k} \neq 0 \wedge g_{j,\mathfrak{w}_k} \neq 0\}$$

is finite, whence $(f_i g_j)_{(i,j)\in I\times J}$ is a Noetherian family. Given $\mathfrak{m} \in \mathfrak{M}$, we also have

$$\begin{aligned}\left(\sum_{(i,j)\in I\times J} f_i g_j\right)_{\mathfrak{m}} &= \sum_{(i,j)\in I\times J} \sum_{k=1}^{n} f_{i,\mathfrak{v}_k} g_{j,\mathfrak{w}_k} \\ &= \sum_{k=1}^{n} \left(\sum_{i\in I} f_i\right)_{\mathfrak{v}_k} \left(\sum_{j\in J} g_j\right)_{\mathfrak{w}_k} \\ &= \left(\left(\sum_{i\in I} f_i\right)\left(\sum_{j\in J} g_j\right)\right)_{\mathfrak{m}},\end{aligned}$$

with $(\mathfrak{v}_1, \mathfrak{w}_1), \ldots, (\mathfrak{v}_n, \mathfrak{w}_n)$ as above. □

REMARK 3.4. Also, if $(f_i)_{i\in I} \in C[[\mathfrak{M}]]^I$ is a Noetherian family, then so is $(\lambda_i f_i)_{i\in I}$, for each family $(\lambda_i)_{i\in I} \in C^I$ of scalars.

3.3. Extension by strong linearity. Let C be a ring and let $\mathfrak{M}$, $\mathfrak{N}$ be monomial sets. In all what follows, we understand that C operates on the left on C-modules and C-algebras. A linear mapping $L : C[[\mathfrak{M}]] \to C[[\mathfrak{N}]]$ is said to be *strongly additive*, if for all Noetherian families $(f_i)_{i\in I} \in C[[\mathfrak{M}]]^I$, the family $(L(f_i))_{i\in I} \in C[[\mathfrak{N}]]^I$ is also Noetherian and

$$L\left(\sum_{i\in I} f_i\right) = \sum_{i\in I} L(f_i).$$

Notice that this condition implies that L is *strongly linear*, i.e.,

$$L\left(\sum_{i\in I}\lambda_i f_i\right)=\sum_{i\in I}\lambda_i L(f_i),$$

for every Noetherian family $(f_i)_{i\in I}\in C[[\mathfrak{M}]]^I$ and every family $(\lambda_i)_{i\in I}\in C^I$ of scalars. Notice also that the composition of two strongly linear mappings is again strongly linear.

A mapping $\varphi:\mathfrak{M}\to C[[\mathfrak{N}]]$ is said to be *Noetherian*, if $(\varphi(\mathfrak{m}))_{\mathfrak{m}\in\mathfrak{S}}$ is a Noetherian family for every Noetherian subset $\mathfrak{S}$ of $\mathfrak{M}$.

Proposition 3.5. *Let $C[[\mathfrak{M}]]$ and $C[[\mathfrak{N}]]$ be C-modules of Noetherian series. Then any Noetherian mapping $\varphi:\mathfrak{M}\to C[[\mathfrak{N}]]$ extends to a unique strongly linear mapping $\hat{\varphi}:C[[\mathfrak{M}]]\to C[[\mathfrak{N}]]$.*

Proof. Let $f\in C[[\mathfrak{M}]]$. By definition, $(\varphi(\mathfrak{m}))_{\mathfrak{m}\in\operatorname{supp} f}$ is a Noetherian family, and so is $(f_{\mathfrak{m}}\varphi(\mathfrak{m}))_{\mathfrak{m}\in\operatorname{supp} f}$. We will prove that

$$\begin{array}{rcl}\hat{\varphi}:C[[\mathfrak{M}]] & \longrightarrow & C[[\mathfrak{N}]]\\ f & \longmapsto & \displaystyle\sum_{\mathfrak{m}\in\operatorname{supp} f} f_{\mathfrak{m}}\varphi(\mathfrak{m})\end{array}$$

is the unique strongly linear mapping which coincides with φ on $\mathfrak{M}$.

Given $\lambda\in C$ and $f\in C[[\mathfrak{M}]]$ we clearly have $\hat{\varphi}(\lambda f)=\lambda\hat{\varphi}(f)$. Now let $(f_i)_{i\in I}\in C[[\mathfrak{M}]]^I$ be a Noetherian family and let $\mathfrak{S}=\bigcup_{i\in I}\operatorname{supp} f_i$. We claim that $(f_{i,\mathfrak{m}}\varphi(\mathfrak{m}))_{(i,\mathfrak{m})\in I\times\mathfrak{S}}$ is a Noetherian family. First of all,

$$\bigcup_{(i,\mathfrak{m})\in I\times\mathfrak{S}}\operatorname{supp} f_{i,\mathfrak{m}}\varphi(\mathfrak{m})\subseteq\bigcup_{\mathfrak{m}\in\mathfrak{S}}\operatorname{supp}\varphi(\mathfrak{m})$$

is Noetherian. Secondly, given $\mathfrak{n}\in\mathfrak{N}$, the set $\{\mathfrak{m}\in\mathfrak{S}\mid\varphi(\mathfrak{m})_{\mathfrak{n}}\neq 0\}$ is finite, since $(\varphi(\mathfrak{m}))_{\mathfrak{m}\in\mathfrak{S}}$ is a Noetherian family. Finally, for each $\mathfrak{m}\in\mathfrak{S}$ with $\varphi(\mathfrak{m})_{\mathfrak{n}}\neq 0$, the set $\{i\in I\mid f_{i,\mathfrak{m}}\neq 0\}$ is also finite, since $(f_i)_{i\in I}$ is a Noetherian family. Hence, the set $\{(i,\mathfrak{m})\in I\times\mathfrak{S}\mid f_{i,\mathfrak{m}}\varphi(\mathfrak{m})_{\mathfrak{n}}\neq 0\}$ is finite, which proves our claim. Now our claim, together with Proposition 3.1(d) proves that $(\hat{\varphi}(f_i))_{i\in I}=\left(\sum_{\mathfrak{m}\in\mathfrak{S}}f_{i,\mathfrak{m}}\varphi(\mathfrak{m})\right)_{i\in I}$ is a Noetherian family and

$$\begin{aligned}\sum_{i\in I}\hat{\varphi}(f_i) &= \sum_{i\in I}\sum_{\mathfrak{m}\in\mathfrak{S}}f_{i,\mathfrak{m}}\varphi(\mathfrak{m})=\sum_{(i,\mathfrak{m})\in I\times\mathfrak{S}}f_{i,\mathfrak{m}}\varphi(\mathfrak{m})\\ &= \sum_{\mathfrak{m}\in\mathfrak{S}}\left(\sum_{i\in I}f_{i,\mathfrak{m}}\right)\varphi(\mathfrak{m})=\hat{\varphi}\left(\sum_{i\in I}f_i\right).\end{aligned}$$

This establishes the strong linearity of $\hat{\varphi}$.

In order to see that $\hat{\varphi}$ is unique with the desired properties, it suffices to observe that for each $f\in C[[\mathfrak{M}]]$, we must have $\hat{\varphi}(f_{\mathfrak{m}}\mathfrak{m})=f_{\mathfrak{m}}\varphi(\mathfrak{m})$ by linearity and $\hat{\varphi}(f)=\sum_{\mathfrak{m}\in\operatorname{supp} f}f_{\mathfrak{m}}\varphi(\mathfrak{m})$ by strong linearity. □

Actually, the above proposition generalizes to the "strongly multilinear" case. If $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$ and $\mathfrak{N}$ are monomial sets, then we call a multilinear mapping

$$M : C[[\mathfrak{M}_1]] \times \cdots \times C[[\mathfrak{M}_n]] \to C[[\mathfrak{N}]]$$

strongly multilinear (or *strongly multi-additive*), if for all Noetherian families $(f_{1,i_1})_{i_1 \in I_1} \in C[[\mathfrak{M}_1]]^{I_1}, \ldots, (f_{n,i_n})_{i_n \in I_n} \in C[[\mathfrak{M}_n]]^{I_n}$, the family

$$(M(f_{1,i_1}, \ldots, f_{n,i_n}))_{(i_1,\ldots,i_n) \in I_1 \times \cdots \times I_n}$$

is also Noetherian and

$$M\left(\sum_{i_1 \in I_1} f_{1,i_1}, \ldots, \sum_{i_n \in I_n} f_{n,i_n}\right) = \sum_{(i_1,\ldots,i_n) \in I_1 \times \cdots \times I_n} M(f_{1,i_1}, \ldots, f_{n,i_n}).$$

In particular, if $\mathfrak{M}$ is a monomial monoid, then the multiplication on $C[[\mathfrak{M}]]$ is strongly bilinear, by Proposition 3.3. Also, compositions

$$\begin{array}{rcl} N \circ \prod_{i=1}^{m} M_i : \quad \prod_{i=1}^{m} \prod_{j=1}^{n_i} C[[\mathfrak{M}_{i,j}]] & \longrightarrow & C[[\mathfrak{V}]]; \\ ((f_{i,j})_{1 \leqslant j \leqslant n_m})_{1 \leqslant i \leqslant m} & \longmapsto & N(M_1(f_{1,1}, \ldots, f_{1,n_1}), \ldots, \\ & & \quad M_m(f_{m,1}, \ldots, f_{m,n_m})) \end{array}$$

of strongly multilinear mappings $N : C[[\mathfrak{N}_1]] \times \cdots \times C[[\mathfrak{N}_m]] \to C[[\mathfrak{V}]]$ and $M_i : C[[\mathfrak{M}_{i,1}]] \times \cdots \times C[[\mathfrak{M}_{i,n_i}]] \to C[[\mathfrak{N}_i]]$ for $i \in \{1, \ldots, m\}$ are strongly multilinear.

Recall that a mapping $\varphi : \mathfrak{M}_1 \times \cdots \times \mathfrak{M}_n \to C[[\mathfrak{N}]]$ is Noetherian, if $(\varphi(\mathfrak{m}_1, \ldots, \mathfrak{m}_n))_{(\mathfrak{m}_1,\ldots,\mathfrak{m}_n) \in \mathfrak{S}}$ is a Noetherian family for every Noetherian subset $\mathfrak{S}$ of $\mathfrak{M}_1 \times \cdots \times \mathfrak{M}_n$. The following proposition is proved in a similar way as Proposition 3.5:

PROPOSITION 3.6. *Let $C[[\mathfrak{M}_1]], \ldots, C[[\mathfrak{M}_n]]$ and $C[[\mathfrak{N}]]$ be C-modules of Noetherian series. Then any Noetherian mapping $\varphi : \mathfrak{M}_1 \times \cdots \times \mathfrak{M}_n \to C[[\mathfrak{N}]]$ extends to a unique strongly multilinear mapping $\hat{\varphi} : C[[\mathfrak{M}_1]] \times \cdots \times C[[\mathfrak{M}_n]] \to C[[\mathfrak{N}]]$.* □

REMARK 3.7. In a similar way as we identified $C[[\mathfrak{M} \amalg \mathfrak{N}]]$ with $C[[\mathfrak{M}]] \times C[[\mathfrak{N}]]$ in Remark 3.2, we may see $C[[\mathfrak{M} \times \mathfrak{N}]]$ as the *strong tensor product* of $C[[\mathfrak{M}]]$ and $C[[\mathfrak{N}]]$. We have a natural strongly bilinear mapping $P : C[[\mathfrak{M}]] \times C[[\mathfrak{N}]] \to C[[\mathfrak{M} \times \mathfrak{N}]]; (f, g) \mapsto \sum_{(\mathfrak{m},\mathfrak{n}) \in \operatorname{supp} f \times \operatorname{supp} g} f_{\mathfrak{m}} g_{\mathfrak{n}} (\mathfrak{m}, \mathfrak{n})$. Furthermore, for any strongly bilinear mapping $B : C[[\mathfrak{M}]] \times C[[\mathfrak{N}]] \to C[[\mathfrak{V}]]$, there exists a unique strongly linear mapping $L : C[[\mathfrak{M} \times \mathfrak{N}]] \to C[[\mathfrak{V}]]$, such that $B = L \circ P$.

3.4. Applications of strong linearity.

COROLLARY 3.8. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be monomial monoids and let $\varphi : \mathfrak{M} \to C[[\mathfrak{N}]]$ be a Noetherian mapping which preserves multiplication. Then $\hat{\varphi}$ preserves multiplication.*

Proof. The mappings $(f, g) \mapsto \hat{\varphi}(fg)$ and $(f, g) \mapsto \hat{\varphi}(f)\hat{\varphi}(g)$ are both strongly bilinear mappings from $C[[\mathfrak{M}]] \times C[[\mathfrak{M}]]$ into $C[[\mathfrak{N}]]$, which coincide on $\mathfrak{M}^2$. The result now follows from the uniqueness of strongly bilinear extensions in Proposition 3.6. □

COROLLARY 3.9. *Let $\mathfrak{M}$ be a monomial monoid and $\varphi : \mathfrak{M} \to C[[\mathfrak{M}]]$ a Noetherian mapping, such that $\varphi(\mathfrak{m}\mathfrak{n}) = \varphi(\mathfrak{m})\mathfrak{n} + \mathfrak{m}\varphi(\mathfrak{n})$ for all $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$. Then $\hat{\varphi}$ is a (strong) derivation on $C[[\mathfrak{M}]]$.*

Proof. The mappings $(f, g) \mapsto \varphi(fg)$ and $(f, g) \mapsto \varphi(f)g + f\varphi(g)$ are both strongly bilinear mappings from $C[[\mathfrak{M}]] \times C[[\mathfrak{M}]]$ into $C[[\mathfrak{M}]]$, which coincide on $\mathfrak{M}^2$. The result again follows from the uniqueness of strongly bilinear extensions in Proposition 3.6. □

COROLLARY 3.10. *Let $\varphi : \mathfrak{M} \to C[[\mathfrak{N}]]$ and $\psi : \mathfrak{N} \to C[[\mathfrak{V}]]$ be two Noetherian mappings. Then*

$$\widehat{\hat{\psi} \circ \varphi} = \hat{\psi} \circ \hat{\varphi}.$$

Proof. This still follows from the uniqueness of extensions by strong linearity, since $\hat{\psi} \circ \varphi$ and $\hat{\psi} \circ \hat{\varphi}$ coincide on $\mathfrak{M}$. □

Assume that $\mathfrak{M}$ is a monomial monoid. We call a series $f \in C[[\mathfrak{M}]]$ *infinitesimal*, if $\mathfrak{m} \prec 1$ for all $\mathfrak{m} \in \operatorname{supp} f$. Then extension by strong linearity may in particular be used to define the composition $g \circ (f_1, \ldots, f_k)$ of a multivariate power series $g \in C[[z_1, \ldots, z_k]] = C[[z_1^{\mathbb{N}} \cdots z_k^{\mathbb{N}}]]$ with infinitesimal series $f_1, \ldots, f_k \in C[[\mathfrak{M}]]$. Indeed, if $\varphi : z_1^{\mathbb{N}} \cdots z_k^{\mathbb{N}} \to C[[\mathfrak{M}]]$ is the multiplicative mapping which sends each $z_1^{n_1} \cdots z_k^{n_k}$ to $f_1^{n_1} \cdots f_k^{n_k}$, then we define $g \circ (f_1, \ldots, f_k) = \hat{\varphi}(g)$. Then corollaries 3.8 and 3.10 yield the following result:

COROLLARY 3.11. *Let $f_1, \ldots, f_k$ be infinitesimal Noetherian series in $C[[\mathfrak{M}]]$. Then:*

(a) $(gh) \circ (f_1, \ldots, f_k) = g \circ (f_1, \ldots, f_k) h \circ (f_1, \ldots, f_k)$, *for* $g, h \in C[[z_1, \ldots, z_k]]$.
(b) $(h \circ (g_1, \ldots, g_l)) \circ (f_1, \ldots, f_k) = h \circ (g_1 \circ (f_1, \ldots, f_k), \ldots, g_l \circ (f_1, \ldots, f_k))$, *for* $h \in C[[z_1, \ldots, z_l]]$ *and infinitesimal* $g_1, \ldots, g_l \in C[[z_1, \ldots, z_k]]$. □

4. The topological implicit function theorem

4.1. Truncation of Noetherian series. Let $\mathfrak{M}$ be a monomial set and $f \in C[[\mathfrak{M}]]$. Given a subset $\mathfrak{S} \subseteq \mathfrak{M}$, we define the *restriction* $f_{|\mathfrak{S}} \in C[[\mathfrak{S}]] \subseteq C[[\mathfrak{M}]]$ of f to $\mathfrak{S}$ by

$$f_{|\mathfrak{S}} = \sum_{\mathfrak{m} \in \mathfrak{S} \cap \operatorname{supp} f} f_{\mathfrak{m}} \mathfrak{m}.$$

Given two series $f, g \in C[[\mathfrak{M}]]$, we say that f is a *truncation* of g (and we write $f \trianglelefteqslant g$), if there exists an initial segment $\mathfrak{I}$ of $\operatorname{supp} g$, such that $f = g_{|\mathfrak{I}}$. Thus $\trianglelefteqslant$ is an ordering on $C[[\mathfrak{M}]]$.

Let $(f_i)_{i \in I} \in C[[\mathfrak{M}]]^I$ be a non-empty family of series. A *common truncation* of the f_i is a series g, such that $g \trianglelefteqslant f_i$ for all $i \in I$. A *greatest common truncation* of the f_i is a common truncation, which is greatest for $\trianglelefteqslant$. Such a greatest truncation actually always exists and we denote it by $\bigtriangleup_{i \in I} f_i$:

Proposition 4.1. *Any non-empty family $(f_i)_{i \in I} \in C[[\mathfrak{M}]]$ admits a greatest common truncation.*

Proof. Fix some $j \in I$ and consider the set $\mathcal{I}$ of initial segments $\mathfrak{I}$ of $\operatorname{supp} f_j$, such that $f_{j|\mathfrak{I}} \trianglelefteqslant f_i$ for all $i \in I$. We observe that arbitrary unions of initial segments of a given ordering are again initial segments. Hence $\mathfrak{I}_{\max} = \bigcup_{\mathfrak{I} \in \mathcal{I}} \mathfrak{I}$ is an initial segment of each $\operatorname{supp} f_i$. Furthermore, for each $i \in I$ and $\mathfrak{m} \in \mathfrak{I}_{\max}$, there exists an $\mathfrak{I} \in \mathcal{I}$ with $f_{j|\mathfrak{I},\mathfrak{m}} = f_{j,\mathfrak{m}} = f_{i,\mathfrak{m}}$. Hence $f_{j|\mathfrak{I}_{\max}} = f_{i|\mathfrak{I}_{\max}} \trianglelefteqslant f_i$ for all $i \in I$. This proves that $f_{|\mathfrak{I}_{\max}}$ is a common truncation of the f_i. It is also greatest for $\trianglelefteqslant$, since any common truncation is of the form $f_{j|\mathfrak{I}}$ for some initial segment $\mathfrak{I} \in \mathcal{I}$ of $\mathfrak{I}_{\max}$ with $f_{j|\mathfrak{I}} \trianglelefteqslant f_{j|\mathfrak{I}_{\max}}$. □

Let $(f_i)_{i \in I} \in C[[\mathfrak{M}]]^I$ again be a family of series. A *common extension* of the f_i is a series g, such that $f_i \trianglelefteqslant g$ for all $i \in I$. A *least common extension* of the f_i is a common extension, which is least for $\trianglelefteqslant$. If such a least common extension exists, then we denote it by $\bigtriangledown_{i \in I} f_i$.

Now consider a directed index set I. In other words, we have an ordering on I, such that for any $i, j \in I$, there exist a $k \in I$ with $i \leqslant k$ and $j \leqslant k$. Let $(f_i)_{i \in I}$ be a $\trianglelefteqslant$-increasing family of series in $C[[\mathfrak{M}]]$, i.e., $f_i \trianglelefteqslant f_j$ whenever $i \leqslant j$. If $\mathfrak{M}$ is Noetherian or totally ordered, then there exists a least common extension of the f_i:

Proposition 4.2. *Assume that $\mathfrak{M}$ is Noetherian or totally ordered. Then any directed $\trianglelefteqslant$-increasing family $(f_i)_{i \in I}$ of series in $C[[\mathfrak{M}]]$ admits a unique least common extension $\bigtriangledown_{i \in I} f_i$, and* $\operatorname{supp} \bigtriangledown_{i \in I} f_i = \bigcup_{i \in I} \operatorname{supp} f_i$.

Proof. Let $\mathfrak{S} = \bigcup_{i \in I} \operatorname{supp} f_i$. We claim that $\mathfrak{S}$ is Noetherian. This is clear if $\mathfrak{M}$ is Noetherian. Assume that $\mathfrak{M}$ is totally ordered and that $\mathfrak{m}_1 \preccurlyeq \mathfrak{m}_2 \preccurlyeq \cdots$ is an infinite sequence of monomials in $\mathfrak{S}$. Since I is directed and $\operatorname{supp} f_i \subseteq \operatorname{supp} f_j$ whenever $i \leqslant j$, there exist $i_1 \leqslant i_2 \leqslant \cdots$ with $\mathfrak{m}_k \in \operatorname{supp} f_{i_k}$ for each

k. But we also have $f_{i_1} \trianglelefteq f_{i_k}$ for each k, so that $\mathfrak{m}_1, \mathfrak{m}_2, \ldots \in \operatorname{supp} f_{i_1}$. Since $\operatorname{supp} f_{i_1}$ is Noetherian, the sequence $\mathfrak{m}_1, \mathfrak{m}_2, \ldots$ therefore stabilizes.

Given $\mathfrak{m} \in \mathfrak{S}$, we claim that the coefficient $g_{\mathfrak{m}} = f_{i,\mathfrak{m}}$ is independent of the choice of $i \in I$, under the condition that $\mathfrak{m} \in \operatorname{supp} f_i$. Indeed, let $i, j \in I$ be such that $\mathfrak{m} \in \operatorname{supp} f_i$ and $\mathfrak{m} \in \operatorname{supp} f_j$. Then there exists a $k \in I$ with $i \leqslant k$ and $j \leqslant k$. Hence, $f_i \trianglelefteq f_k$ and $f_j \trianglelefteq f_k$, so that $f_{i,\mathfrak{m}} = f_{k,\mathfrak{m}} = f_{j,\mathfrak{m}}$. Now the series $g = \sum_{\mathfrak{m} \in \mathfrak{S}} g_{\mathfrak{m}} \mathfrak{m}$ is the least common extension of the f_i. □

4.2. Stationary limits. Let I be a directed index set and $(f_i)_{i \in I} \in C[[\mathfrak{M}]]^I$ a family of series. We call $g \in C[[\mathfrak{M}]]$ a *pseudo-limit* of the f_i, if for each final segment $\mathfrak{F}$ of $\mathfrak{M}$ and for all $i \in I$, we have

$$(\forall j \geqslant i : \operatorname{supp}(f_j - f_i) \subseteq \mathfrak{F}) \;\; \Rightarrow \;\; (\operatorname{supp}(g - f_i) \subseteq \mathfrak{F}).$$

Equivalently, we may require that for each inital segment $\mathfrak{I}$ of $\mathfrak{M}$ and for each $i \in I$, we have

$$(\forall j \geqslant i : f_{j|\mathfrak{I}} = f_{i|\mathfrak{I}}) \;\; \Rightarrow \;\; (g_{|\mathfrak{I}} = f_{i|\mathfrak{I}}).$$

Assume from now on that $\mathfrak{M}$ is either Noetherian or totally ordered. Below, we will show that the *stationary limit* of the f_i, which is defined by

$$\operatorname*{stat\,lim}_{i \in I} f_i = \bigtriangledown_{i \in I} \bigtriangleup_{j \geqslant i} f_j,$$

is in particular a pseudo-limit. We first prove some useful properties of $\bigtriangledown$ and $\bigtriangleup$.

PROPOSITION 4.3. *Let $(f_i)_{i \in I} \in C[[\mathfrak{M}]]^I$ be a family of series and let $\mathfrak{I}$ be an initial segment of $\mathfrak{M}$.*

(a) *If $I \neq \varnothing$, then*

$$\bigtriangleup_{i \in I} f_{i|\mathfrak{I}} \;=\; \left(\bigtriangleup_{i \in I} f_i \right)_{|\mathfrak{I}}.$$

(b) *If $(f_i)_{i \in I}$ is directed and $\trianglelefteq$-increasing, then*

$$\bigtriangledown_{i \in I} f_{i|\mathfrak{I}} \;=\; \left(\bigtriangledown_{i \in I} f_i \right)_{|\mathfrak{I}}.$$

Proof. We first observe that for all $f, g \in C[[\mathfrak{M}]]$ we have $f \trianglelefteq g \Rightarrow f_{|\mathfrak{I}} \trianglelefteq g_{|\mathfrak{I}}$. In particular, this ensures that $\bigtriangledown_{i \in I} f_{i|\mathfrak{I}}$ exists in (b).

Now assume that $I \neq \varnothing$ and let $g = \bigtriangleup_{i \in I} f_i$. Then $g \trianglelefteq f_i$, whence $g_{|\mathfrak{I}} \trianglelefteq f_{i|\mathfrak{I}}$, for all $i \in I$. This shows that $g_{|\mathfrak{I}}$ is a common truncation of the $f_{i|\mathfrak{I}}$. Conversely, assume that $h \in C[[\mathfrak{I}]]$ is such that $h \trianglelefteq f_{i|\mathfrak{I}}$ for all $i \in I$. Then also $h \trianglelefteq f_i$ for all $i \in I$, so that $h \trianglelefteq g$. Hence $h = h_{|\mathfrak{I}} \trianglelefteq g_{|\mathfrak{I}}$. This shows that $g_{|\mathfrak{I}}$ is the greatest common truncation of the $f_{i|\mathfrak{I}}$.

Assume now that $(f_i)_{i \in I}$ is directed and $\trianglelefteq$-increasing and let $g = \bigtriangledown_{i \in I} f_i$. Then $f_i \trianglelefteq g$, whence $f_{i|\mathfrak{I}} \trianglelefteq g_{|\mathfrak{I}}$, for all $i \in I$. Consequently, $g_{|\mathfrak{I}}$ is a common extension of the $f_{i|\mathfrak{I}}$. Furthermore, its support $\operatorname{supp} g_{|\mathfrak{I}} = (\operatorname{supp} g) \cap \mathfrak{I} = (\bigcup_{i \in I} \operatorname{supp} f_i) \cap \mathfrak{I} = \bigcup_{i \in I} \operatorname{supp} f_i \cap \mathfrak{I} = \bigcup_{i \in I} \operatorname{supp} f_{i|\mathfrak{I}}$ is the same as the support of the least common extension of the $f_{i|\mathfrak{I}}$. Hence $g_{|I} = \bigtriangledown_{i \in I} f_{i|\mathfrak{I}}$. □

Proposition 4.4. *Let $(f_i)_{i \in I} \in C[[\mathfrak{M}]]^I$ be a directed family and $i \in I$. Then*

$$\bigtriangledown_{j \in I} \bigtriangleup_{k \geqslant j} f_k = \bigtriangledown_{j \geqslant i} \bigtriangleup_{k \geqslant j} f_k.$$

Proof. Since $I \supseteq \{j \in I \mid j \geqslant i\}$, we have $\bigtriangledown_{j \in I} \bigtriangleup_{k \geqslant j} f_k \trianglerighteq \bigtriangledown_{j \geqslant i} \bigtriangleup_{k \geqslant j} f_k$. On the other hand, given $\mathfrak{m} \in \operatorname{supp} \bigtriangledown_{j \in I} \bigtriangleup_{k \geqslant j} f_k$, we have $\mathfrak{m} \in \bigtriangleup_{k \geqslant j} f_k$ for some $j \in I$. Choosing $l \in I$ with $l \geqslant i$ and $l \geqslant j$, we then have $\mathfrak{m} \in \bigtriangleup_{k \geqslant l} f_k \trianglerighteq \bigtriangleup_{k \geqslant j} f_k$ and $\mathfrak{m} \in \bigcup_{m \geqslant i} \operatorname{supp} \bigtriangleup_{k \geqslant m} f_k = \operatorname{supp} \bigtriangledown_{m \geqslant i} \bigtriangleup_{k \geqslant m} f_k$. □

Proposition 4.5. *For any directed family $(f_i)_{i \in I} \in C[[\mathfrak{M}]]^I$, its stationary limit is a pseudo-limit.*

Proof. Let $\mathfrak{I}$ be an initial segment of $\mathfrak{M}$ and let $i \in I$ be such that $f_{j|\mathfrak{I}} = f_{i|\mathfrak{I}}$ for all $j \geqslant i$. Then Proposition 4.3 implies that

$$\left(\bigtriangledown_{j \geqslant i} \bigtriangleup_{k \geqslant j} f_k \right)_{|\mathfrak{I}} = \bigtriangledown_{j \geqslant i} \bigtriangleup_{k \geqslant j} f_{k|\mathfrak{I}} = \bigtriangledown_{j \geqslant i} \bigtriangleup_{k \geqslant j} f_{i|\mathfrak{I}} = f_{i|\mathfrak{I}}. \tag{4.1}$$

Hence $(\operatorname{stat} \lim_{j \in I} f_j)_{|\mathfrak{I}} = f_{i|\mathfrak{I}}$, by Proposition 4.4. □

Given f and g in $C[[\mathfrak{M}]]$, we will write $f \prec\!\!\cdot\; g$, if for all $\mathfrak{m} \in \operatorname{supp} f$, there exists an $\mathfrak{n} \in \operatorname{supp} g$ with $\mathfrak{m} \prec \mathfrak{n}$. The following properties of $\prec\!\!\cdot$ will be used frequently in the next section:

Proposition 4.6. *Let $f, g, h \in C[[\mathfrak{M}]]$. Then*

(a) *$f \prec\!\!\cdot\; f$ if and only if $f = 0$.*
(b) *$f \prec\!\!\cdot\; g \wedge g \prec\!\!\cdot\; h \Rightarrow f \prec\!\!\cdot\; h$.*
(c) *$f \prec\!\!\cdot\; h \wedge g \prec\!\!\cdot\; h \Rightarrow f + g \prec\!\!\cdot\; h$.*
(d) *If $(f_i)_{i \in I} \in C[[\mathfrak{M}]]^I$ now stands for a directed family, then*

$$(\forall i \in I : f_i - g \prec\!\!\cdot\; h) \;\Rightarrow\; ((\operatorname{stat} \lim_{i \in I} f_i) - g \prec\!\!\cdot\; h).$$

Proof. The first three properties are trivial. Consider the final segment

$$\mathfrak{F} = \{\mathfrak{m} \in \mathfrak{M} \mid \mathfrak{d} \succ \mathfrak{m}, \text{ for some } \preccurlyeq\text{-maximal element } \mathfrak{d} \text{ in } \operatorname{supp} h\}.$$

Then our hypothesis means that $\operatorname{supp}(f_i - g) \subseteq \mathfrak{F}$ for all i. Now

$$\operatorname{supp}((\operatorname{stat} \lim_{i \in I} f_i) - g) \subseteq \mathfrak{F},$$

by Proposition 4.5. But this means that $(\operatorname{stat} \lim_{i \in I} f_i) - g \prec\!\!\cdot\; h$. □

4.3. The implicit function theorem. A final segment $\mathfrak{F}$ of a monomial set $\mathfrak{M}$ is said to be *attractive*, if for each $\mathfrak{m} \in \mathfrak{M}$ there exists an $\mathfrak{n} \in \mathfrak{F}$ with $\mathfrak{m} \succcurlyeq \mathfrak{n}$. If $\mathfrak{M}$ is totally ordered, then all non-empty final segments are attractive. The intersection of two attractive final segments is again an attractive final segment and arbitrary non-empty unions of attractive final segments are again attractive final segments. In other words, the attractive final subsets $\mathfrak{F}$ of $\mathfrak{M}$ together with the empty set are the open sets of a topology on $\mathfrak{M}$.

Now let C be a commutative additive group. The *attractive open subsets* of $C[[\mathfrak{M}]]$ are the subsets of the form $f + C[[\mathfrak{F}]]$, where $f \in C[[\mathfrak{M}]]$ and where $\mathfrak{F}$ is an attractive final segment of $\mathfrak{M}$. These sets form a basis for the open subsets of the *natural* or *attractive topology* on $C[[\mathfrak{M}]]$. We notice that the attractive topology makes $C[[\mathfrak{M}]]$ an additive topological group. Given another monomial set $\mathfrak{N}$, we also notice that the attractive topology on $C[[\mathfrak{M}]] \times C[[\mathfrak{N}]] \cong C[[\mathfrak{M} \amalg \mathfrak{N}]]$ (recall Remark 3.2) coincides with the usual product topology on $C[[\mathfrak{M}]] \times C[[\mathfrak{N}]]$ (if $C[[\mathfrak{M}]]$ and $C[[\mathfrak{N}]]$ are given the attractive topologies).

Consider a mapping $\Phi : C[[\mathfrak{M}]] \to C[[\mathfrak{M}]]$, where $\mathfrak{M} \neq \varnothing$. We call Φ *contracting*, if for all $f, g \in C[[\mathfrak{M}]]$, we have $\Phi(g) - \Phi(f) \prec g - f$. A contracting mapping is in particular continuous at each point $f \in C[[\mathfrak{M}]]$, since for any attractive open neighbourhood $\Phi(f) + C[[\mathfrak{F}]]$ of $\Phi(f)$, the set $f + C[[\mathfrak{F}]]$ is an open neighbourhood of f with $\Phi(f + C[[\mathfrak{F}]]) \subseteq \Phi(f) + C[[\mathfrak{F}]]$.

THEOREM 4.7. *Assume that $\mathfrak{M} \neq \varnothing$ is Noetherian or totally ordered and let $\Phi : C[[\mathfrak{M}]] \times C[[\mathfrak{N}]] \to C[[\mathfrak{M}]]$ be a continuous mapping, such that the mapping $\Phi_g : C[[\mathfrak{M}]] \to C[[\mathfrak{M}]]; f \mapsto \Phi(f, g)$ is contracting for each $g \in C[[\mathfrak{N}]]$. Then there exists a unique mapping $\Psi : C[[\mathfrak{N}]] \to C[[\mathfrak{M}]]$ with $\Psi(g) = \Phi(\Psi(g), g)$ for each $g \in C[[\mathfrak{N}]]$, and Ψ is continuous.*

Proof. Given $g \in C[[\mathfrak{N}]]$, consider the transfinite sequence $(f_\alpha)_\alpha$ defined as follows:

$$\begin{aligned} f_0 &\in C[[\mathfrak{M}]] \text{ (any choice of } f_0 \text{ will do)}\,; \\ f_{\alpha+1} &= \Phi_g(f_\alpha)\,; \\ f_\lambda &= \operatorname*{stat\,lim}_{\alpha<\lambda} f_\alpha, \text{ for limit ordinals } \lambda. \end{aligned}$$

We will show that $(f_\alpha)_\alpha$ converges to a solution of the equation $f = \Phi_g(f)$.

The sequence $f_{\alpha+1} - f_\alpha$ decreases for $\prec$. Let us prove by (weak) transfinite induction over α that $f_{\alpha+1} - f_\alpha \prec f_{\beta+1} - f_\beta$ for all ordinals $\beta < \alpha$. This is clear for $\alpha = 0$. Assume that $\alpha = \beta + 1$ is a successor ordinal. Since Φ_g is contracting, the induction hypothesis then implies that $f_{\alpha+1} - f_\alpha \prec f_{\beta+1} - f_\beta \preccurlyeq f_{\gamma+1} - f_\gamma$ for all $\gamma \leqslant \beta < \alpha$.

If α is a limit ordinal and $\beta < \alpha$, then let us prove by a second (weak) transfinite induction over γ that $f_\gamma - f_{\beta+1} \prec f_{\beta+1} - f_\beta$ for all $\beta + 1 < \gamma < \alpha$.

This is indeed true for $\gamma = \beta + 2$, by the first induction hypothesis. Assuming that $f_\gamma - f_{\beta+1} \prec f_{\beta+1} - f_\beta$, we also have

$$f_{\gamma+1} - f_{\beta+1} = (f_{\gamma+1} - f_\gamma) + (f_\gamma - f_{\beta+1}) \prec f_{\beta+1} - f_\beta,$$

again by the first induction hypothesis and Proposition 4.6(c). If γ is a limit ordinal, then the second induction hypothesis implies that $f_\delta - f_{\beta+1} \prec f_{\beta+1} - f_\beta$ for all $\beta < \delta < \gamma$. Hence,

$$f_\gamma - f_{\beta+1} = (\operatorname{stat}\lim_{\delta<\gamma} f_\delta) - f_{\beta+1} = (\operatorname{stat}\lim_{\beta<\delta<\gamma} f_\delta) - f_{\beta+1} \prec f_{\beta+1} - f_\beta,$$

by Proposition 4.6(d).

At this point, we have proved that $f_\gamma - f_{\beta+1} \prec f_{\beta+1} - f_\beta$ for all $\beta + 1 < \gamma < \alpha$. Now Proposition 4.6(d) implies that

$$f_\alpha - f_{\beta+1} = (\operatorname{stat}\lim_{\gamma<\alpha} f_\gamma) - f_{\beta+1} = (\operatorname{stat}\lim_{\beta+1<\gamma<\alpha} f_\gamma) - f_{\beta+1} \prec f_{\beta+1} - f_\beta.$$

In a similar way, one proves that $f_\alpha - f_{\beta+2} \prec f_{\beta+1} - f_\beta$. Since Φ_g is contracting, $f_\alpha - f_{\beta+1} \prec f_{\beta+1} - f_\beta$ also implies that $f_{\alpha+1} - f_{\beta+2} \prec f_{\beta+1} - f_\beta$. Consequently, $f_{\alpha+1} - f_\alpha = (f_{\alpha+1} - f_{\beta+2}) + (f_{\beta+2} - f_{\beta+1}) + (f_{\beta+1} - f_\alpha) \prec f_{\beta+1} - f_\beta$, by Proposition 4.6(c).

Existence and uniqueness. Having shown that the sequence $f_{\alpha+1} - f_\alpha$ is decreasing for $\prec$, we now claim that we must have $f_{\alpha+1} - f_\alpha = 0$ for some sufficiently large α. Otherwise, each of the sets $\mathfrak{d}(f_{\alpha+1} - f_\alpha)$ of $\preccurlyeq$-maximal monomials of $f_{\alpha+1} - f_\alpha$ would be non empty, so that $\mathfrak{d}(f_{\beta+1} - f_\beta) \cap \mathfrak{d}(f_{\alpha+1} - f_\alpha) \neq \varnothing$ for some $\beta < \alpha$. Indeed, this will happen as soon as the monomials in $\mathfrak{M}$ get exhausted, i.e., for some $\beta < \alpha$ such that the cardinality of α is the one larger than the cardinality of $\mathfrak{M}$. Now let $\mathfrak{m} \in \mathfrak{d}(f_{\beta+1} - f_\beta) \cap \mathfrak{d}(f_{\alpha+1} - f_\alpha)$. Since $f_{\alpha+1} - f_\alpha \prec f_{\beta+1} - f_\beta$, there exists an $\mathfrak{n} \in \operatorname{supp}(f_{\beta+1} - f_\beta)$ with $\mathfrak{n} \succ \mathfrak{m}$. But this contradicts the $\preccurlyeq$-maximality of $\mathfrak{m}$ in $\operatorname{supp} f_{\beta+1} - f_\beta$. This shows our claim and we conclude that the $\Psi(g) \equiv f_\alpha$ with $f_{\alpha+1} - f_\alpha = 0$ satisfies $\Psi(g) = \Phi_g(\Psi(g))$.

Assume now that two Noetherian series f and f' both satisfy $f = \Phi_g(f)$ and $f' = \Phi_g(f')$. Then $f' - f = \Phi_g(f') - \Phi_g(f) \prec f' - f$, since Φ_g is contracting. But we can only have $f' - f \prec f' - f$ if $f' = f$. This establishes the existence and the uniqueness of the mapping Ψ.

Continuity. In order to prove that Ψ is continuous in any given $g_0 \in C[[\mathfrak{N}]]$, let $W = \Psi(g_0) + C[[\mathfrak{H}]]$ be an attractive open neighbourhood of $\Psi(g_0)$. Then there exists an attractive open subset of $C[[\mathfrak{M}]] \times C[[\mathfrak{N}]]$ of the form $U \times V = (\Psi(g_0) + C[[\mathfrak{F}]]) \times (g_0 + C[[\mathfrak{G}]])$, such that $\Phi(U \times V) \subseteq W$. We claim that $\Psi(V) \subseteq W$. Indeed, let $g \in V$. Taking $f_0 = \Psi(g_0)$ in our sequence above, it suffices to prove that $f_\alpha \in W$ for all α. We prove this by transfinite induction.

For $\alpha = 0$ and $\alpha = 1$, we are already done. If $\alpha = \beta + 1 > \gamma \geqslant 0$, then $f_\alpha - f_\beta \prec f_{\gamma+1} - f_\gamma \in C[[\mathfrak{H}]]$ implies that $f_\alpha - f_\beta \in C[[\mathfrak{H}]]$, whence $f_\alpha \in W$. If α is a limit ordinal, then we have seen above that $f_\alpha - f_{\beta+1} \prec f_{\beta+1} - f_\beta$

for all $\beta < \alpha$. Taking any such β, we also have $f_{\beta+1} - f_\beta \in C[[\mathfrak{H}]]$ by the induction hypothesis, whence again $f_\alpha - f_{\beta+1} \in C[[\mathfrak{H}]]$ and $f_\alpha \in W$. This completes the induction and the proof of the theorem. □

Remark 4.8. The theorem still holds for monomial sets $\mathfrak{M}$ without "infinite combs" [PCR93]. Our proof also generalizes to this setting, because it can be shown in this case that the stationary limit of a sequence $(f_\alpha)_{\alpha<\beta} \in C[[\mathfrak{M}]]^\beta$ exists, whenever $f_{\alpha+1} - f_\alpha$ is strictly decreasing for $\prec$.

Remark 4.9. Although the above topological implicit function theorem may be very useful to solve certain parameterized functional equations over Noetherian series, one of its major drawbacks is that we needed the very strong Noetherianity assumption on $\mathfrak{M}$ in the partial context. Even the slightly weaker condition about the absence of infinite combs is usually not satisfied. The functional equation

$$f(z_1, z_2) = 1 + (z_1 + z_2) f\left(\sqrt{z_1}, \sqrt{z_2}\,\right)$$

with $\mathfrak{M} = \{z_1^{\alpha_1} z_2^{\alpha_2} \mid \alpha_1, \alpha_2 \in \mathbb{Q}^{\geqslant 0} \wedge \alpha_1 + \alpha_2 < 2\}$ is an example which shows that there is not much hope for a stronger implicit function theorem in the same spirit. Indeed, the natural "solution" to this equation, which is obtained by recursively replacing the left hand side by the right hand side in the equation, does not have a Noetherian support.

Remark 4.10. Another drawback of Theorem 4.7 is that it does not provide us with any additional information about the solutions. The solutions may even be quite pathological: consider the monomial group $x^{\mathbb{R}}$ with $x^\alpha \succcurlyeq x^\beta \Leftrightarrow \alpha \geqslant \beta$. Given $f \in \mathbb{R}[[x^{\mathbb{R}}]]$, we denote $f^\uparrow = \sum_{\alpha>0} f_{x^\alpha} x^\alpha$. We define a linear (but not strongly linear) operator $L : \mathbb{R}[[x^{\mathbb{R}}]] \to \mathbb{R}[[x^{\mathbb{R}}]]$ by

$$\begin{aligned} L(f(x)) &= f^\uparrow(\sqrt{x}) + f^\uparrow(1/\sqrt{x}), \text{ if } \operatorname{supp} f \text{ is finite;} \\ L(f(x)) &= f^\uparrow(\sqrt{x}), \text{ otherwise.} \end{aligned}$$

Then it is easily verified that L is contracting (whence continuous) on $\mathbb{R}[[x^{\mathbb{R}}]]$. The equation

$$f(x) = x + L(f(x))$$

will therefore admit a unique solution, which happens to be $f(x) = x + \sqrt{x} + \sqrt{\sqrt{x}} + \cdots$. However, we do *not* have $f(x) = x + L(x) + L(L(x)) + \cdots$.

5. Noetherian operators and combinatorial representations

5.1. Noetherian operators. Let $\mathfrak{M}$ and $\mathfrak{N}$ be sets of monomials. A *Noetherian operator* is a mapping $\Phi : C[[\mathfrak{M}]] \to C[[\mathfrak{N}]]$, such that there exists a family $(M_i)_{i\in I}$ of strongly multilinear mappings $M_i : C[[\mathfrak{M}]]^{|i|} \to C[[\mathfrak{N}]]$

with

$$\Phi\left(\sum_{k\in K} f_k\right) = \sum_{\substack{i \in I \\ k_1,\ldots,k_{|i|} \in K}} M_i(f_{k_1},\ldots,f_{k_{|i|}}), \tag{5.1}$$

for all Noetherian families $(f_k)_{k\in K} \in C[[\mathfrak{M}]]^K$. In particular, this assumes that the family of summands $M_i(f_{k_1},\ldots,f_{k_{|i|}})$ is Noetherian. We will call $(M_i)_{i\in I}$ a *multilinear decomposition* of Φ. The number $|i| \in \mathbb{N}$ is the *arity* of M_i.

By regrouping the M_i of the same arity, it actually suffices to consider the case when $I = \mathbb{N}$ and there is exactly one M_i for each arity $i \in \mathbb{N}$. In this case, we may write $\Phi = \Phi_0 + \Phi_1 + \cdots$, with $\Phi_i(f) = M_i(f,\ldots,f)$ for all f and i. In Section 5.4, we will see that this representation is unique, under the assumption that $C \supseteq \mathbb{Q}$ and that the M_i are symmetric (we may always take the M_i to be symmetric if $C \supseteq \mathbb{Q}$). However, for the purpose of combinatorial representations in the next section, it is natural to consider more general multilinear decompositions. Notice also that the space of Noetherian operators from $C[[\mathfrak{M}]] \to C[[\mathfrak{N}]]$ has a natural strong group structure.

Remark 5.1. The formula (5.1) should hold in particular for families that consist of only one element. In other words, we should have

$$\Phi(f) = \sum_{i\in I} M_i(f,\ldots,f),$$

for all $f \in C[[\mathfrak{M}]]$. However, the more complicated assumption (5.1) is essential, as one can see in Example 5.5 below.

Remark 5.2. In view of Remark 3.2 the present definition of Noetherian operators also provides a definition of multivariate Noetherian operators.

Example 5.3.

- Each constant mapping $\Phi : C[[\mathfrak{M}]] \to C[[\mathfrak{N}]]; f \mapsto c$ is a Noetherian operator.
- Any strongly linear or strongly multilinear operator L resp. M is a Noetherian operator.
- Addition $+ : C[[\mathfrak{M}]]^2 \to C[[\mathfrak{M}]]; (f,g) \mapsto f+g$ is a Noetherian operator.
- If $\mathfrak{M}$ is a monomial monoid, then multiplication on $C[[\mathfrak{M}]]$ is a Noetherian operator.

Example 5.4. Let $\Phi, \Psi : C[[\mathfrak{M}]] \to C[[\mathfrak{N}]]$ be Noetherian operators.

- $\Phi + \Psi : f \mapsto \Phi(f) + \Psi(f)$ is a Noetherian operator.

- If $\mathfrak{N}$ is a monomial monoid, then $\Phi\Psi : f \mapsto \Phi(f)\Psi(f)$ is a Noetherian operator.

EXAMPLE 5.5. Let $\Phi : C[[\mathfrak{M}]] \to C[[\mathfrak{N}]]$ and $\Psi : C[[\mathfrak{N}]] \to C[[\mathfrak{V}]]$ be two Noetherian operators. Then we claim that $\Psi \circ \Phi$ is also a Noetherian operator. Indeed, let $(M_i)_{i \in I}$ resp. $(N_j)_{j \in J}$ be multilinear decompositions of Φ and Ψ. Then for each Noetherian family $(f_k)_{k \in K} \in C[[\mathfrak{M}]]^K$ we have

$$\begin{aligned} \Psi \circ \Phi\left(\sum_{k \in K} f_k\right) &= \Psi\left(\sum_{\substack{i \in I \\ k_1, \ldots, k_{|i|} \in K}} M_i(f_{k_1}, \ldots, f_{k_{|i|}})\right) \\ &= \sum_{\substack{j \in J \\ i_1, \ldots, i_{|j|} \in I \\ k_{1,1}, \ldots, k_{1,|i_1|} \in K \\ \vdots \\ k_{|j|,1}, \ldots, k_{|j|,\left|i_{|j|}\right|} \in K}} \begin{array}{l} N_j(M_{i_1}(f_{k_{1,1}}, \ldots, f_{k_{1,|i_1|}}), \ldots, \\ \qquad M_{i_{|j|}}(f_{k_{|j|,1}}, \ldots, f_{k_{|j|,\left|i_{|j|}\right|}})). \end{array} \end{aligned}$$

This establishes our claim, since the operators $N_j \circ \prod_{l=1}^{|j|} M_{i_l}$ are strongly multilinear. Notice that Example 5.4 may be regarded as a combination of the present example and the last two cases in Example 5.3.

One obtains interesting subclasses of Noetherian operators by restricting the strongly multilinear mappings involved in the multilinear decompositions to be of a certain type. More precisely, let $\mathfrak{M}$ be a monomial monoid and let $\mathcal{M}$ be a set of strongly multilinear mappings $M : C[[\mathfrak{M}]]^{|M|} \to C[[\mathfrak{M}]]$. We say that $\mathcal{M}$ is a *multilinear type* if

MT1. The constant mapping $\{0\} \mapsto f$ is in $\mathcal{M}$ for each $f \in C[[\mathfrak{M}]]$.
MT2. The i-th projection mapping $\pi_i : C[[\mathfrak{M}]]^{|M|} \to C[[\mathfrak{M}]]$ is in $\mathcal{M}$ for $i = 1, \ldots, |M|$.
MT3. The multiplication mapping from $C[[\mathfrak{M}]]^2$ into $C[[\mathfrak{M}]]$ is in $\mathcal{M}$.
MT4. If $M, N_1, \ldots, N_{|M|} \in \mathcal{M}$, then $M \circ \prod_{i=1}^{|M|} N_i \in \mathcal{M}$.

Given subsets $\mathfrak{V}_1, \ldots, \mathfrak{V}_v, \mathfrak{W}_1, \ldots, \mathfrak{W}_w$ of $\mathfrak{M}$, we say that a strongly multilinear mapping

$$M : C[[\mathfrak{V}_1]] \times \cdots \times C[[\mathfrak{V}_v]] \to C[[\mathfrak{W}_1]] \times \cdots \times C[[\mathfrak{W}_w]]$$

is *of type* $\mathcal{M}$, if for $i = 1, \ldots, w$, there exists a mapping $N_i : C[[\mathfrak{M}]]^v \to C[[\mathfrak{M}]]$ in $\mathcal{M}$, such that $\pi_i \circ M$ coincides with the restriction of the domain and image of N_i to $C[[\mathfrak{V}_1]] \times \cdots \times C[[\mathfrak{V}_v]]$ resp. $C[[\mathfrak{W}_i]]$. We say that a Noetherian operator

$$\Phi : C[[\mathfrak{V}_1]] \times \cdots \times C[[\mathfrak{V}_v]] \to C[[\mathfrak{W}_1]] \times \cdots \times C[[\mathfrak{W}_w]]$$

is of type $\mathcal{M}$, if it admits a multilinear decomposition consisting of strongly multilinear mappings of type $\mathcal{M}$ only. In Examples 5.4 and 5.5, we may then replace "Noetherian operator" by "Noetherian operator of type $\mathcal{M}$".

EXAMPLE 5.6. For any set $\mathcal{S}$ of strongly linear mappings $C[[\mathfrak{M}]] \to C[[\mathfrak{M}]]$, there exists a smallest multilinear type $\mathcal{M} = \langle \mathcal{S} \rangle$ which contains $\mathcal{S}$. Taking $\mathbb{T} = C[[\mathfrak{M}]]$ to be the field of transseries whose logarithmic and exponential depths are bounded by ω, interesting special cases are obtained when taking $\mathcal{S} = \{\partial\}$ or $\mathcal{S} = \{\int\}$. Noetherian operators of type $\langle\{\partial\}\rangle$ resp. $\langle\{\int\}\rangle$ may then simply be called *differential* resp. *integral Noetherian operators*. Given a finite subset $g_1, \ldots, g_n$ of positive infinitely large transseries in $\mathbb{T}$, another interesting case is obtained by taking $\mathcal{S} = \{\circ_{g_1}, \ldots, \circ_{g_n}\}$, where $\circ_{g_i}$ stands for right composition with g_i.

5.2. Combinatorial representations of Noetherian operators. Let $\Phi : C[[\mathfrak{M}]] \to C[[\mathfrak{N}]]$ be a Noetherian operator with a multilinear decomposition $(M_i)_{i \in I}$. Then Φ is uniquely determined by the action of the M_i on monomials in $\mathfrak{M}$. For the deeper theory of Noetherian operators, it is convenient to represent this action in a combinatorial way.

Abstractly speaking, a set of $\mathfrak{M}$-*labeled structures* is a set Σ, together with a map that assigns to each $\sigma \in \Sigma$ a *labeling* $\sigma[\cdot] : \{1, \ldots, |\sigma|\} \to \mathfrak{M}; p \mapsto \sigma[p]$, where $|\sigma| \in \mathbb{N}$ stands for the size or arity of σ; for simplicity, we denote such a set of $\mathfrak{M}$-labeled structures also by Σ. For each subset $\mathfrak{S}$ of $\mathfrak{M}$, we denote the subset of $\mathfrak{S}$-labeled structures in Σ by

$$\Sigma_{\mathfrak{S}} = \{\sigma \in \Sigma \mid \operatorname{im} \sigma[\cdot] \subseteq \mathfrak{S}\}.$$

We strictly order couples in $\Sigma \times \mathfrak{M}$ by $(\sigma, \mathfrak{m}) \succ (\sigma', \mathfrak{m}') \Leftrightarrow \mathfrak{m} \succ \mathfrak{m}'$. A mapping $\theta : \Sigma \to \mathcal{P}(\mathfrak{N})$ is called a *choice operator*. We say that θ is *Noetherian*, if for any Noetherian subset $\mathfrak{S}$ of $\mathfrak{M}$, the subset

$$\{(\sigma, \mathfrak{n}) \mid \sigma \in \Sigma_{\mathfrak{S}} \wedge \mathfrak{n} \in \theta(\sigma)\}$$

of $\Sigma \times \mathfrak{N}$ is Noetherian.

EXAMPLE 5.7. Let $f : \mathfrak{M}^m \to \mathfrak{M}$ be a strictly increasing m-ary operation and let $\Sigma = \mathfrak{M}^m$, with $(x_1, \ldots, x_m)[p] = x_p$ for all $x_1, \ldots, x_m \in \mathfrak{M}$ and $1 \leqslant p \leqslant m$. Then $\theta : \Sigma \to \mathcal{P}(\mathfrak{M}); (x_1, \ldots, x_m) \mapsto \{f(x_1, \ldots, x_m)\}$ is a Noetherian choice operator.

Returning to our Noetherian operator Φ, each tuple $\sigma = (i, \mathfrak{m}_1, \ldots, \mathfrak{m}_{|i|})$ may be seen as an $\mathfrak{M}$-labeled combinatorial structure with $|\sigma| = |i|$ and $\sigma[p] = \mathfrak{m}_p$ for all $1 \leqslant p \leqslant |\sigma|$. Let $\Sigma = \Sigma^{\Phi}$ denote the set of such structures. We get a natural Noetherian choice operator $\theta = \theta^{\Phi} : \Sigma \to \mathcal{P}(\mathfrak{N})$ by taking $\theta(\sigma) = \operatorname{supp} M_i(\mathfrak{m}_1, \ldots, \mathfrak{m}_{|i|})$. Graphically speaking (see Figure 1 below), we may represent the action of θ on σ by a box with (a tuple of) "inputs" in $\mathfrak{M}$ and (a set of) "outputs" in $\mathfrak{N}$.

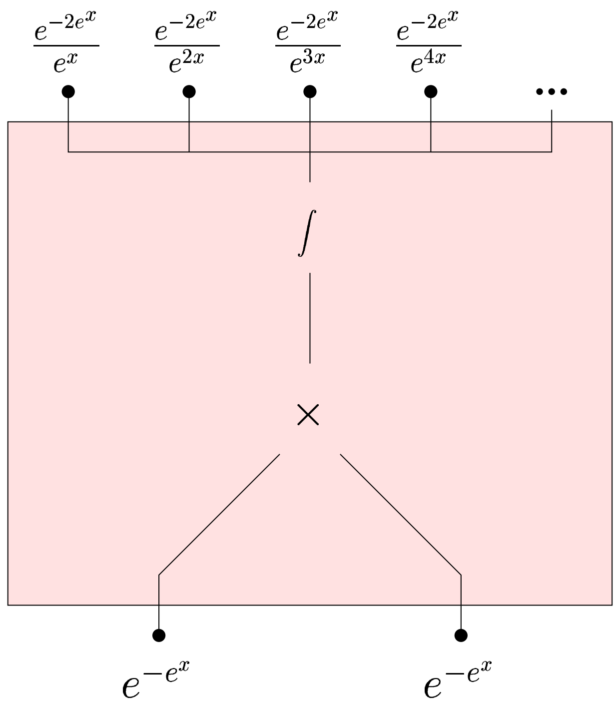

FIGURE 1. Graphical representation of the action of θ^M on the structure $\sigma \in \Sigma^M$ with input (e^{-e^x}, e^{-e^x}), for the strongly bilinear operator $M : (f, g) \mapsto \int fg$. Notice that $\int e^{-2e^x} = e^{-2e^x}(-\frac{1}{2e^x} + \frac{1}{4e^{2x}} - \frac{1}{4e^{3x}} + \frac{3}{8e^{4x}} + \cdots)$.

Conversely, given a Noetherian choice operator $\theta : \Sigma \to \mathcal{P}(\mathfrak{N})$ and an operator $\Theta : \Sigma \to C[[\mathfrak{N}]]$ with $\operatorname{supp} \Theta(\sigma) \subseteq \theta(\sigma)$ for all $\sigma \in \Sigma$, we define a Noetherian operator by

$$\Phi(f) = \sum_{\sigma \in \Sigma} \left(\prod_{p=1}^{|\sigma|} f_{\sigma[p]} \right) \Theta(\sigma). \tag{5.2}$$

As to its multilinear decomposition, we associate an $M_\sigma : C[[\mathfrak{M}]]^{|\sigma|} \to C[[\mathfrak{N}]]$ to each $\sigma \in \Sigma$ by

$$M_\sigma(f_1, \ldots, f_{|\sigma|}) = \left(\prod_{p=1}^{|\sigma|} f_{p,\sigma[p]} \right) \Theta(\sigma).$$

For Noetherian families $(f_i)_{i\in I} \in C[[\mathfrak{M}]]^I$, we indeed have

$$\begin{aligned}\Phi\left(\sum_{i\in I} f_i\right) &= \sum_{\sigma\in\Sigma}\left(\prod_{p=1}^{|\sigma|}\sum_{i\in I} f_{i,\sigma[p]}\right)\Theta(\sigma)\\ &= \sum_{\substack{\sigma\in\Sigma\\ i_1,\ldots,i_{|\sigma|}\in I}}\left(\prod_{p=1}^{|\sigma|} f_{i_p,\sigma[p]}\right)\Theta(\sigma)\\ &= \sum_{\substack{\sigma\in\Sigma\\ i_1,\ldots,i_{|\sigma|}\in I}} M_\sigma(f_{i_1},\ldots,f_{i_p}),\end{aligned}$$

since for each $\sigma\in\Sigma$, there are only finitely many tuples $(i_1,\ldots,i_{|\sigma|}) \in I^{|\sigma|}$, such that $\prod_{p=1}^{|\sigma|} f_{i_p,\sigma[p]} \neq 0$.

5.3. Composition of choice operators. In Example 5.5, we have shown that the composition of two Noetherian operators $\Phi : C[[\mathfrak{M}]] \to C[[\mathfrak{N}]]$ and $\Psi : C[[\mathfrak{N}]] \to C[[\mathfrak{V}]]$ is again Noetherian. Let us now show how to interpret the composition $\Psi\circ\Phi$ in a combinatorial way. Denote the natural choice operators associated to Φ and Ψ by $\theta : \Sigma \to \mathcal{P}(\mathfrak{N})$ resp. $\xi : \mathrm{T} \to \mathcal{P}(\mathfrak{V})$. We first define the composition $\xi\circ\theta : \Upsilon \to \mathcal{P}(\mathfrak{V})$ of the choice operators ξ and θ. Then Φ, Ψ and $\Psi\circ\Phi$ will be given by (5.2) and similar formulas, for certain mappings $\Theta : \Sigma \to C[[\mathfrak{N}]]$, $\Xi : \mathrm{T} \to C[[\mathfrak{V}]]$ resp. $\Xi\circ\Theta : \Upsilon \to C[[\mathfrak{V}]]$. Here we may assume that Θ and Ξ are given and we have to construct $\Xi\circ\Theta$.

Let $\tau\in\mathrm{T}$ be given together with a tuple $\sigma = (\sigma_1,\ldots,\sigma_{|\tau|}) \in \Sigma^{|\tau|}$, such that $\tau[q]\in\theta(\sigma_q)$ for each $1\leqslant q\leqslant|\tau|$. Then these data determine a unique $\mathfrak{M}$-labeled structure $\upsilon = \tau[\sigma]$, with $|\upsilon| = \sum_{q=1}^{|\tau|}|\sigma_q|$ and $\upsilon[p+\sum_{r=1}^{q-1}|\sigma_r|] = \sigma_q[p]$, for all $1\leqslant q\leqslant|\tau|$ and $1\leqslant p\leqslant|\sigma_q|$. We define Υ to be the set of all such combinatorial structures (see Figure 2 below). Then we claim that the choice operator $\xi\circ\theta : \Upsilon\to\mathcal{P}(\mathfrak{V}); \tau[\sigma]\mapsto\xi(\tau)$ is Noetherian.

So let $\mathfrak{S}$ be a Noetherian subset of $\mathfrak{M}$. We will prove that for any sequence $x_1 = (\tau_1[\sigma_1],\mathfrak{v}_1), x_2 = (\tau_2[\sigma_2],\mathfrak{v}_2),\ldots$ of elements in the set

$$\{(\tau[\sigma],\mathfrak{v}) | \tau[\sigma]\in\Upsilon_{\mathfrak{S}} \wedge \mathfrak{v}\in\xi(\tau)\},$$

there exist $i<j$ with $(\tau_i[\sigma_i],\mathfrak{v}_i) \succcurlyeq (\tau_j[\sigma_j],\mathfrak{v}_j)$. Since θ is Noetherian, $\mathfrak{T} = \bigcup_{\sigma\in\Sigma}\theta(\sigma)$ is a Noetherian subset of $\mathfrak{N}$, and we observe that $\tau\in\mathrm{T}_{\mathfrak{T}}$ for each $\tau[\sigma]\in\Upsilon_{\mathfrak{S}}$. Since ξ is Noetherian, we may therefore assume that $(\tau_i,\mathfrak{v}_i)\succcurlyeq(\tau_j,\mathfrak{v}_j)$, modulo the extraction of a subsequence. If $\mathfrak{v}_i\succ\mathfrak{v}_j$ for some $i<j$, then we have $(\tau_i[\sigma_i],\mathfrak{v}_i)\succ(\tau_j[\sigma_j],\mathfrak{v}_j)$ and we are done. Hence, we may assume that $(\tau_1,\mathfrak{v}_1) = (\tau_2,\mathfrak{v}_2) = \cdots$. We conclude by the observation that given $\tau\in\mathrm{T}$ there exist only a finite number of $(\sigma_1,\ldots,\sigma_{|\tau|})\in\Sigma^{|\tau|}$, such that $\tau[\sigma]\in\Upsilon_{\mathfrak{S}}$. Indeed, for each q, there are only a finite number of $\sigma_q\in\Sigma_{\mathfrak{S}}$ with $\tau[q]\in\theta(\sigma_q)$, since θ is Noetherian.

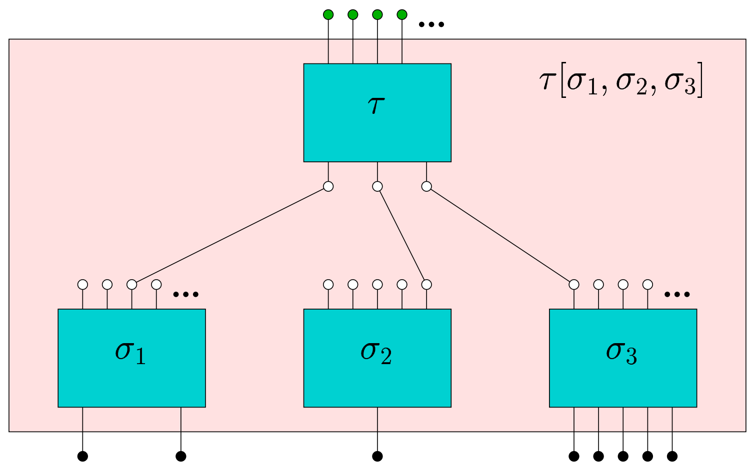

FIGURE 2. Illustration of the action of $\xi \circ \theta$ on a structure $\tau[\sigma_1, \sigma_2, \sigma_3]$ in Υ. For each σ_i that we attach to τ, we require the "output" of σ_i to coincide with the "input" of τ.

Now consider the operator

$$\Xi \circ \Theta : \Upsilon \to C[[\mathfrak{V}]]; \tau[\sigma] \mapsto \left(\prod_{q=1}^{|\tau|} \Theta(\sigma_q)_{\tau[q]} \right) \Xi(\tau).$$

Clearly, $\operatorname{supp}(\Xi \circ \Theta)(\upsilon) \subseteq (\xi \circ \theta)(\upsilon)$ for all $\upsilon \in \Upsilon$. We claim that

$$(\Psi \circ \Phi)(f) = \sum_{\upsilon \in \Sigma^{\xi \circ \theta}} \left(\prod_{r=1}^{|\upsilon|} f_{\upsilon[r]} \right) (\Xi \circ \Theta)(\upsilon), \tag{5.3}$$

for all $f \in C[[\mathfrak{M}]]$. Indeed,

$$\begin{aligned}
(\Psi \circ \Phi)(f) &= \sum_{\tau \in \mathrm{T}} \left(\prod_{q=1}^{|\tau|} \Phi(f)_{\tau[q]} \right) \Xi(\tau) \\
&= \sum_{\tau \in \mathrm{T}} \left[\prod_{q=1}^{|\tau|} \sum_{\substack{\sigma_q \in \Sigma_\sigma \\ \tau[q] \in \theta(\sigma_q)}} \left(\prod_{p=1}^{|\sigma_q|} f_{\sigma_q[p]} \right) \Theta(\sigma_q)_{\tau[q]} \right] \Xi(\tau) \\
&= \sum_{\tau[\sigma] \in \Upsilon} \left[\prod_{q=1}^{|\tau|} \left(\prod_{p=1}^{|\sigma_q|} f_{\sigma_q[p]} \right) \Theta(\sigma_q)_{\tau[q]} \right] \Xi(\tau)
\end{aligned}$$

$$= \sum_{\upsilon\in\Upsilon}\left(\prod_{r=1}^{|\upsilon|} f_{\upsilon[r]}\right)(\Xi\circ\Theta)(\upsilon).$$

This yields the desired combinatorial description of the composition $\Psi\circ\Phi$.

5.4. Canonical multilinear decompositions. We already noticed that each Noetherian operator $\Phi : C[[\mathfrak{M}]] \to C[[\mathfrak{N}]]$ has a multilinear decomposition of the form $(M_i)_{i\in\mathbb{N}}$, such that M_i has arity i for each $i \in \mathbb{N}$. Setting $\Phi_i = M_i(f,\ldots,f)$ for all f and i, we then have

$$\Phi = \Phi_0 + \Phi_1 + \Phi_2 + \cdots \tag{5.4}$$

Now assume that $C \supseteq \mathbb{Q}$ (so that C is in particular torsion-free). Then, modulo replacing each Φ_i by the operator $\tilde{\Phi}_i$ with

$$\tilde{\Phi}_i(f_1,\ldots,f_i) = \frac{1}{i!}\sum_{\sigma\in\mathfrak{S}_i} \Phi_i(f_{\sigma(1)},\ldots,f_{\sigma(i)}),$$

we may assume without loss of generality that the Φ_i are symmetric. Under this additional symmetry assumption, the decomposition (5.4) is actually unique, and we call Φ_i *the* homogeneous part of Φ of degree i.

Proposition 5.8. *Let $\Phi : C[[\mathfrak{M}]]^i \to C[[\mathfrak{N}]]$ be a Noetherian operator with a multilinear decomposition $(M_i)_{i\in\mathbb{N}}$, such that M_i is symmetric and of arity i for each $i \in \mathbb{N}$. If C is torsion-free and $\Phi = 0$, then $M_i = 0$ for each $i \in \mathbb{N}$.*

Proof. We observe that it suffices to prove that $\Phi_i = 0$ for each $i \in \mathbb{N}$, since the M_i are symmetric and C is torsion-free. Assume the contrary and let $f \in C[[\mathfrak{M}]]$ be such that $\Phi_i(f) \neq 0$ for some i. Choose $\mathfrak{m} \in \mathfrak{S} = \bigcup_{i\in I} \operatorname{supp} \Phi_i(f) \neq \varnothing$ is Noetherian. The Noetherianity of $(\Phi_i(f))_{i\in\mathbb{N}}$ implies that there exist only a finite number of indices i, such that $\mathfrak{m} \in \operatorname{supp} \Phi_i(f)$. Let $i_1 < \cdots < i_n$ be those indices.

Let $c_k = \Phi_{i_k}(f)_{\mathfrak{m}}$ for all $k \in \{1,\ldots,n\}$. For any $l \in \{1,\ldots,n\}$, we have $\Phi_{i_k}(lf)_{\mathfrak{m}} = l^{i_k} c_k$, by multilinearity. On the other hand, $\Phi(lf)_{\mathfrak{m}} = \Phi_{i_1}(lf)_{\mathfrak{m}} + \cdots + \Phi_{i_n}(lf)_{\mathfrak{m}} = 0$ for each l, so that

$$\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ n^{i_1} & \cdots & n^{i_n} \end{pmatrix}\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = 0.$$

The matrix on the left hand side admits an inverse with rational coefficients. (Indeed, by the sign rule of Descartes, a real polynomial $\alpha_1 x^{i_1} + \cdots + \alpha_n x^{i_n}$ cannot have n distinct positive zeros unless $\alpha_1 = \cdots = \alpha_n = 0$.) Consequently, an integer multiple of the vector on the right hand side vanishes. We infer that $c_1 = \cdots = c_n = 0$, since C is torsion-free. This contradiction completes the proof. □

6. The algebraic implicit function theorem

Let $\mathfrak{M}$ and $\mathfrak{N}$ be monomial sets and let

$$\Phi : C[[\mathfrak{M}]] \times C[[\mathfrak{N}]] \to C[[\mathfrak{M}]], (f, g) \mapsto \Phi(f, g)$$

be a Noetherian operator. We call Φ *strictly extensive in* f if there exists a multilinear decomposition $(M_i)_{i\in I}$ of Φ, such that for all i, $(\mathfrak{v}_1, \ldots, \mathfrak{v}_{|i|}) \in (\mathfrak{M} \amalg \mathfrak{N})^{|i|}$, $1 \leqslant j \leqslant |i|$ and $\mathfrak{m} \in \operatorname{supp} M_i(\mathfrak{v}_1, \ldots, \mathfrak{v}_{|i|})$, we have $\mathfrak{v}_j \in \mathfrak{M} \Rightarrow \mathfrak{m} \prec \mathfrak{v}_j$. In particular, such a Φ is contracting in f. The main objective of this section will be to prove the following theorem:

THEOREM 6.1. *Let $\Phi : C[[\mathfrak{M}]] \times C[[\mathfrak{N}]] \to C[[\mathfrak{M}]], (f, g) \mapsto \Phi(f, g)$ be a Noetherian operator, which is strictly extensive in f. Then for each $g \in C[[\mathfrak{N}]]$ the operator $\Phi(\cdot, g)$ on $C[[\mathfrak{M}]]$ has a unique fixed point $\Psi(g)$, and the operator $\Psi : C[[\mathfrak{N}]] \to C[[\mathfrak{M}]]$ is Noetherian.*

6.1. Iteration of choice operators with parameters. Let $\Phi : C[[\mathfrak{M}]]\times C[[\mathfrak{N}]] \to C[[\mathfrak{M}]]$ be as in Theorem 6.1 and let $\theta : \Sigma \to \mathcal{P}(\mathfrak{M})$ be the natural Noetherian choice operator associated to Φ. The fact that Φ is strictly extensive in f implies that θ may be assumed to be *strictly extensive* on $\mathfrak{M}$, i.e.,

$$\forall \sigma \in \Sigma, \forall \mathfrak{m} \in (\operatorname{im} \sigma[\cdot] \cap \mathfrak{M}), \forall \mathfrak{n} \in \theta(\sigma), \quad \mathfrak{n} \prec \mathfrak{m}.$$

Also, let $\iota : \Delta_{\mathfrak{N}} \to \mathcal{P}(\mathfrak{N})$ be the natural Noetherian choice operator associated to the identity mapping $\mathrm{Id}_{\mathfrak{N}} : C[[\mathfrak{N}]] \to C[[\mathfrak{N}]]$. Actually, we take $\Delta_{\mathfrak{N}} = \{\delta_{\mathfrak{n}} \mid \mathfrak{n} \in \mathfrak{N}\}$, with $|\delta_{\mathfrak{n}}| = 1, \delta_{\mathfrak{n}}[1] = \mathfrak{n}$ and $\iota(\delta_n) = \{\mathfrak{n}\}$ for all $\mathfrak{n} \in \mathfrak{N}$.

Now consider the sets $\mathrm{T} = \amalg_{h\in\mathbb{N}} \mathrm{T}_h$ of $(\mathfrak{M} \amalg \mathfrak{N})$-labeled combinatorial structures, where the T_d are defined by

$$\begin{aligned} \mathrm{T}_0 &= \Sigma_{\mathfrak{N}}\,; \\ \mathrm{T}_{d+1} &= (\Sigma\backslash\Sigma_{\mathfrak{N}}) \circ (\mathrm{T}_d \amalg \Delta_{\mathfrak{N}}). \end{aligned}$$

For each $\tau \in \mathrm{T}$, the minimal $d \in \mathbb{N}$ with $\tau \in \mathrm{T}_d$ is called the *depth* of τ. We have a natural choice operator $\xi : \mathrm{T} \to \mathcal{P}(\mathfrak{M})$, which is defined componentwise by

$$\begin{aligned} \xi_{|\mathrm{T}_0} &= \theta_{|\Sigma_{\mathfrak{N}}}\,; \\ \xi_{|\mathrm{T}_{d+1}} &= \theta_{|\Sigma\backslash\Sigma_{\mathfrak{N}}} \circ (\xi_{|\mathrm{T}_d} \amalg \iota_{|\Delta_{\mathfrak{N}}}). \end{aligned}$$

Here $\xi_{|\mathrm{T}_d} \amalg \iota_{|\Delta_{\mathfrak{N}}} : \mathrm{T}_d \amalg \Delta_{\mathfrak{N}} \to \mathcal{P}(\mathfrak{M} \amalg \mathfrak{N})$ stands for the choice operator which coincides with ξ on T_d and with ι on $\Delta_{\mathfrak{N}}$. Similarly, the componentwise definition of ξ means that we take $\xi = \coprod_{d\in\mathbb{N}} \xi_{|\mathrm{T}_d}$. In Figure 3 below one finds an illustration of the action of ξ on a structure in T. We will also call $\theta^{*,\mathfrak{N}}$ the *iteration of* θ *with parameters in* $\mathfrak{N}$.

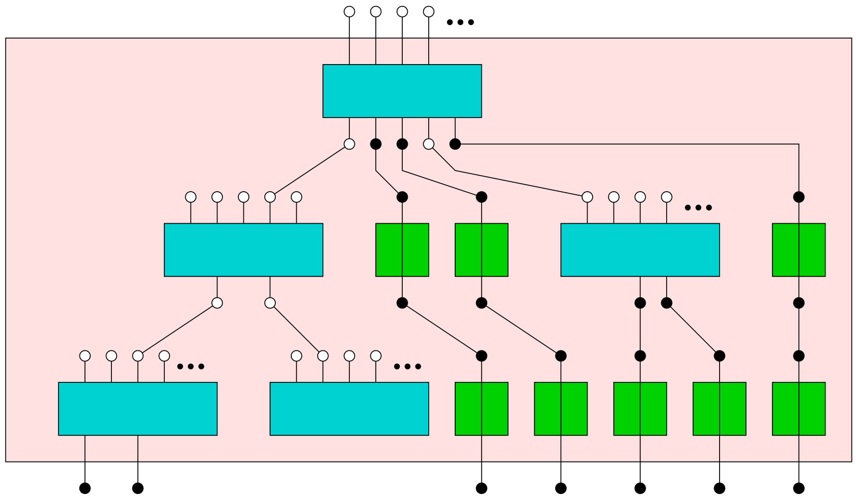

FIGURE 3. Illustration of the action of the iterated choice operator $\xi = \theta^{*,\mathfrak{N}}$ on a structure in $\mathrm{T} = \Sigma^{*,\mathfrak{N}}$. The connected "inputs" and "outputs" should match in a similar way as in Figure 2. The white and black dots correspond to monomials in $\mathfrak{M}$ resp. $\mathfrak{N}$.

THEOREM 6.2. *Let Σ be a set of $(\mathfrak{M} \amalg \mathfrak{N})$-labeled structures and $\theta : \Sigma \to \mathcal{P}(\mathfrak{M})$ a Noetherian choice operator which is extensive on $\mathfrak{M}$. Then $\theta^{*,\mathfrak{N}}$ is Noetherian.*

Proof. Let $\mathfrak{A}$ be a Noetherian subset of $\mathfrak{N}$. Assume that there exists a bad sequence

$$(\upsilon_1, \mathfrak{m}_1), (\upsilon_2, \mathfrak{m}_2), \ldots, \tag{6.1}$$

with $\upsilon_i \in \mathrm{T}_{\mathfrak{A}}$ and $\mathfrak{m}_i \in \xi(\tau_i)$ for each i. We may assume that we have chosen this bad sequence minimally in the sense that the depth of each υ_i is minimal in the set of all bad sequences with fixed $(\upsilon_1, \mathfrak{m}_1), \ldots, (\upsilon_{i-1}, \mathfrak{m}_{i-1})$. Writing $\upsilon_i = \sigma_i[\tau_{i,1}, \ldots, \tau_{i,|\sigma_i|}]$ for each i, we claim that the induced ordering on $\check{\mathfrak{B}} = \{(\tau_{i,j}, \mathfrak{w}_{i,j}) \mid i \in \mathbb{N} \wedge 1 \leqslant j \leqslant |\tau_i| \wedge \mathfrak{w}_{i,j} \in \xi(\tau_{i,j})\}$ is Noetherian.

Indeed, suppose for contradiction that the claim is false, and let

$$(\tau_{i_1,j_1}, \mathfrak{w}_{i_1,j_1}), (\tau_{i_2,j_2}, \mathfrak{w}_{i_2,j_2}), \ldots$$

be a bad sequence. Notice that $(\tau_{i_k,j_k}, \mathfrak{w}_{i_k,j_k}) \prec (\upsilon_{i_k}, \mathfrak{m}_{i_k})$ for all k, since θ is strictly extensive on $\mathfrak{M}$. Hence, taking k such that i_k is minimal, the sequence

$$(\upsilon_1, \mathfrak{m}_1), \ldots, (\upsilon_{i_k-1}, \mathfrak{m}_{i_k-1}), (\tau_{i_k,j_k}, \mathfrak{w}_{i_k,j_k}), (\tau_{i_{k+1},j_{k+1}}, \mathfrak{w}_{i_{k+1},j_{k+1}}), \ldots$$

is also bad. This contradicts the minimality of (6.1).

At this point we have proved that $\check{\mathfrak{B}}$ is Noetherian. In particular, $\mathfrak{B} = \{\mathfrak{w} \mid (\upsilon, \mathfrak{w}) \in \check{\mathfrak{B}}\}$ is Noetherian. Hence, there exist $i_1 > i_2 > \cdots$ with $(\sigma_{i_1}, \mathfrak{m}_{i_1}) \not\succcurlyeq (\sigma_{i_2}, \mathfrak{m}_{i_2}) \not\succcurlyeq \cdots$, since $\sigma_1, \sigma_2, \ldots \in \Sigma_{|\mathfrak{B} \amalg \mathfrak{A}}$. If $\mathfrak{m}_{i_m} \succ \mathfrak{m}_{i_n}$ for some $m > n$, then $(\upsilon_{i_m}, \mathfrak{m}_{i_m}) \succ (\upsilon_{i_n}, \mathfrak{m}_{i_n})$ and we are done. Otherwise, $(\sigma_{i_1}, \mathfrak{m}_{i_1}) = (\sigma_{i_2}, \mathfrak{m}_{i_2}) = \cdots$. Now for every $1 \leqslant p \leqslant |\sigma_{i_1}|$, the $(\tau, \mathfrak{w}) \in \check{\mathfrak{B}} \amalg \{(\delta_{\mathfrak{n}}, \mathfrak{n}) \mid \mathfrak{n} \in \mathfrak{A}\}$ with $\mathfrak{w} = \sigma_{i_1}[p]$ are finite in number, since they form an antichain. Consequently, $\upsilon_{i_1}, \upsilon_{i_2}, \ldots$ can only take a finite number of values and there exist $m < n$ with $(\upsilon_{i_m}, \mathfrak{m}_{i_m}) = (\upsilon_{i_n}, \mathfrak{m}_{i_n})$. This contradicts the badness of (6.1). □

6.2. Proof of the implicit function theorem. With the notations from the previous section, let $\Theta : \Sigma \to C[[\mathfrak{M}]]$ be a mapping, such that $\operatorname{supp} \Theta(\sigma) \subseteq \theta(\sigma)$ for all $\sigma \in \Sigma$, and such that (5.2) holds for all $f \in C[[\mathfrak{M}]] \times [[\mathfrak{N}]]$. We now define $\Xi : \mathrm{T} \to C[[\mathfrak{M}]]$ componentwise as follows:

$$\begin{aligned} \Xi_{|\mathrm{T}_0} &= \Theta_{|\Sigma_{\mathfrak{N}}} ; \\ \Xi_{|\mathrm{T}_{d+1}} &= \Theta_{|\Sigma \setminus \Sigma_{\mathfrak{N}}} \circ (\Xi_{|\mathrm{T}_d} \amalg \mathrm{I}_{|\Delta_{\mathfrak{N}}}), \end{aligned}$$

where $\mathrm{I}_{|\Delta_{\mathfrak{N}}} : \Delta_{\mathfrak{N}} \to C[[\mathfrak{N}]]; \delta_{\mathfrak{n}} \mapsto \mathfrak{n}$. Theorem 6.2 implies that we may define a function $\Psi : C[[\mathfrak{N}]] \to C[[\mathfrak{M}]]$ by the formula

$$\Psi(g) = \sum_{\tau \in \mathrm{T}} \left(\prod_{p=1}^{|\tau|} g_{\tau[p]} \right) \Xi(\tau). \tag{6.2}$$

We can now prove the following more explicit version of the implicit function theorem.

Theorem 6.3. *Let $\Phi : C[[\mathfrak{M}]] \times C[[\mathfrak{N}]] \to C[[\mathfrak{M}]], (f, g) \mapsto \Phi(f, g)$ be a Noetherian operator, which is strictly extensive in f. Then the Noetherian operator $\Psi : C[[\mathfrak{N}]] \to C[[\mathfrak{M}]]$ defined by* (6.2) *is unique with the property that $\Psi(g) = \Phi(\Psi(g), g)$ for all $g \in C[[\mathfrak{N}]]$.*

Proof. Identifying $C[[\mathfrak{M}]] \times C[[\mathfrak{N}]]$ and $C[[\mathfrak{M} \amalg \mathfrak{N}]]$ via the natural isomorphism, we have

$$(\Psi(g), g) = \Psi(g) + g = \sum_{\tau \in \mathrm{T} \amalg \Delta_{\mathfrak{N}}} \left(\prod_{q=1}^{|\tau|} g_{\tau[q]} \right) (\Xi \amalg \mathrm{I})(\tau),$$

for all $g \in C[[\mathfrak{N}]]$. Similarly, for all $(f, g) \in C[[\mathfrak{M}]] \times C[[\mathfrak{N}]]$, we have

$$\Phi_{\mathrm{rest}}(f, g) = \Phi(f, g) - \Phi(0, g) = \sum_{\sigma \in \Sigma \setminus \Sigma_{\mathfrak{N}}} \left(\prod_{p=1}^{|\sigma|} (f + g)_{\sigma[p]} \right) (\Theta_{|\Sigma \setminus \Sigma_{\mathfrak{N}}})(\sigma).$$

Applying (5.3), we conclude that

$$\begin{aligned}\Psi(g) &= \sum_{\tau\in \mathrm{T}_0}\left(\prod_{q=1}^{|\tau|} g_{\tau[q]}\right)\Xi(\tau)+\sum_{\tau\in \mathrm{T}\setminus \mathrm{T}_0}\left(\prod_{q=1}^{|\tau|} g_{\tau[q]}\right)\Xi(\tau)\\ &= \sum_{\tau\in \mathrm{T}_0}\left(\prod_{q=1}^{|\tau|} g_{\tau[q]}\right)\Xi(\tau)+\\ &\quad \sum_{\upsilon\in(\Sigma\setminus\Sigma_{\mathfrak{N}})\circ(\mathrm{T}\amalg\Delta_{\mathfrak{N}})}\left(\prod_{r=1}^{|\upsilon|} g_{\upsilon[r]}\right)(\Theta_{|\Sigma\setminus\Sigma_{\mathfrak{N}}}\circ(\Xi_{|\mathrm{T}}\amalg \mathrm{I}_{|\Delta_{\mathfrak{N}}}))(\upsilon)\\ &= \Phi(0,g)+\Phi_{\mathrm{rest}}(\Psi(g),g)\\ &= \Phi(\Psi(g),g),\end{aligned}$$

for all $g\in C[[\mathfrak{N}]]$. The uniqueness of Ψ follows in the same way as in the proof of Theorem 4.7, since Φ is contracting in f. □

Corollary 6.4. *Let $\mathcal{M}$ be a multilinear type. If Φ is of type $\mathcal{M}$ in Theorem 6.1, then so is Ψ.* □

6.3. Applications.

Example 6.5. We first show that the classical implicit function theorem for bivariate power series follows from Theorem 4.7. So let $f=\sum_{i,j} f_{i,j}v^i u^j\in C[[v,u]]$ be a bivariate power series with $f_{0,0}=0$ and $f_{1,0}\neq 0$. Then we have to prove that there exists a unique power series $g\in uC[[u]]$ with

$$f(g(u),u)=0.$$

Modulo division of f by $f_1=\sum_j f_{1,j}u^j$ and passing f_1 to the other side of the equation, the problem can be reduced to solving the equation

$$g(u)=f(g(u),u) \tag{6.3}$$

for $f\in C[[v,u]]$ with $f_{0,0}=f_{1,0}=0$. Under these assumptions, the series f corresponds to an operator $\Phi: uC[[u]]\times\{0\}\to uC[[u]]; (g,0)\mapsto f(g,u)=\sum_{i,j} f_{i,j}g(u)^i u^j$. Theorem 4.7 then provides us with a unique mapping $\Psi:\{0\}\to vC[[v]]$ with $\Psi(0)=\Phi(\Psi(0),0)$. Taking $g=\Psi(0)$, we thus find the unique solution to (6.3).

Moreover, Theorem 6.3 actually tells us that the "natural solution" to (6.3), which is obtained by recursively plugging in the left hand side of the equation in the right hand side, is indeed a solution. We also notice that by applying Theorem 6.3 to the operator

$$\begin{aligned}\Phi: uC[f_i][[u]]\times\{0\} &\longrightarrow uC[f_i][[u]];\\ (g,0) &\longmapsto f(g,u)=\sum_i f_i g(u)^i\end{aligned}$$

instead of the previous Φ, we actually get a solution $g(u)$ in terms of the coefficients of f.

EXAMPLE 6.6. The above example naturally generalizes to the multivariate case. What is more, we may consider non-commutative power series in several variables. Given symbols $u_1, \ldots, u_n$, order the free monomial monoid $\{u_1, \ldots, u_n\}^\star$ in $u_1, \ldots, u_n$ by the ordering $\succcurlyeq$ from Example 2.1. Then the ring of non-commutative power series in $u_1, \ldots, u_n$ over C is given by

$$C\langle\langle u_1, \ldots, u_n\rangle\rangle = C[[\{u_1, \ldots, u_n\}^\star]].$$

Now consider the equation

$$g(u_1, \ldots, u_n) = f(g(u_1, \ldots, u_n), u_1, \ldots, u_n), \tag{6.4}$$

for $f \in C[[v, u_1, \ldots, u_n]]$ with $f_1 = f_v = 0$. Then it may be proved in a similar way as in the previous example that this equation admits a unique infinitesimal solution. Again, this solution is equal to the natural expression which is obtained when repeatedly plugging in the left hand side of (6.4) into the right hand side. Again, the solution may be expressed naturally in terms of the coefficients of the equation.

EXAMPLE 6.7. Let $\mathbb{T} = C[[\mathfrak{M}]]$ be the field of transseries in x, whose logarithmic and exponential depths are bounded by some integer $d \in \mathbb{N}$ [vdH97]. The transseries $e^{-x^2} + e^{-e^x} + e^{-e^x/x} + \cdots$ is an example of an element in $\mathbb{T}$ if $d = 2$. Now consider the integral equation

$$f = g + \int f^2, \tag{6.5}$$

for $f, g \in \mathbb{T}$ and where $f, g \prec e^{-x}$. Taking $\mathfrak{N} = \{\mathfrak{m} \in \mathfrak{M} \mid \mathfrak{m} \prec e^{-x}\}$ we may consider the operator $\Phi : C[[\mathfrak{N}]] \times C[[\mathfrak{N}]] \to C[[\mathfrak{N}]]; (f, g) \mapsto g + \int f^2$. Theorem 4.7 then implies that there exists a unique function $\Psi : C[[\mathfrak{N}]] \to C[[\mathfrak{N}]]$, such that $f = \Psi(g)$ satisfies (6.5) for all $g \in C[[\mathfrak{N}]]$. Theorem 6.3 and its corollary imply that Ψ is actually an integral Noetherian operator. Modulo regrouping terms, this means that the series

$$f = g + \int g^2 + 2\int g \int g^2 + 4 \int g \int g \int g^2 + \int \left(\int g^2\right)^2 + \cdots$$

is indeed a solution to (6.5) for all $g \in C[[\mathfrak{N}]]$.

EXAMPLE 6.8. Let $\mathbb{T} = C[[\mathfrak{M}]]$ now be the field of transseries in x, whose exponential and logarithmic depths are bounded by ω. Consider the functional equation

$$f(x) = g(x) + h(x) f(x^2) + f'(e^{\log^2 x}). \tag{6.6}$$

for $f, g, h \in \mathbb{T}$ and $f, g, h \prec e^{-x}$. Taking $\mathfrak{N} = \{\mathfrak{m} \in \mathfrak{M} \mid \mathfrak{m} \prec e^{-x}\}$, Theorem 6.3 yields a Noetherian operator $\Psi : C[[\mathfrak{N}]] \times C[[\mathfrak{N}]] \to C[[\mathfrak{N}]]; (g, h) \mapsto$

$\Psi(g, h)$, such that $f(x) = \Psi(g, h)$ is a solution to (6.6). Moreover, Ψ is what one could call a "differential compositional Noetherian operator".

EXAMPLE 6.9. For independent infinitely large variables $x, y \succ 1$ consider the monomial group

$$\mathfrak{M} = x^{\mathbb{R}} y^{\mathbb{R}} e^{x\mathbb{R}} e^{y\mathbb{R}} e^{xe^{x+y}\mathbb{R}}$$

and its subset

$$\mathfrak{N} = x^{\mathbb{R}} y^{\mathbb{R}} e^{x\mathbb{R}} e^{y\mathbb{R}} e^{-xe^{x+y}\mathbb{R}_*^+}.$$

Then the equation

$$f = e^{-xe^{x+y}} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + e^{-x-3y}\frac{\partial^3 f}{\partial x^3}\frac{\partial^2 f}{\partial x \partial y} \tag{6.7}$$

admits a unique solution $f \in \mathbb{R}[[\mathfrak{N}]]$, which can be expressed as a "partial differential series". Theorem 4.7 can not be directly applied in this case.

REFERENCES

[DG86] B. I. Dahn and P. Göring, *Notes on exponential-logarithmic terms*, Fund. Math. **127** (1986), 45–50.

[É92] J. Écalle, *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*, Actualités mathématiques, Hermann, Paris, 1992.

[Hah07] H. Hahn, *Über die nichtarchimedischen Größensysteme*, Sitz. Akad. Wiss. Wien **116** (1907), 601–655.

[Hig52] G. Higman, *Ordering by divisibility in abstract algebras*, Proc. London Math. Soc. **2** (1952), 326–336.

[Mil85] E. C. Milner, *Basic wqo- and bqo-theory*, Graphs and orders (Rival, ed.), D. Reidel Publ. Comp., Dordrecht, 1985, pp. 487–502.

[NW63] C. St. J. A. Nash-Williams, *On well-quasi-ordering finite trees*, Proc. Cambridge Philos. Soc. **59** (1963), 833–835.

[PC90] S. Prieß-Crampe, *Der Banachsche Fixpunktsatz für ultrametrische Räume*, Results Math. **18** (1990), 178–186.

[PCR93] S. Priess-Crampe and P. Ribenboim, *Fixed points, combs and generalized power series*, Abh. Math. Sem. Hamburg **63** (1993), 227–244.

[Pou85] M. Pouzet, *Applications of well quasi-ordering and better quasi-ordering*, Graphs and orders (Rival, ed.), D. Reidel Publ. Comp., Dordrecht, 1985, pp. 503–519.

[vdH97] J. van der Hoeven, *Automatic asymptotics*, Ph.D. thesis, École polytechnique, France, 1997.

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