THE LIMIT LAMINATION METRIC FOR THE COLDING-MINICOZZI MINIMAL LAMINATION

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ABSTRACT. We prove that the singular set $S(\mathcal{L})$ of convergence in a Colding-Minicozzi limit minimal lamination $\mathcal{L}$ is a $C^{1,1}$-curve which is orthogonal to leaves of the limit minimal lamination $\mathcal{L}$ in some neighborhood of $\mathcal{S}(\mathcal{L})$. We also obtain useful information on the related limit lamination metric.

1. Introduction

In a series of papers, Colding and Minicozzi [1], [2], [4], [3], [5] developed a theory for minimal surfaces with compactness and regularity results, one of whose end goals is to prove: Every sequence of properly embedded minimal surfaces in a homogeneously regular Riemannian three-manifold N, each of which intersects small balls of fixed size radii in simply-connected components, has a subsequence that converges to a minimal lamination $\mathcal{L}$ of N, which we call a *Colding-Minicozzi limit minimal lamination*, when it exists. Such a sequence of minimal surfaces is called *locally-simply-connected* and we refer to this compactness result as the Limit Lamination Theorem when it holds (see Section 11.1 of [12] for some additional partial results in the case when $N = \mathbb{R}^3$). We believe that this theorem always holds when the minimal surfaces are complete and N is complete with nonnegative curvature but fails in general if N does not have nonnegative curvature.

Furthermore, Colding and Minicozzi prove that the convergence of such a locally-simply-connected sequence of minimal surfaces to a minimal lamination $\mathcal{L}$ is $C^{\alpha}, 0 < \alpha < 1$, in the complement of a locally finite collection $S(\mathcal{L})$ of Lipschitz curves that are properly embedded in N and along which the convergence is not C^1. They also prove that if $S(\mathcal{L}) \neq \emptyset$, then $\mathcal{L}$ is a foliation when restricted to a neighborhood of $S(\mathcal{L})$ and the singular curves in $S(\mathcal{L})$

are transverse to $\mathcal{L}$. Note that $S(\mathcal{L}) \neq \emptyset$ if the curvature of the sequence of surfaces blows up at some point in N.

When N is $\mathbb{R}^3$ and the sequence of minimal surfaces is locally-simply-connected and they are planar domains, then there is always a subsequence of the surfaces which converges to a minimal lamination $\mathcal{L}$ [5], [12]. Meeks and Rosenberg [13] have applied this compactness and regularity theorem in [4], as well as a related one-sided curvature estimate [4] and in the case the surfaces in the sequence are simply connected, to prove that the plane and the helicoid are the only properly embedded simply-connected minimal surfaces in $\mathbb{R}^3$. These results imply that in a small neighborhood of any point of almost maximal curvature on an embedded minimal disk in a Riemannian three-manifold, the disk has the appearance of an almost perfectly formed homothetically shrunk helicoid with many sheets. Motivated by this local picture of embedded minimal disks at points of large curvature, the author conjectured that the Lipschitz curves $S(\mathcal{L})$ should be $C^{1,1}$-curves which are orthogonal to the leaves of $\mathcal{L}$. A main goal of this paper is to present the author's original proof of this conjecture; a second later proof appears in [8].

The presentation of the proof of Theorem 1 below given in [8] was based on the suggestion by Colding and Minicozzi to have it more closely follow the proof of their original Lipschitz regularity theorem for the singular set, while the proof presented here uses a nonstandard blow-up procedure and gives some different insight into the convergence of the minimal surfaces to $\mathcal{L}$ in a neighborhood of $S(\mathcal{L})$.

THEOREM 1 (Regularity Theorem).   *Let $\{\mathcal{O}_i\}_{i \in I}$ be an open cover of small geodesic balls in a Riemannian three-manifold N which forms a basis for the topology of N. Let $\Sigma(n)$ be a sequence of properly embedded compact minimal surfaces such that for all $i \in I$ and for n sufficiently large, $\Sigma(n) \cap \mathcal{O}_i$ consists of simply-connected components. If $\mathcal{L}$ is a limit minimal lamination for the $\Sigma(n)$ and the convergence of $\Sigma(n)$ is $C^\alpha, \alpha \in (0,1)$, outside of a nonempty set $S(\mathcal{L})$ of locally finite collection of Lipschitz curves which are transverse to $\mathcal{L}$, then:*

(1) *$S(\mathcal{L})$ consists of a locally-finite collection of integral curves of the unit Lipschitz normal vector field to their $C^{1,1}$-foliation neighborhoods in $\mathcal{L}$.*

(2) *The singular set of convergence $S(\mathcal{L})$ is a locally-finite set of $C^{1,1}$-curves which are orthogonal to the leaves of $\mathcal{L}$.*

(3) *The geodesic curvature of the curves in $S(\mathcal{L})$ is defined almost everywhere and where the curvature is defined, it is locally bounded. If N is compact, then the estimate for the bound on the curvature of the curves in $S(\mathcal{L})$ can be chosen to depend only on N and the covering $\{\mathcal{O}_i\}_{i \in I}$.*

The above theorem and its proof are useful in applications of the Limit Lamination Theorem of Colding and Minicozzi. For example, we use the above theorem in the proof of a related theorem, which we call the Lamination Metric Theorem. The Lamination Metric Theorem describes how the intrinsic metrics on minimal surfaces in Theorem 1 converge to a natural metric space structure of the limit lamination; this application appears in Section 3 and it has applications [9], [10], [11] to classifying properly embedded minimal surfaces of finite genus in $\mathbb{R}^3$. Recently, Meeks and Rosenberg [13] have used this result in an important way to prove that the closure of a complete embedded minimal surface of positive injectivity radius in a three-manifold has the structure of a $C^{1,\alpha}$ lamination.

Note that the $C^{1,1}$-regularity of the singular set given in Theorem 1 implies that the singular set has almost everywhere defined curvature, which is locally bounded. In [16], Meeks and Weber give an example of a sequence of properly embedded minimal disks in the unit ball in $\mathbb{R}^3$ which converges to a minimal foliation $\mathcal{L}$ of the unit ball and such that the singular set $S(\mathcal{L})$ of convergence passes though the origin and is an arc on a circle of radius 1. This example shows that the although the curvature of the singular curve is locally bounded it need not be zero. More generally, Meeks and Weber prove that the $C^{1,1}$-regularity of $S(\mathcal{L})$ given in Theorem 1 is the best possible result by proving that any properly embedded $C^{1,1}$-curve Γ in an open set in $\mathbb{R}^3$ has a regular neighborhood in the open set with a Colding-Minicozzi limit minimal lamination $\mathcal{L}$ with $S(\mathcal{L}) = \Gamma$.

Based on all the above results, the author makes the following conjecture which, if correct, is sharp by the above mentioned example of Meeks and Weber where the singular set is a circle of curvature 1 at the origin.

CONJECTURE 1.1. *Suppose $\{D_n \mid n \in \mathbb{N}\}$ is a sequence of embedded minimal disks in the unit ball B centered at the origin in $\mathbb{R}^3$ with $\partial D_n \subset \partial B$, which converge to a minimal lamination $\mathcal{L}$ of the interior of B. Suppose that $S(t)$ is a component of $S(\mathcal{L})$ parametrized by arc length and such that $S(0)$ is the origin of B. Then $\limsup_{t \to 0} \|S'(t) - S'(0)\|/t \leq 1$.*

2. The proof of the regularity theorem

In this section we will prove Theorem 1. As mentioned in the Introduction this theorem states that the singular set $S(\mathcal{L})$ of convergence of Colding-Minicozzi is a $C^{1,1}$-locally-finite collection of integral curves of the unit normal to a minimal lamination limit $\mathcal{L}$ of a convergent sequence of minimal surfaces $\Sigma(n)$ which is locally-simply-connected in some Riemannian three-manifold.

Because of the local nature of Theorem 1, the proof of this theorem in the special case where the Riemannian three-manifold is $\mathbb{R}^3$ can be easily adapted to prove the general case and so we will restrict our attention to the $\mathbb{R}^3$ case. Let B be the open unit ball centered at the origin. Suppose

$D(n) \subset B$ is a sequence of properly embedded minimal disks with a sequence of points $p_n \in D(n)$ converging to the origin $O = (0, 0, 0)$ and where the absolute Gaussian curvature of $D(n)$ at p_n is diverging to infinity. Assume that a subsequence of the $D(n)$ converges to a minimal foliation $\mathcal{L}$ of B and the convergence is $C^\alpha, \alpha \in (0, 1)$, outside of a transverse Lipschitz curve $S(\mathcal{L})$ passing through O. (By a result of Solomon [17], a codimension-one minimal foliation is of class $C^{1,1}$.) After choosing a subsequence, assume that the original sequence converges to $\mathcal{L}$. We will prove that $S(\mathcal{L})$ is a $C^{1,1}$-curve orthogonal to the leaves of $\mathcal{L}$, by proving that $S(\mathcal{L})$ is a C^1-curve orthogonal to the leaves of $\mathcal{L}$. It then follows that $S(\mathcal{L})$ is an integral curve of the unit normal vector field to $\mathcal{L}$ which is $C^{0,1}$ in a neighborhood of $S(\mathcal{L})$ (see [17] for this regularity of the unit normal vector field) and so $S(\mathcal{L})$ is of class $C^{1,1}$.

Let $\widetilde{W}$ be a compact product neighborhood of a compact arc $\widetilde{\alpha} \colon [-\varepsilon', \varepsilon'] \to S(\mathcal{L})$ which parametrizes a neighborhood of the origin $O \in S(\mathcal{L})$ with $\widetilde{\alpha}(0) = O$. Here we may assume that the $C^{1,1}$-coordinates in $\widetilde{W}$ are chosen so that $\widetilde{W} = D \times [-\varepsilon', \varepsilon']$, where $D \times \{t\}$ is the open geodesic disk of radius ε' in the leaf of $\mathcal{L}$ containing $\widetilde{\alpha}(t)$ and $\widetilde{\alpha}(t) = \widetilde{\alpha}([-\varepsilon', \varepsilon']) \cap (D \times \{t\})$.

Since $\widetilde{\alpha}$ is a Lipschitz curve, for a positive choice of ε much smaller than ε', the interior of $\alpha = \widetilde{\alpha}|_{[-\varepsilon, \varepsilon]}$ is contained in the interior of $W \subset \widetilde{W}$ which is the intersection of $\widetilde{W}$ with the solid cylinder $C(\varepsilon)$ containing α with axis being the unit normal line to the leaf $D \times \{0\} \subset \widetilde{W}$ at O; here we assume that the top disk in W is a disk in the leaf $D \times \{\varepsilon\}$ containing $\alpha(\varepsilon)$ and the bottom disk in W is a disk in $D \times \{-\varepsilon\}$ containing $\alpha(-\varepsilon)$. Note that for n large and ε small, the cylinder sides of ∂W intersect $D(n)$ transversely in curves that have positive geodesic curvature from the point of view of the outward pointing vector to the cylinder.

The end points $\alpha(-\varepsilon)$ and $\alpha(\varepsilon)$ are limits, respectively, of points $p(n, -)$ and $p(n, +)$ in $D(n)$ of almost maximal curvature. By almost maximal curvature, we mean the following: the absolute Gaussian curvature of $D(n)$ at these points goes to infinity as $n \to \infty$ and after translating $D(n)$ so that such a point is at the origin and applying a large homothety so that the Gaussian curvature at the origin is -1, then this new sequence of minimal disks converges smoothly on compact subsets of $\mathbb{R}^3$ to a properly embedded simply-connected minimal surface of Gaussian curvature bounded from below by -1; we refer the reader to [13] for further discussion on the existence of points of almost maximal curvature. In our case, for any point $p \in S(\mathcal{L})$, we can find a sequence of points of almost maximal curvature on $D(n)$ converging to p and so we can find the points $p(n, -)$ and $p(n, +)$.

The main theorem in [13] states that a properly embedded simply-connected minimal surface in $\mathbb{R}^3$ with nonzero Gaussian curvature at the origin is a helicoid. A simple consequence of this result is that, when appropriately scaled

by curvature, the disk $D(n)$ is approximated by a well-formed homothetically-shrunk large compact region of a helicoid in small neighborhoods of $p(n,-)$ and of $p(n,+)$ (see Proposition 2 in [8] for the proof); let $E(n,-)$ and $E(n,+)$ denote these neighborhoods of $p(n,-)$ and $p(n,+)$ where $D(n)$ has the appearance of well-formed helicoids.

From the multigraph picture of $D(n)$ near $\alpha(-\varepsilon)$ and $\alpha(\varepsilon)$ presented in [1], [2], [4], [13] and the fact that $E(n,-)$ and $E(n,+)$ are almost helicoids, there exist geodesic arcs $\widetilde{\beta}(n,-)$ and $\widetilde{\beta}(n,+)$ in $E(n,-)$ and $E(n,+)$ passing through $p(n,-)$ and $p(n,+)$ respectively and "orthogonal" to the forming helicoid axes on $E(n,-)$ and on $E(n,+)$. The results in these papers show that the beginning multigraphs in $E(n,-)$ and $E(n,+)$ extend sideways from the forming helicoid axes for some fixed positive distance for n large. In particular, for n large, $\widetilde{\beta}(n,-)$ and $\widetilde{\beta}(n,+)$ can be extended to larger compact geodesics in this local multigraph picture of $D(n)$ near $p(n,-)$ and $p(n,+)$. By choosing ε sufficiently small, we may assume that the extended geodesics $\beta(n,-)$ and $\beta(n,+)$ each have their end points on the cylinder sides of $\partial C(\varepsilon)$. For n large and ε small, the cylinder sides of ∂W intersect $D(n)$ almost orthogonally in two almost flat highly sheeted "helical" type compact arcs $\widetilde{\delta}(n,1),\widetilde{\delta}(n,2)$, which, under a possible enlargement of W, each intersect $\beta(n,-)$ and $\beta(n,+)$ in single points. Let $\delta(n,1),\delta(n,2)$ be the respective subarcs of $\widetilde{\delta}(n,1)$ and $\widetilde{\delta}(n,2)$, each with boundary end points being the just described points of intersection. Define $E(n) \subset D(n)$ to be the subdisk with boundary $\beta(n,-) \cup \beta(n,+) \cup \delta(n,1) \cup \delta(n,2)$. Note that the sequence $E(n)$ converges to the product foliation $\mathcal{L} \cap W$ of W with singular set α (we now consider α to lie in W). After choosing a subsequence, $\beta(n,-)$ and $\beta(n,+)$ converge C^1 to geodesics $\beta(-)$ and $\beta(+)$ on the top and bottom disks of W.

Since the boundary of $E(n)$ is geodesically convex and $E(n)$ has nonpositive Gaussian curvature, there exists a unique geodesic $\sigma_n \subset E(n)$ of least length joining the points $p(n,-)$ to $p(n,+)$; assume that σ_n is parametrized to have unit speed. We now prove that σ_n converges as a point set to α as $n \to \infty$. Recall that for n large the Gaussian curvature of $E(n)$ near $\delta(n,1) \cup \delta(n,2)$ is uniformly bounded and $\partial E(n)$ is geodesically convex. Thus, any limit point of σ_n in $\mathcal{L} - S(\mathcal{L})$ must stay a positive distance from the cylinder sides of W. Hence, σ_n stays away from $\partial E(n)$ except at its end points. Suppose that σ_n does not limit as a point set to α. In this case there is a limit point $p \notin S(\mathcal{L})$ of these geodesics and p lies on a disk $F(t) = (D \times \{t\}) \cap W$ for some $t, -\varepsilon \le t \le \varepsilon$. Let $\sigma_n(t(n))$ be a sequence of points converging to p. Since in a small neighborhood of p the disks $E(n)$ converge smoothly to the leaves of $\mathcal{L}$, after choosing a subsequence, we may assume that the tangent vectors to $\sigma_n(t(n))$ also converge to a tangent vector $T(p)$ at p.

Since the exponential map for geodesics emanating from p and defined on the leaf $D \times \{t\}$ of $\mathcal{L}$ is injective as long as it is defined, the geodesic $\Gamma(t)$ passing

through p with tangent vector $T(p)$ on $F(t) - \alpha(t)$ must leave $F(t) - \alpha(t)$. Since the exponential map is injective, at least one of the end points of this geodesic must lie on the boundary of $F(t)$. But $\Gamma(t)$ is itself a limit of the geodesics σ_n which stay a positive distance from the boundary of $F(t)$ when $t \neq \pm \varepsilon$. Thus when $t \neq \pm \varepsilon$ we arrive at a contradiction. In the case $t = \pm \varepsilon$, the disk $F(t)$ is a half disk with geodesically convex boundary with $\beta(+)$ or $\beta(-)$ on part of the boundary; in this case the previous argument with slight modifications gives a contradiction. Thus, the limit set of the σ_n is contained in α and so clearly equals α since α is a connected arc and the limit set of the σ_n contains the end points of this arc.

Recall that the minimal foliation $\mathcal{L}$ of B is of class $C^{1,1}$ and the unit normal vector field N to $\mathcal{L}$ is Lipschitz. Therefore, the integral curves of N are of class $C^{1,1}$ and foliate W. We will let L denote the unoriented line field associated to N. Our strategy to complete the proof of the theorem is to prove that α is an integral curve of L passing through O. We will accomplish this by showing that the tangent line field to σ_n converges to L as $n \to \infty$. But a unit speed C^1-curve τ_n of fixed length whose tangent line field is ε_n-close to some ambient Lipschitz line field must be C^1-close to any integral curve of the line field which has a point C^0-close to τ_n, where the closeness depends only on ε_n, the length of the curve and the Lipschitz constant of the line field. Thus, for n large, σ_n would be ε_n-close in the C^1-norm to the integral curve γ of L passing through O with $\varepsilon_n \to 0$ as $n \to \infty$. (Basically, this result follows directly from the uniqueness and existence of ordinary differential equations for an associated Lipschitz line field L in an open set in $\mathbb{R}^3$. From the flow associated to L, one constructs $C^{1,1}$ coordinates in a neighborhood of O and in these coordinates $O = (0,0,0)$, L corresponds to the line field E_3 parallel to $e_3 = (0,0,1)$. In these coordinates let $\widetilde{\gamma}_n$ denote the related γ_n curves, whose tangent line fields converge to E_3. In this classical setting, we can express $\widetilde{\gamma}_n$ as small graphs over the x_3-axis and these graphs converge C^1 to it. This argument also shows that the lengths of the γ_n are uniformly bounded in any sufficiently small coordinate neighborhood of O.)

Since the σ_n limit as a point set to α, would be converging to γ, and the σ_n and the set of integral curves of N are both compact in the C^1-norm, then the σ_n would converge in the C^1-norm to the integral curve γ of L passing through O. Hence, α would be contained in the integral curve γ of N. We remark that we are using in an essential way that the unit normal vector field to a minimal foliation is Lipschitz [17] and so, by the uniqueness of solutions to ordinary differential equations defined by Lipschitz functions, have unique integral curves. It remains to prove that the tangent line field to σ_n converges to L as $n \to \infty$.

For each σ_n, let $\theta(\sigma_n(t))$ be the unoriented angle that the tangent line to σ_n at $\sigma_n(t)$ makes with the line $L(\sigma_n(t))$. We need to prove that for all $\delta > 0$, there exists a positive integer N_0 such that for $n \geq N_0$, $\theta(\sigma_n(t)) < \delta$ for t in

the domain of σ_n. Since our goal is to prove that near $\alpha(0)$, α is an integral curve of N, then it suffices to demonstrate that $\theta(\sigma_n(t)) < \delta$ for $\sigma_n(t)$ close to $\alpha(0)$. Reasoning by contradiction, suppose that there exists a $\delta > 0$ and a sequence of parameter values $t(n)$, each one in the domain of σ_n, such that for some $s \in (-\varepsilon/2, \varepsilon/2)$, $\sigma_n(t(n))$ converges to the point $\alpha(s)$ as $n \to \infty$ and $\theta(\sigma_n(t(n))) \geq \delta$ for all n. Without loss of generality, we will assume that $s = 0$. Clearly, δ can be chosen arbitrarily small, which will be used later on.

We now review a basic geometrical property on which our proof depends. Suppose $q_n \in E(n)$ is a point of almost maximal curvature. Then in a small neighborhood of q_n, $E(n)$ has the appearance of an almost perfectly formed homothetically scaled down helicoid with many sheets. Note that σ_n must pass through the core around the axis of this forming helicoid; otherwise, σ_n would not limit as a point set to the singular set of $\mathcal{L}$. The least-length property of σ_n guarantees that in the expanded scale of the limit helicoid, the corresponding curves $\lambda_n(\sigma_n - q_n)$, where $\lambda_n = \sqrt{|K(q_n)|}$, must approximate the axis of the forming helicoid axis since this property would hold on a genuine helicoid. More precisely, the least-length property that we are referring to is the following: given two sequences of points $f(n), g(n)$ on a fixed vertical helicoid with $x_3(f(n)) \to \infty$ and $x_3(g(n)) \to -\infty$, then the least-length geodesics arcs γ_n joining $f(n)$ and $g(n)$ converge C^1 on compact subsets of the helicoid to the axis of the helicoid. The proof that σ_n approximates in a C^1-sense the axis of the forming helicoid can be understood by the following blow-up argument. Translate to the origin and scale by curvature neighborhoods of q_n to obtain a sequence $H(n)$ of minimal disks of bounded curvature which, after taking a subsequence, converge to a helicoid H in $\mathbb{R}^3$. It is important to note that although we need to take a subsequence of the $H(n)$ to get convergence to H, the axis of H, which is parallel to $L(\alpha(0))$, is independent of the choice of convergent subsequence; this uniqueness of axis follows from a fixed distance extension of the beginning multigraph proved in [2]. Suppose the sequence $H(n(i))$ converges to H and let $\widetilde{\sigma}_{n(i)}$ be the related geodesics. Note that the $\widetilde{\sigma}_{n(i)}$ must pass near the origin and pass through the core of the forming helicoid H. By our previous observations, the $\widetilde{\sigma}_{n(i)}$ must converge to C^1 on balls in $\mathbb{R}^3$ to the axis of H; this proves the assertion that σ_n approximates in a C^1-sense (in the scale of the limit helicoid) the axis of the forming helicoid at $\widetilde{q}_n$. Since the axis of the limit helicoid that appears at q_n is the line $L(q_n)$, the tangent lines to σ_n very near q_n must be converging to $L(q)$ as well, where q is the limit of the q_n. We will refer to this result as "the C^1-basic approximation to L property" that σ_n satisfies near points of almost maximal curvature. This property will now be applied to derive a contradiction.

Since there is a sequence of points $q_n \in E(n)$ with almost maximal curvature converging to $\alpha(0)$ and near these points $E(n)$ has the appearance

of a helicoid with axis almost parallel to $L(\alpha(0))$, the C^1-basic approxima-
tion property implies that there exist times $t(n,*)$ in the domain of σ_n with
$\sigma_n(t(n,*)) \to \alpha(0)$ and such that the tangent line $\sigma_n(t(n,*))$ is converging
to the line $L(\alpha(0))$; we are using here the property that the line field L is
continuous to prove the existence of the numbers $t(n,*)$. After choosing a
subsequence, we may assume that either $t(n) < t(n,*)$ or $t(n) > t(n,*)$ for all
n, where the $t(n)$ were defined previously. Without loss of generality in our
arguments, we will assume that $t(n) < t(n,*)$. Note that $\theta(\sigma_n(t(n,*)) \to 0$ as
$n \to \infty$ and so we may assume that $\theta(\sigma_n(t(n,*))) < \delta/2$ for n large. Let $r(n)$
be the largest number in $[t(n), t(n,*)]$ such that $\theta(\sigma_n(r(n)) = \delta$ and let $s(n)$
be the smallest number in the interval $[r(n), t(n,*)]$ such that $\theta(\sigma_n(s(n))) =
\delta/2$. From this point on in the proof, we will assume that δ is chosen less than
$\pi/10$.

Now consider the following new sequence of disks $G(n)$ obtained by trans-
lating $D(n)$ by $-\sigma_n(r(n))$ and then homothetically scaling by the factor
$1/(s(n) - r(n))$:

$$G(n) = \frac{1}{s(n) - r(n)}(D(n) - \sigma_n(r(n)))$$

We claim that the homothety factor $1/(s(n) - r(n)) \to \infty$ as $n \to \infty$. If
the sequence of positive numbers $s(n) - r(n)$ does not converge to zero, then
for some $\eta > 0$, there exist arbitrarily large n such that $s(n) - r(n) \geq \eta$.
Let $\widetilde{\sigma}_n$ be the image of the arc related to $\sigma_n|_{[r(n),s(n)]}$ but in the disk $G(n)$.
Since $\delta < \pi/10$ and θ lies between δ and $\delta/2$ on the interval $[r(n), s(n)]$, the
end points of $\widetilde{\sigma}_n$ must be at least a fixed positive distance $d(\eta)$ from each
other which is independent of n and can be estimated from below in terms of
η. Since σ_n converges to α, it follows that there exists a sequence of points
$\widetilde{q}_n \in G(n)$ of almost maximal curvature such that the $(d(\eta)/10)$-neighborhood
of $\widetilde{q}_n$ in $\mathbb{R}^3$ intersects $\widetilde{\sigma}_n$ only in its interior and $\widetilde{\sigma}_n$ passes through the core of
the helicoid that is forming at the point $\widetilde{q}_n$. Thus, the angle that the tangent
line to a point $\widetilde{\sigma}_n(t'(n))$ of $\widetilde{\sigma}_n$ which is closest to $\widetilde{q}_n$ makes with the axis of
the forming helicoid at $\widetilde{q}_n$ is converging to zero as $n \to \infty$. But the original
line field at the related point $\widehat{q}_n$ on $E(n)$ is converging to the axis of the
helicoid which is forming on $E(n)$ at $\widehat{q}(n)$. Since these forming helicoids are
parallel ($G(n)$ and $D(n)$ are essentially translated surfaces), we see that for
n large there is a time $t'(n) \in [r(n), s(n)]$ such that $\theta(\sigma_n(t'(n))) < \delta/2$. This
contradicts the property that $\theta(\sigma_n(t))$ lies between δ and $\delta/2$ on $[r(n), s(n)]$.
This contradiction proves that $s(n) - r(n) \to 0$ as $n \to \infty$. Since σ_n has unit
speed and $s(n) - r(n) \to 0$ as $n \to \infty$, the arcs $\sigma_n([r(n), s(n)])$ converge to
$\alpha(0)$ as a point set.

Since $s(n) - r(n) \to 0$, the boundaries of the disks $G(n)$ eventually lie
outside of any compact set in $\mathbb{R}^3$ and so the compactness theorem in [4]
implies that a subsequence of the $G(n)$ converges on compact subsets of $\mathbb{R}^3$

to a minimal lamination $\widetilde{\mathcal{L}}$ of $\mathbb{R}^3$ with possibly a new singular set $S(\widetilde{\mathcal{L}})$. After choosing this subsequence, assume that $G(n)$ converges to $\widetilde{\mathcal{L}}$.

There are several cases to consider depending upon how the $G(n)$ limit to $\widetilde{\mathcal{L}}$. In the first case suppose that the Gaussian curvature of the collection $G(n)$ is uniformly bounded in the closed ball $B(1)$ of radius 1 centered at the origin. Again let $\widetilde{\sigma}_n$ denote the arc related to the image arc $\sigma_n([r(n), s(n)])$. Note that by our normalization, the beginning point of $\widetilde{\sigma}_n$ is at the origin and $\widetilde{\sigma}_n$ is a geodesic of length one on $G(n)$ and so $\widetilde{\sigma}_n$ is contained in $B(1)$.

After choosing another subsequence, we may assume that the $\widetilde{\sigma}_n$ converge in the C^1-norm to a geodesic $\widetilde{\sigma}$ on a leaf of $\widetilde{\mathcal{L}}$. The lines $L(\sigma_n(t))$, considered to be lines passing through the origin, converge to $L(\alpha(0))$ for all $t \in [r(n), s(n)]$ because the sequence of geodesic arcs $\sigma_n([r(n), s(n)])$ converges to $\alpha(0)$ as we remarked earlier. Since the lines $L(\sigma_n(t))$ are converging to $L(\alpha(0))$ for each $t \in [r(n), s(n)]$ and $\theta(\sigma_n(r(n))) = \delta$ and $\theta(\sigma_n(s(n)) = \delta/2$, the tangent vectors to the end points of $\widetilde{\sigma}$ are not parallel in $\mathbb{R}^3$. In particular the metric on the leaf $H(\widetilde{\sigma})$ of $\widetilde{\mathcal{L}}$ containing $\widetilde{\sigma}$ is not flat since $\widetilde{\sigma}$ is not a straight line. It follows that the Gaussian curvature of the sequence $G(n)$ is uniformly bounded on compact subsets of $\mathbb{R}^3$; otherwise, $\widetilde{\mathcal{L}}$ would have a singular point of convergence which in turn would imply that $\widetilde{\mathcal{L}}$ is a foliation by planes but the metric on $H(\widetilde{\sigma})$ is not flat. Since the limit leaves of $\widetilde{\mathcal{L}}$ are planes (see the proof of Theorem 1.6 in [13]), $H(\widetilde{\sigma})$ is not a limit leaf of $\widetilde{\mathcal{L}}$. Once one knows that $H(\widetilde{\sigma})$ is not a limit leaf, then a standard argument using Jacobi vector fields shows that the convergence of $G(n)$ to give $H(\widetilde{\sigma})$ is of multiplicity one in area. (One just takes a normalized difference between highest and lowest sheets of $G(n)$ which converge to $H(\widetilde{\sigma})$ to obtain a positive Jacobi function on $H(\widetilde{\sigma})$, which can be assumed to be simply-connected by taking covering spaces, and then apply the results in [6] or [7] to conclude $H(\widetilde{\sigma})$ is a plane.) Since the convergence of the $G(n)$ which give $H(\widetilde{\sigma})$ is of multiplicity one on $H(\widetilde{\sigma})$, the standard lifting argument of closed curves on $H(\widetilde{\sigma})$ to $G(n)$ shows that $H(\widetilde{\sigma})$ is simply-connected. Since $H(\widetilde{\sigma})$ is simply-connected and has locally bounded Gaussian curvature in $\mathbb{R}^3$, its closure is a lamination and Theorem 1.6 in [13] implies that $H(\widetilde{\sigma})$ is properly embedded in $\mathbb{R}^3$. Hence, by the main result in [13], $H(\widetilde{\sigma})$ is a helicoid and it follows that $\widetilde{\mathcal{L}} = H(\widetilde{\sigma})$. But then the length minimizing property of our original σ_n implies that the associated $\widetilde{\sigma}_n$ in $G(n)$ would converge C^1 on balls in $\mathbb{R}^3$ to the axis of the helicoid $\widetilde{\mathcal{L}}$ which is a straight line. The reason for this is that $\widetilde{\sigma}_n$ converges to the axis of $\widetilde{\mathcal{L}}$ because the rescaling $G(n)$ of $D(n)$ is essentially by curvature, since $\widetilde{\mathcal{L}}$ is a nonflat surface. This contradicts the earlier observation that the tangent vectors at end points of $\widetilde{\sigma} \subset H(\widetilde{\sigma})$ are not parallel. This contradiction implies that the Gaussian curvature of the surfaces $G(n)$ is not uniformly bounded in $B(1)$.

What we have proven so far is that the Gaussian curvature of the $G(n)$ is not uniformly bounded in the ball $B(1)$. Hence, $\widetilde{\mathcal{L}}$ is a foliation of $\mathbb{R}^3$ by

parallel planes whose singular set is a Lipschitz curve $S(\widetilde{\mathcal{L}})$ which intersects $B(1)$ at some point. At this point in the proof it is helpful to remark that the normal line to the planes in $\widetilde{\mathcal{L}}$ is $L(\alpha(0))$; this follows from the fact that L varies continuously in the ball B and $\widetilde{\mathcal{L}}$ is the scaled limit of portions of $E(n)$ from smaller and smaller neighborhoods of $\alpha(0)$. For convenience assume that the planes in $\widetilde{\mathcal{L}}$ are horizontal. Since the tangent vectors to planes in $\widetilde{\mathcal{L}}$ are horizontal and the tangent vectors to $\widetilde{\sigma}_n$ are bounded away from the horizontal, the $\widetilde{\sigma}_n$ must converge to a portion of $S(\widetilde{\mathcal{L}})$ as $n \to \infty$. On the other hand, by our choice of small δ, tangent vectors to $\widetilde{\sigma}_n$ make a small angle with $L(\alpha(0))$ and so, by our normalization, $\widetilde{\sigma}_n$ is a curve whose end points have differing heights of at least $1/2$. Our previous arguments show that there exist points $g(n) \in G(n)$ of almost maximal curvature whose extrinsic heights converge to $1/4$ and whose intrinsic distance to $\widetilde{\sigma}_n$ is converging to zero. In this case, the length minimizing property of σ_n implies that $\widetilde{\sigma}_n$ must pass through the core of the forming vertical helicoid at $g(n)$. Hence, for n large, there must be a $t'(n) \in [r(n), s(n)]$ where the tangent line to $\widetilde{\sigma}_n$ makes an angle of less than $\delta/4$ with $L(\alpha(0))$. Hence, for n large, $\theta(\sigma_n(t'(n)))$ is less than $\delta/2$ but $\theta \circ \sigma_n([r(n), s(n)]) = [\delta/2, \delta]$. This contradiction completes our proof of Theorem 1.

3. The lamination metric theorem

As an application of our proof of the $C^{1,1}$-regularity of the singular curve $S(\mathcal{L})$ in the Colding-Minicozzi lamination theorem, we now prove an interesting and useful compactness theorem. This theorem is helpful in deriving curvature estimates for properly embedded minimal surfaces with finite genus in $\mathbb{R}^3$ or in $M \times \mathbb{R}$, where M is a compact Riemannian surface (see [9], [10], [11], [14], [15]). It is also used in an important manner in proving a recent result by Meeks and Rosenberg [15], which states that the closure of any complete embedded minimal surface with positive injectivity radius in a Riemannian three-manifold has the structure of a minimal lamination. The metric defined in the next theorem gives rise to the natural one on minimal parking garage structures of $\mathbb{R}^3$, which are important examples of Colding-Minicozzi limit minimal laminations (see [12]).

THEOREM 2 (Lamination Metric Theorem). *Suppose N is a Riemannian three-manifold. Suppose $\Sigma(k)$ is a sequence of connected properly embedded minimal surfaces in N that is locally-simply-connected and which converges to a minimal lamination $\mathcal{L}$ of N with non empty singular set of convergence being a transverse, possibly disconnected, Lipschitz curve $S(\mathcal{L})$. Let Γ be a component of the singular set $S(\mathcal{L})$. Let $\mathcal{L}(1, \Gamma)$ be the union of the leaves of $\mathcal{L}$ that intersect Γ and let $S(1, \Gamma) = S(\mathcal{L}) \cap \mathcal{L}(1, \Gamma)$. Define inductively, for*

$n \geq 2, \mathcal{L}(n,\Gamma)$ *to be the union of the leaves of* $\mathcal{L}$ *that intersect* $S(n-1,\Gamma)$ *and define* $S(n,\Gamma)$ *to be* $S(\mathcal{L}) \cap \mathcal{L}(n,\Gamma)$. *Then the following statements hold:*

(1) *For all* $n, \mathcal{L}(n,\Gamma)$ *is a connected open set of* N.

(2) $\mathcal{L}(\infty,\Gamma) = \bigcup_{i=1}^{\infty} \mathcal{L}(n,\Gamma)$ *is a connected open set of* N, *and every component of* $S(\mathcal{L})$ *that intersects* $\mathcal{L}(\infty,\Gamma)$ *is contained in it.*

(3) *If* N *is compact, then, for some* n, $\mathcal{L}(\infty,\Gamma) = \mathcal{L}(n,\Gamma)$.

(4) *For each* $n \in \mathbb{N} \cup \{\infty\}, \mathcal{L}(n,\Gamma)$ *is a path connected metric space with respect to the following distance function. Define* $d_{(n,\Gamma)}(p,q)$ *to be equal to the infimum of the lengths of continuous piecewise* C^1*-arcs in* $\mathcal{L}(n,\Gamma)$ *such that every* C^1*-arc component of the path lies entirely in a leaf of* $\mathcal{L}$ *or in* $S(\mathcal{L})$. *If the metric on* N *is complete, then there exists a shortest length path in* $\mathcal{L}(n,\Gamma)$ *joining* p *and* q *which consists of a finite number of geodesics segments in the leaves of* $\mathcal{L}$ *together with a finite number of segments in* $S(\mathcal{L})$; *such a shortest length path we will call a minimizing geodesic.*

(5) *If* N *is complete, then the Riemannian distance functions* d_k *on* $\Sigma(k) \cap \mathcal{L}(\infty,\Gamma)$ *converge to the restriction of* $d_{(\infty,\Gamma)}$ *to* $\Sigma(k) \cap \mathcal{L}(\infty,\Gamma)$ *in a manner which we now describe. Suppose* $p_k, q_k \in \Sigma(k)$ *converge to points* $p, q \in \mathcal{L}(\infty,\Gamma)$, *respectively, and* γ_k *is a minimizing geodesic on* $\Sigma(k)$ *joining* p_k *and* q_k. *Then the lengths of the* γ_k *are uniformly bounded and a subsequence of these minimizing geodesics converges to a piecewise-smooth curve* γ *in* $\mathcal{L}(\infty,\Gamma)$ *with end points* p, q. *The curve* γ *consists of a finite number of geodesic segments in leaves of* $\mathcal{L}$ *together with a finite number of segments and corners in* $S(\mathcal{L})$. *Furthermore, if* p *and* q *are joined by a minimizing geodesic in* $\mathcal{L}(\infty,\Gamma)$ *that intersects* $S(\mathcal{L})$, *then* γ *is a length minimizing geodesic. In particular, if* p, q *lie on distinct leaves of* $\mathcal{L}(\infty,\Gamma)$, *then for every* $\varepsilon > 0$, *there exists a positive integer* I, *such that if* $k \geq I$, *then* $d_{(\infty,\Gamma)}(p,q) - \varepsilon \leq d_k(p_k,q_k) \leq d_{(\infty,\Gamma)}(p,q) + \varepsilon$. *Conversely, given* $p, q \in \mathcal{L}(\infty,\Gamma)$, *there exist points* $p_k, q_k \in \Sigma(k)$ *(carefully chosen) converging to* p, q, *respectively, such that if* γ_k *is any sequence of least length geodesics in* $\Sigma(k)$ *joining* p_k *to* q_k, *then a subsequence converges to a minimizing geodesic in* $\mathcal{L}(\infty,\Gamma)$.

(6) *If* N *is complete, then* $(\mathcal{L}(\infty,\Gamma), d_{(\infty,\Gamma)})$ *is a complete path connected metric space.* $(\mathcal{L}(\infty,\Gamma), d_{(\infty,\Gamma)})$ *does not have a countable basis, and so this metric space is not compact.*

Proof. Statements 1 and 2 follow immediately from the Colding-Minicozzi Lamination Theorem and elementary topology. Since N being compact implies $S(\mathcal{L})$ is a finite collection of simple closed curves, the definition of $\mathcal{L}(n,\Gamma)$ implies that $\mathcal{L}(\infty,\Gamma) = \mathcal{L}(n,\Gamma)$ for some integer n, which proves statement 3. Statement 4 follows immediately from local considerations in Riemannian

geometry and the arguments contained in the proof of our regularity theorem (Theorem 1) for the singular set. This proof of regularity shows that any sequence of geodesics on $\Sigma(k)$ of fixed length L with some limit point $p \in \mathcal{L}(n, \Gamma)$, contains a subsequence which converges to a continuous curve in $\mathcal{L}(n, \Gamma)$ of length L consisting of geodesic segments in leaves of $\mathcal{L}$ together with arcs and corners in $S(\mathcal{L})$.

It is natural to consider a continuous piecewise C^1 curve σ in $\mathcal{L}(\infty, \Gamma)$ to be a *geodesic* if it is a limit of geodesics on the $\Sigma(k)$ parametrized by arc length. Note that such a limit geodesic σ is locally length minimizing on $\mathcal{L}$ unless for some t_0, $\sigma(t_0)$ lies in the interior of σ and for all sufficiently small $\varepsilon > 0$, $\sigma(t_0 - \varepsilon)$ and $\sigma(t_0 + \varepsilon)$ lie on the same leaf of $\mathcal{L}(\infty, \Gamma)$ and σ has a nontrivial *corner point* at $\sigma(t_0)$. The existence of such limit geodesics in $\mathcal{L}(\infty, \Gamma)$ which are not locally length minimizing but which are the limits of length minimizing geodesics in the $\Sigma(k)$ presents some difficulties which must be overcome in the proof of statement 5.

We now prove statement 5. By statement 2, $\mathcal{L}(\infty, \Gamma)$ is a connected open set of N and every component of $S(\mathcal{L})$ that intersects $\mathcal{L}(\infty, \Gamma)$ is contained in it. It follows that the geodesic completion $\overline{\mathcal{L}(\infty, \Gamma)}$ in N of this open set is a complete Riemannian three-manifold whose boundary (which can be considered to lie in N) consists of complete minimal surfaces, which are disjoint from $S(\mathcal{L})$ by the curvature estimates in [4]. Suppose γ_k are length minimizing geodesics in $\Sigma(k)$ with end points p_k, q_k converging to p, q. By the arguments in the proof of the $C^{1,1}$-regularity of $S(\mathcal{L})$, a subsequence of the γ_k converges to a possibly disconnected geodesic γ in $\mathcal{L}(\infty, \Gamma)$ consisting of geodesic arcs in leaves of $\mathcal{L}$ and arcs and corners in $S(\mathcal{L}) \cap \mathcal{L}(\infty, \Gamma)$. A lifting argument, similar to the one which we will use shortly, shows that there exist paths in $\Sigma(k)$ joining p_k to q_k of uniformly bounded length, and so, γ is a compact connected geodesic with end points p and q.

By statement 4, there exist length minimizing geodesics $\widehat{\gamma}_k$ in $\mathcal{L}(\infty, \Gamma)$ with the same end points as γ_k. Note that a subsequence $\widehat{\gamma}_{k_i}$ converges to a minimizing geodesic $\widehat{\gamma}$ of length $L(\widehat{\gamma}) = d_{(\infty, \Gamma)}(p, q)$ with end points p and q.

Suppose now that $\widehat{\gamma}$ intersects $S(\mathcal{L})$ at some point. By the proof of the $C^{1,1}$-regularity of $S(\mathcal{L})$ and a lifting argument, for k_i large, the compact connected least length geodesic $\widehat{\gamma}$ can be approximated by curves $\widehat{\gamma}_{k_i}$ on $\Sigma(k_i)$ with the same end points as γ_{k_i} and with the lengths of the $\widehat{\gamma}_{k_i}$ converging to the length of $\widehat{\gamma}$ as $k_i \to \infty$. We remark that the hypothesis that $\widehat{\gamma} \cap S(\mathcal{L}) \neq \emptyset$ is essential in order to define the lifts $\widehat{\gamma}_{k_i}$ of $\widehat{\gamma}$ with the correct end points, which is not possible if p and q lie on the same leaf of $\mathcal{L}$ but lie on different nearby lifted intervals in $\Sigma(k_i)$ over a $\widehat{\gamma}$ which are disjoint from $S(\mathcal{L})$.

It follows that the lengths of the geodesics γ_k are essentially bounded by $L(\widehat{\gamma}) = d_{(\infty, \Gamma)}(p, q)$. Hence, the limit geodesic γ in $\mathcal{L}(\infty, \Gamma)$ joining p to q of length at most $L(\widehat{\gamma}) = d_{(\infty, \Gamma)}(p, q)$. By definition of $d_{(\infty, \Gamma)}$, the length of the embedded geodesic γ is equal to the $d_{(\infty, \Gamma)}$ distance between its end points.

It remains to prove that the last sentence in statement 5 holds. By the above discussion, the last sentence in statement 5 holds if some minimizing geodesic of $\mathcal{L}(\infty, \Gamma)$ joining p, q intersects $S(\mathcal{L})$, independently of the choice of the converging points p_k, q_k. On the other hand, if a least length embedded geodesic α joining p and q fails to intersect $S(\mathcal{L})$, then for k large, α lifts to paths α_k on $\Sigma(k)$, which converge C^1 to α. So the length minimizing geodesics on $\Sigma(k)$ with the same end points as α_k have a subsequence converging to a minimizing geodesic in $\mathcal{L}(\infty, \Gamma)$ joining p to q. This completes the proof of statement 5.

The first statement in statement 6 follows immediately from statements 4 and 5. Since $\mathcal{L}(\infty, \Gamma)$ is locally a foliation of N, it does not have a countable basis as a $d_{(\infty, \Gamma)}$ metric space, which implies that this metric space is not compact. This completes the proof of the theorem. $\qquad\square$

QUESTION 1. *If N is a compact Riemannian three-manifold with a Colding-Minicozzi limit minimal lamination $\mathcal{L}$ and $S(\mathcal{L}) \neq \varnothing$, then is $N = \mathcal{L}(\infty, \Gamma)$?*

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