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BROWNIAN MOTION IN RIEMANNIAN ADMISSIBLE COMPLEXES

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Abstract. The purpose of this work is to construct a *Brownian motion* with values in simplicial complexes with piecewise differential structure. In order to state and prove the existence of such a Brownian motion, we define a family of continuous Markov processes with values in an admissible complex. We call a process in this family an *isotropic transport process*. We first show that the family of isotropic processes contains a subsequence which converges weakly to a measure which we call the *Wiener measure*. Then, using the finite dimensional distributions of this Wiener measure, we construct a new admissible complex valued continuous Markov process, the Brownian motion. We conclude with a geometric analysis of this Brownian motion and determine the recurrent or transient behavior of such a process.

0. Introduction

It has been shown in [27], [21] and [26] that, on a wide class of Riemannian manifolds, Brownian motion can be approximated in law by a Markov process which generalizes the isotropic scattering transport process on the Euclidean space [31]. On the other hand, Brownian motion has been used as a tool to prove important results in Riemannian geometry and potential theory. This is not surprising since Brownian motion is intimately connected with harmonic functions [15], the Laplacian, and other fundamental objects in mathematics. For instance, a complete Riemannian manifold is hyperbolic exactly when the Brownian motion is transient.

The purpose of this work is to consider the problem of defining the concept of a continuous random walk in the admissible Riemannian complexes and, in particular, to construct a Brownian motion in singular spaces where second order differential calculus is not available.

The first section gives some preliminaries on Riemannian admissible complexes [3], [7] and concludes with a brief survey of the theory of general Markov processes [14].

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The second section is devoted firstly to the construction of a Markov process with values in an admissible complex which we call the *isotropic transport process*, and secondly to the proof that this process is a strong Markov process.

In the third section we construct a family of isotropic processes and we show that this family contains a subsequence which converges weakly to a measure which we call the *Wiener measure*.

The aim of the fourth section is to construct a Brownian motion with values in an admissible complex by using the finite dimensional distributions of the Wiener measure we have constructed.

Finally, in the fifth section, we study the transience and recurrence properties of this Brownian motion. In particular, we show that, in the 2-dimensional case, if the complex is complete, simply connected, of non-positive curvature, and the number of branching faces is always greater than or equal to 3, then the Brownian motion is transient although (surprisingly) the Euclidean Brownian motion in dimension 2 is recurrent.

There is an interesting study by M. Brin and Y. Kifer [8] of Brownian motion in singular spaces; to our knowledge, this is the only such study. In their work, Brin and Kifer consider the case of 2-dimensional simplicial complexes whose simplices are flat Euclidean. They define Brownian motion in such a complex as the planar Brownian motion inside faces which, after hitting an edge, goes into each of the adjacent faces "with equal probability". Our work is, in fact, the first one that shows the existence of Brownian motion, not only in the case of 2-dimensional complexes with flat simplices, but also in the general case of admissible Riemannian complexes.

We remark that the construction of admissible complex valued Brownian motion can be extended to the case of general Hadamard spaces if we assume a given uniform probability (or sub-probability) measure on the link of each point of the space.

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1. Preliminaries

1.1 General theory. (See [1], [2], [11], [17], [18].) Let X be a metric space with a metric d. A curve $c : I \to X$ is called a *geodesic* if there is a number $v \geq 0$, called the speed, such that every $t \in I$ has a neighborhood $U \subset I$ with $d(c(t_1), c(t_2)) = v|t_1 - t_2|$ for all $t_1, t_2 \in U$. If the above equality holds for all $t_1, t_2 \in I$, then c is called a *minimal geodesic*.

The space X is called a *geodesic space* if any two points in X are connected by a minimal geodesic. We assume from now on that X is a complete geodesic space.

A triangle Δ in X is a triple $(\sigma_1, \sigma_2, \sigma_3)$ of geodesic segments whose end points match in the usual way. Denote by H_k the simply connected complete surface of constant Gauss curvature k. A *comparison triangle* $\bar{\Delta}$ for a triangle $\Delta \subset X$ is a triangle in H_k whose sides have the same lengths as Δ. A comparison triangle in H_k exists and is unique up to congruence if the lengths of the sides of Δ satisfy the triangle inequality and, in the case $k > 0$, if the perimeter of Δ is $< 2\pi/\sqrt{k}$. Let $\bar{\Delta} = (\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3)$ be a comparison triangle for $\Delta = (\sigma_1, \sigma_2, \sigma_3)$. Then for every point $x \in \sigma_i$, $i = 1, 2, 3$, we denote by $\bar{x}$ the unique point on $\bar{\sigma}_i$ which lies at the same distances to the ends as x.

Let d denote the distance functions in both X and H_k. A triangle Δ in X is called a CAT_k *triangle* if the sides satisfy the triangle inequality, the perimeter of Δ is $< 2\pi/\sqrt{k}$ for $k > 0$, and if $d(x, y) \leq d(\bar{x}, \bar{y})$, for any two points $x, y \in X$.

We say that X has curvature at most k, and write $k_X \leq k$, if every point $x \in X$ has a neighborhood U such that any triangle in X with vertices in U and minimizing sides is CAT_k. Note that we do not define k_X. If X is a Riemannian manifold, then $k_X \leq k$ iff k is an upper bound for the sectional curvature of X.

A geodesic space X is called geodesically complete iff every geodesic can be stretched in the two directions.

We say that a geodesic space X is without conjugate points if any two points in X are connected by a unique geodesic.

1.2 Riemannian admissible complexes. (See [25], [30].) Let K be a locally finite simplicial complex, endowed with a piecewise smooth Riemannian metric g; i.e., g is a family of smooth Riemannian metrics g_Δ on simplices Δ of K such that the restriction $g_\Delta|_{\Delta'}$ equals $g_{\Delta'}$ for any simplices Δ' and Δ with $\Delta' \subset \Delta$.

Let K be a finite dimensional simplicial complex which is connected and locally finite. A map f from $[a, b]$ to K is called a broken geodesic if there is a subdivision $a = t_0 < t_1 < \cdots < t_{p+1} = b$ such that $f([t_i, t_{i+1}])$ is contained in some cell and the restriction of f to $[t_i, t_{i+1}]$ is a geodesic inside that cell. We define the length of the broken geodesic map f as

$$L(f) = \sum_{i=0}^{i=p} d(f(t_i), f(t_{i+1})).$$

The length inside a cell is measured with respect to the metric of the cells. We define $\tilde{d}(x, y)$, for any two points x, y in K, as the lower bound of the lengths of broken geodesics from x to y. $\tilde{d}$ is a pseudo-distance.

If K is connected and locally finite, then the space $(K, \tilde{d})$ is a length space and hence a geodesic space if is complete (see also [6]).

An l-simplex in K is called a *boundary simplex* if it is adjacent to exactly one $(l+1)$-simplex. The complex K is called *boundaryless* if there are no boundary simplices in K.

We say that the complex K is *admissible* if for every connected open subset U of K the open set $U \setminus \{U \cap \{\text{the } (k-2) - \text{skeleton}\}\}$ is connected (where k is the dimension of K).

Let $x \in K$ be a vertex of K so that x is in the l-simplex Δ_l. We view Δ_l as an affine simplex in $\mathbb{R}^l$, that is, $\Delta_l = \bigcap_{i=0}^{l} H_i$, where $H_0, H_1, \ldots, H_l$ are closed half spaces in general position, and we suppose that x is in the topological interior of H_0. The Riemannian metric g_{Δ_l} is the restriction to Δ_l of a smooth Riemannian metric defined in an open neighborhood V of Δ_l in $\mathbb{R}^l$. The intersection $T_x\Delta_l = \bigcap_{i=1}^{l} H_i \subset T_xV$ is a cone with apex $0 \in T_xV$, and $g_{\Delta_l}(x)$ turns it into a Euclidean cone. Let $\Delta_m \subset \Delta_l$ $(m < l)$ be another simplex adjacent to x. Then the face of $T_x\Delta_l$ corresponding to Δ_m is isomorphic to $T_x\Delta_m$ and we view $T_x\Delta_m$ as a subset of $T_x\Delta_l$.

Set $T_xK = \bigcup_{\Delta_i \ni x} T_x\Delta_i$. We call T_xK the *tangent cone* of K at x. Let $S_x\Delta_l$ denote the subset of all unit vectors in $T_x\Delta_l$ and set $S_x = S_xK = \bigcup_{\Delta_i \ni x} S_x\Delta_i$. The set S_x is called the *link* of x in K. If Δ_l is a simplex adjacent to x, then $g_{\Delta_l}(x)$ defines a Riemannian metric on the $(l-1)$-simplex $S_x\Delta_l$. The family g_x of Riemannian metrics $g_{\Delta_l}(x)$ turns $S_x\Delta_l$ into a simplicial complex with a piecewise smooth Riemannian metric such that the simplices are spherical.

As was shown in [4], a two dimensional complete locally finite simplicial complex (K, g) is curvature bounded by k $(k_K \leq k)$ iff the following three conditions hold:

(1) The Gauss curvature of the open faces is bounded from above by k.
(2) For every edge e of K, any two faces f_1, f_2 adjacent to e, and every interior point $x \in e$, the sum of the geodesic curvatures $k_1(x)$, $k_2(x)$ of e with respect to f_1, f_2 is nonpositive.
(3) For every vertex x of K, every simple loop in S_xK has length at least 2π (i.e., S_xK is a CAT_1 space).

1.3 The Liouville measure for the geodesic flow. We assume that K is an admissible n-dimensional Riemannian complex. We denote by $K^{(i)}$ the i-skeleton of K and by K' the set of points $x \in K$ such that x is contained in the interior of an $(n-1)$-simplex.

Let $x \in K'$. Then x is contained in the interior of an $(n-1)$-simplex Δ'. For any n-simplex Δ whose boundary $\partial\Delta$ contains x, let $S'_x\Delta$ denote the open hemisphere of unit tangent vectors at x pointing towards the inside of Δ. Let $\Delta_1, \ldots, \Delta_m$, $m \geq 2$, be the n-simplices containing Δ'. We set $S'_x = \bigcup_{i=1}^{m} S'_x\Delta_i$, $S' = \bigcup_{x \in K'} S'_x$ and $S'\Delta = \bigcup_{x \in \partial\Delta \cap K'} S'_x\Delta$.

For $v \in S'_x\Delta$ denote by $\theta(v)$ the angle between v and the interior normal $\nu_\Delta(x)$ of Δ' with respect to Δ at x. Let dx be the volume element on K' and let λ_x be the Lebesgue measure on S'_x. We define the *Liouville measure* on

S' by $d\mu'(x,v) = \cos\theta(v) d\lambda_x(v) \otimes dx$. Note that $d\mu'(x,v) \otimes dt$ is the ordinary Liouville measure that is invariant under the geodesic flow on each n-simplex Δ of K. Therefore, for μ'-*a.e.* $v \in S'\Delta$, the geodesic γ_v in Δ determined by $\dot\gamma_v(0) = v$ meets $\partial\Delta \cap K^{(n-1)} \setminus K^{(n-2)}$ after a finite time $t_v > 0$ so that $I(v) = -\dot\gamma_v(t_v) \in S'\Delta$. Note that $\gamma_v(t_v) \in K'$ since K is boundaryless. The measure μ' is invariant under the involution I.

Let $I(v) = u + \cos\theta(I(v))\nu_{\Delta_n}(\gamma_v(t_v))$, where u is tangent to K', and set $F(v) = \bigcup_i \{-u + \cos\theta(I(v))\nu_{\Delta_n^i}(\gamma_v(t_v))\}$, where the union is taken over all n-simplices containing $\gamma_v(t_v)$ except Δ. Thus there is a subset $S_1 \subset S'$ of full μ'-measure such that $F(v)$ is defined for any $v \in S_1$. We set recursively $S_{i+1} = \{(x,v) \in S_1 \backslash F(v) \subset S_i\}$ and define $S_\infty = \bigcap_{i=0}^\infty S_i$, $V = S_\infty \cap I(S_\infty)$. By construction, V has full μ'-measure.

We define the geodesic flow on the space SK (or TK) in the following way: For $(x,v) \in V$ we let

$$\begin{cases} g^t(x,v) &= (X_{(x,v)}(t), \dot X_{(x,v)}(t)), \\ g^0(x,v) &= (x,v), \end{cases}$$

where g^t is the ordinary geodesic flow in the interior of every n-simplex and in the case when $X_{(x,v)}(t_0) \in K'$ for $t_0 \in \mathbb{R}^+$ we set $\dot X_{(x,v)}(t_0) = \dot X_{(x,v)}(t_0+)$ (so that $\dot X_{(x,v)}(t_0) \in F(\dot X_{(x,v)}(t_0-))$.

1.4 General Markov processes. Assume that K is an admissible n-dimensional Riemannian complex with metric g and corresponding distance function d. When K is not compact, let $K_D = K \cup \{D\}$ be the one-point compactification of K. Then we can define a metric δ on K_D such that the topology on K generated by δ is the same as the topology generated by d. In case K is already compact, we simply adjoin D as an isolated point and define the metric δ on K_D by letting $d = \delta$ on $K \times K$ and $\delta(p, D) = 1$ for $p \in K$. Therefore, the restriction of δ to $K \times K$ is uniformly continuous with respect to d.

Let $C(K)$ be the space of bounded continuous real-valued functions on K, $C_0(K)$ the subspace of $C(K)$ consisting of functions that have limit zero at infinity, and $C_c(K)$ the space of functions in $C(K)$ with compact support. Clearly, these three spaces are the same if K is compact. The space $C(K)$ endowed with sup-norm is a (real) Banach space and $C_0(K)$ and $C_c(K)$ are Banach subspaces of $C(K)$. The space $C_c(K)$ is dense in the space $C_0(K)$.

Finally, whenever the term measurable is used, it will refer to the basic σ-algebra of Borel sets in K (or K_D).

The usual setup for the theory of *temporally homogeneous Markov processes* defined on a measurable space $(\Omega \times [0,\infty[, \mathfrak{M} \times \mathfrak{R})$ (where $\mathfrak{R}$ is the Borel σ-algebra in $[0,\infty[$) with values in a topological measurable space $(E, \mathfrak{B})$ consists of the following objects:

(1) A point D adjoined to the space E. We write $E_D = E \cup \{D\}$ and let $\mathfrak{B}_D$ be the σ-algebra in E_D generated by $\mathfrak{B}$.
(2) For each $x \in E_D$, a probability measure P_x on $(\Omega, \mathfrak{M})$.
(3) An increasing family (a *filtration*) $(\mathfrak{M}_t)_{t \geq 0}$ of sub-σ-algebras of $\mathfrak{M}$ and a distinguished point ω_D of Ω.
(4) For each $t \in [0, \infty[$ a measurable map $Y_t : (\Omega, \mathfrak{M}) \to (E_D, \mathfrak{B}_D)$ such that if $Y_t(\omega) = D$ then $Y_s(\omega) = D$ for all $s \geq t$, $Y_\infty(\omega) = D$ for all ω and $Y_0(\omega_D) = D$.
(5) For each $t \in [0, \infty[$ a *translation operator* $\theta_t : \Omega \to \Omega$ such that $\theta_\infty \omega = \omega_D$ for all ω.

We call the collection $Y = (\Omega, \mathfrak{M}, \mathfrak{M}_t, Y_t, \theta_t, P_x)$ a (temporally homogeneous) *Markov process* with state space $(E, \mathfrak{B})$ if and only if the following axioms hold:

(1) For each $t \geq 0$ and fixed $\Gamma \in \mathfrak{B}$, the function $x \mapsto P(t, x, \Gamma) = P_x\{Y_t \in \Gamma\}$ is $\mathfrak{B}$ measurable.
(2) For all $x \in E$, $P(0, x, E \setminus \{x\}) = 0$ and $P_D\{X_0 = D\} = 1$.
(3) For all $t, h \geq 0$, $Y_t \circ \theta_h = Y_{t+h}$ (homogeneity).
(4) For all $s,\ t \in \mathbb{R}^+$, $x \in E_D$ and $\Gamma \in \mathfrak{B}_D$, $P_x\{X_{t+s} \in \Gamma | \mathfrak{M}_t\} = P(s, X_t, \Gamma)$ (Markov property).

The point D may be thought of as a "cemetery" when we regard $t \mapsto Y_t(\omega)$ as the trajectory of particle moving randomly in the space E. With this interpretation in mind, we call the random variable $\xi(\omega) = \inf\{t; X_t(\omega) = D\}$ the *lifetime*.

2. Isotropic transport processes

In this section, K will denote a complete admissible Riemannian complex with dimension n, and we will use all notations of the first section.

2.1 An intuitive approach. Let ΣK denote the space of links of the complex K. Choose a point (x_0, v_0) from the space ΣK and assume that the point x_0 is in the topological interior of a maximal simplex Δ_0. Intuitively, a particle starting from the point x_0 travels geodesically, in a direction v_0 chosen randomly, during an exponentially distributed waiting time s_1 to a new position x_1 assumed to be in the interior of Δ_0. At x_1, the particle chooses a new direction v_1 in the link S_{x_1} over x_1 with the uniform probability $\mathbb{P}[v_1 \in d\lambda] = \lambda_{x_1}(d\lambda)$, where λ denotes the normalized Lebesgue measure on S_{x_1}. From the point x_1 and in the direction v_1, the particle travels geodesically during an exponentially distributed waiting time s_2 to a position x_2 in the interior of the simplex Δ_0. The particle continues its motion in the interior of Δ_0 until it hits transversally (because of the construction of the generalized geodesic flow on the admissible complexes) the border of the simplex Δ_0 at an interior point of an $(n-1)$-simplex adjacent to Δ_0. Call this point x_n.

Starting now from x_n and choosing randomly a new direction in the link over x_n, the particle travels geodesically during an exponentially distributed waiting time s_n to a new position in the interior of a maximal simplex (which may be Δ_0), and so on.

2.2 Mathematical approach. We now give a precise mathematical definition of the random walk described above.

Consider the product space $L = \Sigma K \times \mathbb{R}^+$ and the product σ-algebra $\mathfrak{F} = \mathfrak{E} \times \mathfrak{B}$, where $\mathfrak{E}$ and $\mathfrak{B}$ are, respectively, the Borel σ-algebra of ΣK and the Borel σ-algebra of $\mathbb{R}^+$. Set $\Omega = L^{\mathbb{N}}$ and $\mathfrak{G} = \mathfrak{F}^{\mathbb{N}}$, where $\mathbb{N}$ is the set of positive integers. Thus $(L, \mathfrak{F})$ and $(\Omega, \mathfrak{G})$ are measurable spaces and the points $\omega \in \Omega$ are sequences $\{((x_l, v_l), t_l) \in \Sigma K \times \mathbb{R}^+; l \in \mathbb{N}\}$.

Next, let $\{((x_l, v_l), t_l) \in \Sigma K \times \mathbb{R}^+; l \in \mathbb{N}\}$ be a point of Ω and set $\tilde{Y}_k(\omega) = ((x_k, v_k), t_k)$, $Z_k(\omega) = (x_k, v_k)$ and $\tau_k(\omega) = t_k$. The functions $\tilde{Y}_k : (\Omega, \mathfrak{G}) \to (L, \mathfrak{F})$, $Z_k : (\Omega, \mathfrak{G}) \to (\Sigma K, \mathfrak{E})$ and $\tau_k : (\Omega, \mathfrak{G}) \to (\mathbb{R}^+, \mathfrak{B})$ are measurable.

Finally, consider the space of events

$$\Omega' = \{\omega \in \Omega \mid \forall k \in \mathbb{N}, Z_{k+1}(\omega) \neq Z_k(\omega), \tau_0 = 0, \tau_{k+1}(\omega) > \tau_k(\omega)\}.$$

Put $\xi(\omega) = \lim_{n\to\infty} \tau_n(\omega)$ (lifetime) and let $K_D = K \cup \{D\}$ denote the one point compactification of K. The space K is assumed to be semi-compact, so we can endow K_D with a metric d' such that the space (K_D, d') is compact and the restriction of d' to K coincides with the original metric on K.

We now define the K-valued geodesical random walk by setting, for $t \geq 0$,

$$Y_t(\omega) = \begin{cases} X_{Z_i(\omega)}(t - \tau_i(\omega)) & \text{if } \tau_i(\omega) \leq t \leq \tau_{i+1}(\omega)\ , \\ D & \text{if } \xi(\omega) \leq t\ , \end{cases}$$

where X is the K-projection of the generalized geodesic flow on the complex K. According to this definition, we have $Y_\infty(\omega) = D$ for every $\omega \in \Omega$.

2.3 The Markov property. We complete the preceding construction by defining an admissible complex valued *isotropic transport process* and then showing that this process is a strong Markov process.

Let K denote an admissible Riemannian complex and define the transition density on the measurable space $(L, \mathfrak{F})$ as

$$N(z, t; dz, ds) = \begin{cases} 0 & \text{if } t < s\ , \\ \lambda_x(dz) e^{-(s-t)} ds & \text{if } s \leq t\ , \end{cases}$$

where $z = (x, v)$, $dz = (x, dv)$, and λ_x is the uniform measure on the link $S_x K$.

Proposition 2.1. *Let γ denote a probability measure on the measurable space $(L, \mathfrak{F})$. Then there exists a probability measure P^γ on the measurable space $(\Omega, \mathfrak{G})$ such that the coordinate mappings $\{\tilde{Y}_n; n \in \mathbb{N}\}$ form a temporally* *homogeneous Markov process on the measure space* $(\Omega, \mathfrak{G}, P^\gamma)$*, with* γ *as initial distribution and* N *the transition function, i.e., we have*

$$P^\gamma(\tilde{Y}_{n+1} \in A \mid \tilde{Y}_0, \ldots, \tilde{Y}_n) = \int_A N(Z_n, \tau_n; dz, ds),$$

for all A *belonging to* $\mathfrak{F}$ *and* $n \in \mathbb{N}$*.*

Proof. The proposition is an immediate corollary of I. Tulcea's Theorem (see [12, pp. 613–615]). □

If γ is the measure $\lambda_x \otimes \delta_0$, with δ_0 the Dirac mass at $0 \in \mathbb{R}$, then we will write P^{λ_x} or P^x for P^γ. Consequently, we have, for every $x \in K$, $P^x(\Omega') = 1$, and the process $\{\tilde{Y}_n; n \in \mathbb{N}\}$ is Markov on the measure space $(\Omega', \mathfrak{G}', P^x)$. We let Ω be the set of sequences $\{(z_n, t_n) \in L; n \geq 0\}$ such that $z_{n+1} \neq z_n$ and $0 = t_0 < t_1 < \cdots < t_n < \cdots$, and $\mathfrak{G}$ the σ-algebra of Ω generated by $\{\tilde{Y}_n; n \in \mathbb{N}\}$. In the sequel we will use the probability space(s) $(\Omega, \mathfrak{G}, P^x)$.

Let $(Y_t)_{t\geq 0}$ denote the K-valued random walk constructed above. For all $\omega \in \Omega$, the map $t \mapsto Y_t(\omega)$ is continuous on $\mathbb{R}^+$ and has left-hand limits on $[0, \xi(\omega)[$. We complete the σ-algebra $\mathfrak{G}$ by adjoining a point ω_D to Ω such that $Y_t(\omega_D) = D$ for all t, $\{\omega_D\} \in \mathfrak{G}$ and $P^x(\{\omega_D\}) = 0$ for all $x \in K$. We set $Z_n(\omega_D) = D$ and $\tau_n(\omega_D) = \infty$ for all $n \in \mathbb{N}$ and denote by P^D the Dirac mass at ω_D.

Next we define translation operators $(\theta_t)_{t\geq 0}$ as follows: for all $t \geq 0$, $\theta_t \omega_D = \omega_D$; if $t \geq \xi(\omega)$, then $\theta_t \omega = \omega_D$, while if $t_k \leq t < t_{k+1}$, $k \geq 0$, then $\theta_t \omega = \{(z_{n+k}, (t_{n+k} - t) \vee 0); n \geq 0\}$, where $\omega = \{(z_n, t_n); n \geq 0\}$. Thus, we have $Y_s \circ \theta_t = Y_{s+t}$ for all $s, t \in \mathbb{R}^+$.

Definition 2.2. We call the stochastic process $Y = (\Omega, \mathfrak{G}, Y_t, \theta_t, P^x)$ the (an) isotropic transport process (motion) with values in the admissible Riemannian complex K.

Let $\mathfrak{G}_n := \sigma\{\tilde{Y}_i; 0 \leq i \leq n\}$ and $\mathfrak{F}_t^0 := \sigma\{Y_s; s \leq t\}$ denote, respectively, the σ-algebra of Ω generated by $\{\tilde{Y}_i; 0 \leq i \leq n\}$ and the one generated by $\{Y_s; s \leq t\}$.

Lemma 2.3. *Let* $\Lambda \in \mathfrak{F}_t^0$*. Then, for all* $n \geq 0$*, there exists* $\Lambda_n \in \mathfrak{G}_n$ *such that*

$$\Lambda \cap \{\tau_n \leq t < \tau_{n+1}\} = \Lambda_n \cap \{t < \tau_{n+1}\}.$$

Proof. Set

$$\mathfrak{G}_t := \sigma\big\{\Lambda \in \mathfrak{F}_t^0 \mid (\forall n \geq 0)(\exists \Lambda_n \in \mathfrak{G}_n), \\ \Lambda \cap \{\tau_n \leq t < \tau_{n+1}\} = \Lambda_n \cap \{t < \tau_{n+1}\}\big\}.$$

We can easily check that, for all $A \in \mathfrak{E}_D$, the sets $\{Y_s \in A\}_{s \leq t}$ belong to the σ-algebra $\mathfrak{G}_t$. This completes the proof since the sets $\{Y_s \in A\}_{s \leq t}$ generate the σ-algebra $\mathfrak{F}_t^0$. □

We define, for real functions $g \in C_0(\Sigma K)$ and $f \in C_0(K)$ (or simply measurable functions):

(1) $Pg(x) := \int_{\Sigma_x K} g(x, \eta) d\lambda_x(\eta)$.
(2) For all $t > 0, T_t^0 f(x) := \int_{\Sigma_x K} f(X_{(x,\eta)}(t)) d\lambda_x(\eta)$ and $T_t f(x) := E^x[f(Y_t)]$, the expectation with respect to Y_t.
(3) For all $\lambda > 0$, $R_\lambda^0 f(x) := \int_{\mathbb{R}^+} e^{-\lambda t} T_t^0 f(x) dt$ and $R_\lambda f(x) := \int_{\mathbb{R}^+} e^{-\lambda t} T_t f(x) dt$, the resolvent operators of T_t^0 and T_t, respectively.

Proposition 2.4. *Let $f \in C^0(K)$. Then for all $\lambda > 0$ we have*

$$R_\lambda f = \sum_{n=0}^{\infty} (R_{\lambda+1}^0)^{n+1} f,$$

where $(R_{n+1}^0)^0 := \mathrm{Id}$ is the identity map.

Proof. First we write

$$R_\lambda f(x) = \left[\int_0^{\tau_1} + \sum_{i=1}^{\infty} \int_{\tau_i}^{\tau_{i+1}} \right] e^{-\lambda t} f(Y_t) dt.$$

Taking into account the distribution of τ_1 and the initial distribution of the process Y, the first integral becomes

$$\int_0^\infty e^{-(1+\lambda)s} T_s^0 f(x) ds = R_{1+\lambda}^0 f(x).$$

For the second part of the decomposition, we will prove by induction that for all $i \geq 1$ we have

$$\text{(R)} \qquad \left[\int_{\tau_i}^{\tau_{i+1}} e^{-\lambda t} f(Y_t) dt \right] = (R_{\lambda+1}^0)^{i+1} f(x).$$

Let us first check the case $i = 1$:

$$\left[\int_{\tau_1}^{\tau_2} e^{-\lambda t} f(Y_t) dt \right] = \left[e^{-\lambda \tau_1} \int_0^{\tau_2 - \tau_1} e^{-\lambda t} f(Y_{t+\tau_1}) dt \right],$$

which is equal to

$$\left[e^{-\lambda \tau_1} (R_{\lambda+1}^0) f(X_{Z_1}(0)) \right] = \left[e^{-\lambda \tau_1} (P R_{\lambda+1}^0) f(X_{Z_0}(\tau_1)) \right].$$

Using the distribution of τ_1, we obtain

$$(R_{\lambda+1}^0)(R_{\lambda+1}^0) f(x).$$

Assume that property (R) holds up to order l. We will verify the property for order $l+1$. We have

$$\left[\int_{\tau_{l+1}}^{\tau_{l+2}} e^{-\lambda t} f(Y_t)dt\right] = \left[e^{-\lambda\tau_{l+1}} \int_0^{\tau_{l+2}-\tau_{l+1}} e^{-\lambda t} f(Y_{t+\tau_{l+1}})dt\right],$$

which is equal to

$$\left[e^{-\lambda\tau_{l+1}}(R^0_{\lambda+1})f(X_{Z_{l+1}}(0))\right] = \left[e^{-\lambda\tau_l} e^{-\lambda(\tau_{l+1}-\tau_l)}(PR^0_{\lambda+1})f(X_{Z_l}(\tau_{l+1}-\tau_l))\right].$$

Using the distribution of $(\tau_{l+1}-\tau_l)$, this becomes

$$\left[e^{-\lambda\tau_l}(R^0_{\lambda+1})(R^0_{\lambda+1})f(X_{Z_l}(0))\right],$$

which is equal to

$$\left[\int_{\tau_l}^{\tau_{l+1}} e^{-\lambda t} R^0_{\lambda+1} f(Y_t)dt\right].$$

Hence, applying the induction hypothesis to the function $R^0_{\lambda+1}f$, we obtain (R) for order $l+1$.

To complete the proof, note that the series $\sum_{n=0}^{\infty}(R^0_{\lambda+1})^{n+1}f$ converges uniformly since for all function $f \in C^0(K)$ we have $\|R^0_{\lambda+1}\| \le 1/(\lambda+1)$ (where $\|.\|$ is the sup norm). □

Lemma 2.5. *Let f be a measurable real (positive) function on $(K, \mathfrak{B})$. Then we have, for all $t \ge 0$ and $\lambda > 0$,*

$$E\left\{\int_t^{\infty} e^{-\lambda u} f(Y_u)du \mid \mathfrak{F}^0_t\right\} = e^{-\lambda t} R_\lambda f(Y_t).$$

Remark 2.6. By Lemma 2.3, to establish Lemma 2.5 it suffices to show that this equality holds on sets $\Lambda_n \in \mathfrak{F}^0_t$ with

$$\Lambda_n \cap \{\tau_n \le t < \tau_{n+1}\} = \Lambda_n \cap \{t < \tau_{n+1}\},$$

i.e., it suffices to show that

$$(\dagger) \qquad E\left\{\int_t^{\infty} e^{-\lambda u} f(Y_u)du \mid \Lambda_n\right\} = E\left\{e^{-\lambda t} R_\lambda f(Y_t) \mid \Lambda_n\right\}.$$

Proof of Lemma 2.5. Consider the left side of the equality (†)and write it in the form

$$E\left\{\int_t^{\infty} e^{-\lambda u} f(Y_u)du \mid \Lambda_n\right\} = \left[\left(\int_t^{\tau_{n+1}} + \sum_{i=n+1}^{\infty} \int_{\tau_i}^{\tau_{i+1}}\right) e^{-\lambda u} f(Y_u)du \mid \Lambda_n\right].$$

Using the Markov property of the process $\{\tilde{Y}_n; n \ge 0\}$, the fact that $\Lambda_n \subset \{\tau_n \le t \le \tau_{n+1}\}$ and the exponential distribution of the random variable $\tau_{n+1} - t \wedge \tau_{n+1} - \tau_n$, the first integral of the decomposition becomes

$$\left[e^{-\lambda t} e^{-(t-\tau_n)} \int_0^{\infty} e^{-(\lambda+1)u} Pf(X_{Z_n}(u+(t-\tau_n)))du \mid \Lambda_n\right],$$

which is equal to

$$\left[e^{-\lambda t}e^{-(t-\tau_n)}R^0_{\lambda+1}f(X_{Z_n}(t-\tau_n)) \mid \Lambda_n\right].$$

For the second part of the decomposition, we will show by induction that for all $i \geq 1$ we have

$$(\mathrm{R})^t_n \quad \left[\int_{\tau_{n+i}}^{\tau_{n+i+1}} e^{-\lambda u}f(Y_u)du \mid \Lambda_n\right] = \left[e^{-\lambda t}e^{-(t-\tau_n)}(R^0_{\lambda+1})^{i+1}f(X_{Z_n}(t-\tau_n)) \mid \Lambda_n\right].$$

Let us first check the case $i=1$: We have

$$\left[\int_{\tau_{n+1}}^{\tau_{n+2}} e^{-\lambda u}f(Y_u)du \mid \Lambda_n\right] = \left[e^{-\lambda\tau_{n+1}}\int_0^{\tau_{n+2}-\tau_{n+1}} e^{-\lambda u}f(Y_{u+\tau_{n+1}})du \mid \Lambda_n\right],$$

which is equal to

$$\left[e^{-\lambda\tau_{n+1}}(R^0_{\lambda+1})f(X_{Z_{n+1}}(0)) \mid \Lambda_n\right] = \left[e^{-\lambda t}e^{-\lambda(\tau_{n+1}-t)}(R^0_{\lambda+1})f(X_{Z_n}(\tau_{n+1}-\tau_n)) \mid \Lambda_n\right],$$

which is the same as

$$\left[e^{-\lambda t}e^{-\lambda(\tau_{n+1}-t)}(R^0_{\lambda+1})f(X_{Z_n}((\tau_{n+1}-t)+(t-\tau_n))) \mid \Lambda_n\right].$$

Using the Markov property of $\{\tilde{Y}_n; n\geq 0\}$ and the distribution of $(\tau_{n+1}-t)\wedge(\tau_{n+1}-\tau_n)$, we obtain

$$\left[e^{-\lambda t}e^{-(t-\tau_n)}\int_0^\infty e^{-(\lambda+1)u}P(R^0_{\lambda+1})f(X_{Z_n}(u+(t-\tau_n)))du \mid \Lambda_n\right],$$

which is equal to

$$\left[e^{-\lambda t}e^{-(t-\tau_n)}R^0_{\lambda+1}(R^0_{\lambda+1})f(X_{Z_n}(t-\tau_n)) \mid \Lambda_n\right].$$

Now assume that property $(\mathrm{R})^t_n$ holds up to order l. We will show that it holds for order $l+1$. We have

$$\left[\int_{\tau_{n+(l+1)}}^{\tau_{n+(l+2)}} e^{-\lambda u}f(Y_u)du \mid \Lambda_n\right] = \left[e^{-\lambda\tau_{n+(l+1)}}\int_0^{\tau_{n+(l+2)}-\tau_{n+(l+1)}} e^{-\lambda u}f(Y_{u+\tau_{n+(l+1)}})du \mid \Lambda_n\right],$$

which is equal to

$$\left[e^{-\lambda\tau_{n+(l+1)}}(R^0_{\lambda+1})f(X_{Z_{n+(l+1)}}(0)) \mid \Lambda_n\right],$$

which is

$$\left[e^{-\lambda\tau_{n+l}}e^{-\lambda(\tau_{n+(l+1)}-\tau_{n+l})}(PR^0_{\lambda+1})f(X_{Z_{n+l}}(\tau_{n+(l+1)}-\tau_{n+l}))\mid\Lambda_n\right].$$

Using the distribution of $(\tau_{n+(l+1)}-\tau_{n+l})$, the above becomes

$$\left[e^{-\lambda\tau_{n+l}}(R^0_{\lambda+1})(R^0_{\lambda+1})f(X_{Z_{n+l}}(0))\mid\Lambda_n\right].$$

The latter expectation is equal to

$$\left[\int_{\tau_{n+l}}^{\tau_{n+(l+1)}}e^{-\lambda u}R^0_{\lambda+1}f(Y_u)du\mid\Lambda_n\right].$$

Thus, applying the induction hypothesis to the function $R^0_{\lambda+1}f$, we obtain the equality $(\mathrm{R})^t_n$ for the order $l+1$.

We have now shown that the left side of the equality (†)is equal to

$$\left[e^{-\lambda t}e^{-(t-\tau_n)}\{R^0_{\lambda+1}f(X_{Z_n}(t-\tau_n))+\sum_{i=1}^{\infty}(R^0_{\lambda+1})^{i+1}f(X_{Z_n}(t-\tau_n))\}\mid\Lambda_n\right].$$

By Proposition 2.4 of this section this sum is equal to

$$\left[e^{-\lambda t}e^{-(t-\tau_n)}R_\lambda f(X_{Z_n}(t-\tau_n))\mid\Lambda_n\right].$$

Using once again the Markov property of $\{\tilde{Y}_n; n\geq 0\}$, we get

$$\left[e^{-\lambda t}e^{-(t-\tau_n)}R_\lambda f(X_{Z_n}(t-\tau_n))\mid\Lambda_n\right]=\left[e^{-\lambda t}R_\lambda f(Y_t)\mid\Lambda_n\right],$$

which was to be proved. □

We have now collected all the ingredients to prove the following theorem:

THEOREM 2.7. *Let $Y=(\Omega,\mathfrak{F}^0_t,Y_t,\theta_t,P^x)$ be the isotropic transport process with values in the admissible Riemannian complex K. Then Y is a strong Markov process.*

REMARK 2.8 ([14, I, pp. 97–100]). It suffices to show that the process Y is a Markov process because a right continuous Markov process (with right continuous trajectories) is always strongly Markov for the filtration $\{\mathfrak{F}^0_{t+}\}$. But we know that for continuous stochastic processes the filtration $\{\mathfrak{F}^0_{t+}\}$ is equal to the filtration $\{\mathfrak{F}^0_t\}$, which includes the case of the isotropic transport process (with continuous trajectories).

Proof of Theorem 2.7. By Lemma 2.5 we have

$$E\left\{\int_t^\infty e^{-\lambda u}f(Y_u)du\mid\mathfrak{F}^0_t\right\}=E^{Y_t}\left\{\int_0^\infty e^{-\lambda(t+u)}f(Y_u)du\right\}.$$

Therefore, if the function f is bounded, we have

$$E\left\{\int_t^\infty \varphi(u)f(Y_u)du\mid\mathfrak{F}^0_t\right\}=E^{Y_t}\left\{\int_0^\infty \varphi(t+u)f(Y_u)du\right\},$$

whenever φ is a linear combination of exponentials, and hence, by uniform approximation, whenever φ is continuous and vanishes at infinity. We apply this to the following sequence of functions:

$$\varphi_n(s+t+u) = \begin{cases} 0 & \text{if } 1/n \leq u \ , \\ \frac{1}{n} - u & \text{if } 0 \leq u < 1/n \ . \end{cases}$$

The sequence $(\varphi_n)_{n\geq 0}$ is a sequence of continuous functions vanishing at infinity and converging to the Dirac mass at $s+t$, while the map $u \mapsto f(Y_u)$ is a bounded (right) continuous function. Consequently, if we take the limit as n tends to infinity, we obtain

$$E\left\{f(Y_{t+s})du \mid \mathfrak{F}_t^0\right\} = E^{Y_t}\{f(Y_s)\}.$$

In other words, Y is Markov process. $\square$

3. The Wiener measure

3.1 Construction. Let $Y = (\Omega, \mathfrak{F}_t^0, Y_t, \theta_t, P^x)$ be the isotropic transport process in the complete admissible Riemannian complex K constructed in the last section. For a real $\eta > 0$ and $z = (x, v) \in \Sigma K$ set $\eta z := (x, \eta v)$.

Define a process Y^η from $Y = (\Omega, \mathfrak{F}_t^0, Y_t, \theta_t, P^x)$ by

$$Y_t^\eta(\omega) = \begin{cases} X_{\eta Z_i(\omega)}\left(\frac{t}{\eta^2} - \tau_i(\omega)\right) & \text{if } \tau_i(\omega) \leq t/\eta^2 \leq \tau_{i+1}(\omega) \ , \\ D & \text{if } \xi(\omega) \leq t/\eta^2 \ . \end{cases}$$

The process $Y^\eta = (\Omega, \mathfrak{F}_t^0, Y_t^\eta, \theta_t, P^x)$ has continuous trajectories, and it is, like the process Y, strongly Markov.

Proposition 3.1. *Let K be an admissible Riemannian complex and let $C(\mathbb{R}^+, K)$ be the space of continuous paths in K. Then, for each $\eta > 0$, the process Y^η generates a measure μ_η on the space $C(\mathbb{R}^+, K)$.*

Proof. For $\eta > 0$, define $P_{s,t}^\eta(p, A)$ for $p \in K$ and $A \in \mathfrak{B}(K_D)$ as the transition probability of the process Y^η (i.e., $P_{s,t}^\eta(p, A) := \text{Prob}\{Y_{t+s}^\eta \in A; Y_s^\eta = p\}$).

Consider the finite sets of reals $J = \{t_1 < t_2 < \cdots < t_n\} \subset (\mathbb{R}^+)^n$. For each such finite set $J = \{t_1 < t_2 < \cdots < t_n\}$ define the probability measure

$$P_J^\eta(B) = \int_B P^x(dx_0) \int P_{0,t_1}^\eta(x_0, dx_1) \int \ldots \int P_{t_{n-1},t_n}^\eta(x_{n-1}, dx_n)$$

for $B \subset K_D^n$. Let $\Phi(\mathbb{R}^+)$ denote the set of the finite subsets of $\mathbb{R}^+$. Then, thanks to the Markov property of Y^η, the system $\{P_J^\eta; J \in \Phi(\mathbb{R}^+)\}$ is a projective system on $(K_D, \mathfrak{B}(K_D))$ (i.e., if π_J^I (respectively, π_J) is the natural projection of K^I (respectively, Ω)) to K^J, then $P_I^\eta(\pi_J^I)^{-1} = P_J^\eta$).

On the other hand, the trajectories of Y^η are continuous and the space K is Hausdorff and σ-compact. Consequently, using the Kolmogorov theorem [5], we get a probability measure μ_η on the space $C(\mathbb{R}^+, K)$. □

3.2. The Wiener measure. We now state the main theorem of this section.

THEOREM 3.2. *Let K be an admissible Riemannian complex, consider the family $\{Y^\eta\}_{\eta>0}$ of isotropic transport processes constructed above, and let $(\mu_\eta)_{\eta>0}$ be the family of the generated probability measures on $C(\mathbb{R}^+, K)$. Let the space $C(\mathbb{R}^+, K)$ be endowed with the compact-open topology. Then the family $(\mu_\eta)_{\eta>0}$ has a convergent subsequence.*

To prove this theorem we need the following lemma:

LEMMA 3.3. *Under the hypothesis of Theorem 3.2, the family of the probability measures $(\mu_\eta)_{\eta>0}$ is tight, i.e.,*

$$\lim_{\substack{\eta\to 0\\ c\to 0}} \operatorname{Prob}\left\{ \sup_{\substack{t-c<t_1<t_2<t+c\\ 0\le t_1<t_2\le N}} \min\left[d(Y^\eta_{t_1}, Y^\eta_t); d(Y^\eta_t, Y^\eta_{t_2})\right] > \epsilon \right\} = 0.$$

REMARK 3.4. Before proceeding with the proof of the lemma, we recall the following two facts:

(1) When the space $C(\mathbb{R}^+, K)$ is endowed with the compact-open topology, the tightness property is equivalent to the equality of Lemma 3.3 by a result of Stone [29].
(2) According to a result of Jørgensen [27, Lemma 1.4], if the property
$$\forall \epsilon > 0\, \exists \alpha > 0,\ \sup_{\substack{p\in K_D\\ 0<s}} \frac{1}{s} P^\eta_{0,s}(p, B^c_D(p,\epsilon)) \le \alpha$$
holds, then the equality of Lemma 3.3 also holds.

Proof of Lemma 3.3. By Remark 3.4, if we show that
$$\forall \epsilon > 0\, \exists \alpha > 0,\ \lim_{\eta\to 0} \sup_{\substack{p\in K_D\\ 0<t}} \frac{\operatorname{Prob}\{Y^\eta_t \in B^c_D(p,\epsilon)\}}{t/\eta^2} \le \alpha,$$
then the sequence $(\mu_\eta)_{\eta>0}$ is tight.

We will assume that $\epsilon < \eta$ (otherwise the probability in question is zero, in which case there is nothing to prove) and that $t/\eta^2 < \tau_1$ (see the induction argument in proof of Lemma 2.5), which does not affect the result. We may also assume that ϵ is less than or equal to t/η^2, since otherwise the probability is again zero.

Thus, we have
$$\operatorname{Prob}\big\{Y^\eta_t \in B^c_D(p,\epsilon)\big\} = E\big\{I_{B^c_D(p,\epsilon)}(Y^\eta_t) \mid \epsilon \le t/\eta^2 < \tau_1\big\}.$$

Using the Markov property, we obtain

$$\begin{aligned}&\mathrm{Prob}\,\{Y_t^\eta \in B_D^c(p,\epsilon)\}\\ &\qquad = E\left\{e^{-t/\eta^2}\int_0^\infty PI_{B_D^c(p,\epsilon)}\left(X_{(p,\eta\zeta)}\left(\frac{t}{\eta^2}+s\right)\right)e^{-s}ds \mid \epsilon \le \frac{t}{\eta^2}\right\},\end{aligned}$$

which is equal to

$$E\left\{e^{-t/\eta^2}R_1^0 I_{B_D^c(p,\epsilon)}\left(X_{(p,\eta\zeta)}\left(\frac{t}{\eta^2}\right)\right) \mid \epsilon \le \frac{t}{\eta^2}\right\}.$$

Using the fact that $\|R_1^0\| \le 1$, we obtain

$$E\left\{e^{-t/\eta^2}R_1^0 I_{B_D^c(p,\epsilon)}\left(X_{(p,\eta\zeta)}\left(\frac{t}{\eta^2}\right)\right) \mid \epsilon \le \frac{t}{\eta^2}\right\} \le e^{-t/\eta^2}.$$

So for all $t > 0$ we get

$$\frac{\mathrm{Prob}\{Y_t^\eta \in B_D^c(p,\epsilon)\}}{t/\eta^2} \le \frac{e^{-t/\eta^2}}{t/\eta^2}.$$

Since, for each $t > 0$, $\frac{e^{-t/\eta^2}}{t/\eta^2}$ goes to zero if η goes to zero, the result follows. □

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. Consider the space $C(\mathbb{R}^+, K)$ endowed with the compact-open topology, where K is an admissible Riemannian complex. Let $(\mu_\eta)_{\eta>0}$ be the sequence of probability measures generated by the family of isotropic transport processes $\{Y^\eta\}_{\eta>0}$.

By Lemma 3.3, the sequence $(\mu_\eta)_{\eta>0}$ is tight; moreover, the space $C(\mathbb{R}^+, K)$ endowed with the compact-open topology is separable. Thus, by Prohorov's theorem (see [5]), the sequence $(\mu_\eta)_{\eta>0}$ is relatively compact. The proof is now complete. □

We showed above that the sequence $(\mu_\eta)_{\eta>0}$ has a subsequence which converges to a probability measure. Let W denote this limit. We make the following definition:

Definition 3.5. The measure W on the space $C(\mathbb{R}^+, K)$ is called a Wiener measure.

Example 3.6 (The smooth case). Assume K is a smooth Riemannian manifold of dimension n and let $\triangle$ denote the operator of Laplace-Beltrami on K. Then $\triangle$ is the infinitesimal generator of a Markov process, called the Brownian motion [16], and denoted by $\{B_t^x\}_{t<\zeta'}$. Let $(U_t)_{t>0}$ denote the semigroup associated to the Brownian motion. Suppose that, for all $f \in C_0(K)$, $U_t f \in C_0(K)$. Then we have the following theorem:

THEOREM. *The sequence of processes* $\{Y^\eta\}_{\eta>0}$ *converges weakly to the process* $\{B_t^x\}_{t<\zeta'}$.

Proof. Set $T_t^\eta f(x) = E^x[f(Y_t^\eta)]$. By a result of Pinsky (see [26]), we have, for all $f \in C_0(K)$,

$$\lim_{\eta\to 0} T_t^\eta f = U_{t/n} f,$$

where n is the dimension of K. By Theorem 3.2, there exists a subsequence $(\mu_{\eta'})_{\eta'>0}$ of the sequence of probability measures $(\mu_\eta)_{\eta>0}$, such that $(\mu_{\eta'})_{\eta'>0}$ converges to a probability measure W on the space $C(\mathbb{R}^+, K)$.

Thus, by Stone's theorem [29], W is the classical Wiener measure generated by the Brownian motion $\{B_t^x\}_{t<\zeta'}$. □

4. Brownian motion

We let K denote a complete admissible Riemannian complex and consider $\{Y^\eta\}_{\eta>0}$, the family of the isotropic transport processes, and $(\mu_\eta)_{\eta>0}$, the corresponding sequence of probability measures.

Let $(\mu_{\eta_k})_k$ be a subsequence of the sequence $(\mu_\eta)_{\eta>0}$ which converges to the Wiener measure W.

For $\eta_k > 0$ and for each finite set $J = \{t_1 < t_2 < \cdots < t_n\}$ let $P_J^{\eta_k}$ be the probability measure on the product space K^n defined by

$$P_J^{\eta_k}(B) = \int_B P^x(dx_0) \int P_{0,t_1}^{\eta_k}(x_0, dx_1) \int \dots \int P_{t_{n-1},t_n}^{\eta_k}(x_{n-1}, dx_n)$$

for $B \subset K_D^n$.

PROPOSITION 4.1. *Let* $\Phi(\mathbb{R}^+)$ *denote the set of all finite subsets of* $\mathbb{R}^+$. *Then, for all* J *in the set* $\Phi(\mathbb{R}^+)$, *the sequence of probability measures* $(P_J^{\eta_k})_k$ *has a subsequence converging to a probability measure* μ_J *on the space* $K_D^{|J|}$ *(where* $|J|$ *is the cardinality of* J*). Moreover, the system* $\{\mu_J; J \in \Phi(\mathbb{R}^+)\}$ *is projective on the space* $(K_D, \mathfrak{B}(K_D))$.

Proof. Recall that for all $s \in \mathbb{R}^+$, $t \in \mathbb{R}^+$ and all $p \in K$, the sequence of transition functions $(P_{s,t}^{\eta_k}(p,.))_k$ $(P_{s,t}^{\eta_k}(p, A) := \mathrm{Prob}\{Y_{t+s}^{\eta_k} \in A; Y_s^{\eta_k} = p\}$, where $A \in \mathfrak{B}(K_D))$ defines a sequence of probability measures on the space $(K_D, \mathfrak{B}(K_D))$.

Moreover, the space K_D is σ-compact. Thus, by Prohorov's theorem [5], there exists a probability measure $\mu_{s,t}^p$ and a subsequence $(P_{s,t}^{\eta_k}(p,.))_k$ converging weakly to $\mu_{s,t}^p$.

By a diagonal argument, we obtain, for all $J = \{t_1 < t_2 < \cdots < t_n\}$ in $\Phi(\mathbb{R}^+))$, a probability measure μ_J on the product space $K_D^{|J|}$ given by

$$\mu_J(B) = \int_B P^x(dx_0) \int \mu_{0,t_1}^{x_0}(dx_1) \int \dots \int \mu_{t_{n-1},t_n}^{x_{n-1}}(dx_n)$$

for $B \subset K_D^{|J|}$. The proof is now complete. □

REMARK 4.2. The sequence $(\mu_{\eta_k})_k$ is weakly convergent to the Wiener measure W. Thus, for every set J belonging to $\Phi(\mathbb{R}^+)$, the finite dimensional distribution $W(\pi_J)^{-1}$ coincides with μ_J. In particular, for all $s > 0$, $t > 0$ and $p \in K_D$, we have $W^p(\pi_{\{s,t\}})^{-1} := \mu^p_{s,t}$.

COROLLARY 4.3. *The function which maps a point* $(t, p, \Gamma) \in \mathbb{R}^+ \times K_D \times \mathfrak{B}(K_D)$ *to* $W(t, p, \Gamma) := W^p(\pi_{\{0,t\}})^{-1}(\Gamma)$ *is a transition function on the measurable space* $(K_D, \mathfrak{B}(K_D))$.

Proof. The result is an immediate consequence of Proposition 4.1 and Remark 4.2 . □

We are now ready to state the main theorem of this section.

THEOREM 4.4. *Let* $(t, p, \Gamma) \mapsto W(t, p, \Gamma)$ *denote the transition function on the measurable space* $(K_D, \mathfrak{B}(K_D))$*, corresponding to the Wiener measure on the space* $C(\mathbb{R}^+, K)$ *(see Corollary 4.3). Then there exists a continuous* K_D*-valued Markov process* $\{B^p_t\}_{t\geq 0}$ *with* $W(t, p, \Gamma)$ *as transition function.*

Before proceeding with the proof, we make the following definition:

DEFINITION 4.5. The continuous K_D-valued Markov process $\{B^p_t\}_{t\geq 0}$ is called a *Brownian motion*.

Proof of Theorem 4.4. By a corollary of Kolmogorov's theorem (see [14, I, p. 91, Theorem 3.5]), the conclusion of Theorem 4.4 holds if we show that the transition functions $(t, p, \Gamma) \mapsto W(t, p, \Gamma)$ satisfy the following two conditions for each compact set $\Gamma \subset K_D$:

(1) For all $N > 0$, $\lim_{y\to\infty} \sup_{t\leq N} W(t, y, \Gamma) = 0$.
(2) For all $\epsilon > 0$, $\lim_{t\downarrow 0} \sup_{p\in\Gamma} (1/t) W(t, p, B^c_D(p, \epsilon)) = 0$.

To prove the first condition, consider a compact set $\Gamma \subsetneq K_D$ (if $\Gamma = K_D$, then the first condition is trivially satisfied). Let $(\mu_{\eta_k})_k$ be a sequence of measures associated to the sequence of isotropic processes which converges weakly to the Wiener measure W and let $P^{\eta_k}_t(p, A) := \mathrm{Prob}\{Y^{\eta_k}_t \in A; Y^{\eta_k}_0 = p\}$ denote the associated transition functions.

Recall that, for each $\eta > 0$, all trajectories of the random walk Y^η are concatenations of geodesic segments each of which has length less than or equal to η. Consequently, we have, for each $\eta_k > 0$, $d(p, Y^{\eta_k}_t) \leq \eta_k t$ if $Y^{\eta_k}_0 = p$.

Let $N > 0$ be some (fixed) real number. Then, for all $t \leq N$, if $Y^{\eta_k}_0 = y$, $d(y, Y^{\eta_k}_t) \leq \eta_k N$. Thus, if, for some $\epsilon > 0$, we consider the points $y \in K_D$ for which the distance $d(y, \Gamma)$ is strictly greater than $(\eta_k + \epsilon)N$, then the probability $P^{\eta_k}_t(y, \Gamma)$ vanishes.

In a nutshell, we proved that, for all $\eta_k > 0$, $N > 0$ and $y \in K_D$, we have:

For all $\epsilon > 0$, there exists $\alpha = (\eta_k + \epsilon)N$ such that if $d(y, \Gamma) \geq \alpha$ then $\sup_{t \leq N} P_t^{\eta_k}(y, \Gamma) < \epsilon$.

So if we take the limit of an appropriate subsequence, then the desired conclusion follows.

To prove the second condition, we recall that during the proof of Lemma 3.3 we obtained the inequality

$$\sup_{p \in K_D} \frac{\text{Prob}\{Y_t^{\eta_k} \in B_D^c(p, \epsilon)\}}{t/\eta_k^2} \leq \frac{e^{-t/\eta_k^2}}{t/\eta_k^2}$$

for all $t > 0$ and all $\eta_k > 0$. The second condition follows by letting η_k and t both go to zero.

This completes the proof. $\square$

5. Recurrent and transient behavior of the Brownian motion

In the literature on Brownian motion in the smooth case one usually investigates the *recurrent* or *transient* behavior of this stochastic process. It is known, for example, that the Euclidian Brownian motion is recurrent in the two-dimensional case, and transient in dimensions greater than or equal to three. Moreover, it is known that the noncompact hyperbolic surface valued Brownian motion is transient.

For more results and details, we refer the reader to the papers by H.P Mckean and D. Sullivan [22] and T.J. Lyons and H.P Mckean [20].

5.1 The geometric behavior of the admissible Riemannian complex valued Brownian motion. Let K denote a complete admissible Riemannian complex of dimension n and let $p \in K$. We recall that the K-valued Brownian motion $\{B_t^p\}_{t \geq 0}$, was obtained as a weak limit of the sequence of isotropic transport processes.

On the other hand, we have seen that the trajectories of the isotropic processes are concatenations of geodesic segments. When a trajectory joins (a.e.) transversally the $(n-1)$-skeleton$\backslash(n-2)$-skeleton, it then chooses isotropically a new maximal face (i.e., all adjacent maximal faces have the same probability to be chosen).

Consequently, the K-valued Brownian motion $\{B_t^p\}_{t \geq 0}$ behaves, inside an n-simplex Δ_n, like the standard Brownian motion with values in a Riemannian n-dimensional manifold endowed with the metric g_{Δ_n}.

Moreover, the process hits (a.e.) "transversally" the $(n-1)$-skeleton$\backslash(n-2)$-skeleton, then chooses isotropically a maximal face. Thus we obtain a new discrete random walk corresponding to the isotropic choices of the maximal faces. To give a rigorous mathematical construction of this discrete process, we introduce the following concept.

The *dual graph* X of a complex K is a 1-dimensional simplicial complex defined as follows:

Consider a point inside (i.e., in the topological interior of) each n-simplex of K and, for each $(n-1)$-simplex, a point in its topological interior. Then connect these points with geodesic segments and let $E(X)$ denote the set of such segments. In this way we obtain a graph consisting of a set $V_n(X)$ of vertices of degree $n+1$ (the points inside n-simplexes) and a set $V_{n-1}(X)$ of vertices corresponding to the interior points of the $(n-1)$-simplexes, where the degree of a vertex is equal to the number of the n-simplexes adjacent to it.

Next, consider the Markov chain (discrete Markov process) $\{C_n\}_{n\in\mathbb{N}}$ with transition probabilities given by the function

$$p(x,y) = \begin{cases} \frac{1}{\deg x} & \text{if } x,y \in V_{n-1}(X) \text{ and there exists } z \in V_n(X) \\ & \text{such that } xz,\, zy \in E(X), \\ 0 & \text{otherwise,} \end{cases}$$

where deg x is the degree of x and xz is an edge (geodesic segment) connecting x to z.

Thus this random walk is a discrete "jump" process on the set $V_{n-1}(X)$.

5.2 Brownian motion in an admissible complex with nonpositive curvature and with dimension at the most 2. This subsection is devoted to the study of the transient or recurrent behavior of the Brownian motion in an admissible complex with nonpositive curvature (in the sense of Alexandrov) and with dimension at the most 2.

We first recall the definition of a recurrent or transient process:

DEFINITION 5.2. Let $\{X_t^p\}_t$ denote a stochastic process in a metric space K. Then $\{X_t^p\}_t$ is said to be recurrent if, for every ball B_p containing the point p, the process $\{X_t^p\}_t$ returns to the ball B_p (and does so infinitely often) with probability equal to one. Otherwise, the process is called transient.

REMARK 5.3. When the space K is a discrete space, we consider the point p instead of the ball B_p in the above definition.

THEOREM 5.4. *Let K denote a 2-dimensional (respectively, 1-dimensional) non-compact complete simply connected admissible Riemannian complex with nonpositive curvature. Then, if for every 1-simplex (respectively, vertex) there are at least three 2-simplices (respectively, 1-simplices) adjacent to it, the Brownian motion is transient.*

Before proceeding with the proof of Theorem 5.4, let us first give a short description of a *simple random walk* on a graph.

Let $X = (V(X), E(X))$ denote a connected locally finite graph (i.e., a 1-dimensional admissible Riemannian complex), where $V(X)$ is the set of vertices and $E(X)$ is the set of edges. By a *simple random walk* on the graph

X we mean the Markov chain for which the transition probability $p(x, y)$ from vertex x to vertex y is given by the function

$$p(x, y) = \begin{cases} \frac{1}{\deg x}, & \text{if } xy \in E(X), \\ 0, & \text{otherwise,} \end{cases}$$

where xy is an edge connecting x to y.

We say that X is recurrent (respectively, transient) if the simple random walk is recurrent (respectively, transient).

The *word metric* on the graph X is an intrinsic metric in which each edge has unit length.

Remark 5.5 ([13, Ch. 6]). Let X denote a connected locally finite graph with uncountably many ends. If every vertex has degree greater than or equal to three, then X is transient.

Proof of Theorem 5.4. Let K be an admissible complex and let X denote the dual graph of K. In the following, we will construct a new graph Y from the graph X.

Let x_1 be a vertex belonging to the set $V_1(X)$ and let $z_1 \in V_2(X)$ be such that $x_1 z_1 \in E(X)$. Recall that the degree of z_1 is equal to three. We delete an edge adjacent to z_1, different from $x_1 z_1$. We do the same thing with the other faces adjacent to x_1.

Now consider again z_1. This vertex is connected to another vertex $x_2 \in V_1(X)$ (x_1, z_1 and x_2 are all in the same 2-simplex). We do the same thing with x_2 that we have done with x_1. At the end of this construction, ignoring the vertices of degree equal to two, and using the hypothesis on the complex K, we get a graph Y that is isometrically equivalent to a connected locally finite graph with uncountably many ends and in which each vertex has degree greater than or equal to three. Moreover, the random walk resulting from the isotropic choice of maximal faces by the Brownian motion induces a simple random walk on the graph Y.

Now suppose that the K-valued Brownian motion $\{B_t^p\}_{t \geq 0}$ is recurrent. We can assume that the point p is in the interior of an edge. Take as compact neighborhood of the point p the union of all its adjacent 2-simplices and denote this neighborhood by B_p.

Thus, if $\{B_t^p\}_{t \geq 0}$ returns to the ball B_p with probability equal to one, then the associated simple random walk on Y returns to the point p with probability one. In other words, the graph Y is recurrent. This contradicts Remark 5.5, and so the theorem is now proven. $\square$

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