Illinois Journal of Mathematics
Volume 49, Number 2, Summer 2005, Pages 531–545
S 0019-2082

ON COMPACTNESS OF MEASURES ON POLISH SPACES

PIOTR BORODULIN–NADZIEJA AND GRZEGORZ PLEBANEK

Abstract. We present some results related to the question whether every finite measure μ defined on a σ–algebra $\Sigma \subseteq \mathrm{Borel}[0,1]$ is countably compact. In particular, we show that for every finite measure space (X, Σ, μ), where X is a Polish space and $\Sigma \subseteq \mathrm{Borel}(X)$, there is a regularly monocompact measure space $(\widehat{X}, \widehat{\Sigma}, \widehat{\mu})$ and an inverse–measure–preserving function $f : \widehat{X} \to X$.

1. Introduction

If (X, Σ, μ) is a finite measure space, then we say that the measure μ is countably compact if μ is inner regular with respect to some countably compact family $\mathcal{K} \subseteq \Sigma$ (see Section 2 for precise definitions). The class of countably compact measures was introduced by Marczewski [12] under the name *compact measures*. In the abstract setting (i.e., without referring to topology), such a notion singles out measures which are nice in the sense that they resemble the Lebesgue measure. Every countably compact measure is perfect; in fact, a measure μ on a σ–algebra Σ is perfect if and only if μ is countably compact on every σ–generated $\Sigma_0 \subseteq \Sigma$; see Ryll–Nardzewski [16]. Musiał [13] gave an example of a perfect measure which is not countably compact. Under some mild set–theoretic assumptions there are even perfect measures which are not countably compact and which are of countable Maharam type (i.e., the underlying L_1 space is separable); see Plebanek [15]. Recently David H. Fremlin investigated several other subclasses of perfect measures; his paper [5] presents several subtle results on properties of measures related to infinite games.

In [7], [9] Fremlin posed the following natural question.

Problem FN. *Let μ be a measure defined on a σ–algebra $\Sigma \subseteq \mathrm{Borel}[0,1]$. Is μ countably compact?*

Received December 2, 2004; received in final form May 10, 2005.
2000 *Mathematics Subject Classification.* Primary 28A99, 28C15.
The research partially supported by KBN grant 1 P03A 02827 (2004–07).

It is well-known that, for a Polish space X, every finite measure defined on $\mathrm{Borel}(X)$ is inner regular with respect to compact sets, and hence is countably compact and of countable Maharam type. Measures defined on some $\Sigma \subseteq \mathrm{Borel}(X)$, however, can be more complicated. For instance, Marczewski [11] showed that there is a measure μ defined on some $\Sigma \subseteq \mathrm{Borel}[0,1]$, which contains $\mathfrak{c}$ many stochastically independent sets of measure $1/2$. Such a measure is of Maharam type $\mathfrak{c}$ and cannot be extended to $\mathrm{Borel}[0,1]$ (but is still countably compact).

If μ is a measure on $\Sigma \subseteq \mathrm{Borel}[0,1]$, then it is perfect, and hence countably compact whenever Σ is countably generated. In [7] Fremlin, based on his previous papers [3], [5], proved the following nontrivial generalization of this remark.

THEOREM 1.1 (Fremlin). *If a σ–algebra $\Sigma \subseteq \mathrm{Borel}(X)$, where X is a Polish space, is generated by ω_1 sets, then every finite measure on Σ is countably compact.*

It follows that under CH Problem FN has a positive solution; it is not known if FN can be resolved in ZFC. Let us remark that under CH there is a σ–algebra Σ built from Borel subsets of $[0,1]^2$ and a single non-Borel set $\Delta \subseteq [0,1]^2$, which carries a perfect measure which is not countably compact; see Plebanek [15].

In this paper we present some results related to Problem FN. In Section 3 we give two technical results that are helpful in constructing countably compact families. The following sections discuss properties of finite measures μ defined on $\Sigma \subseteq \mathrm{Borel}(X)$, where X is Polish. In Section 4 we prove that such a measure μ is countably compact under the additional assumption that μ is inner regular with respect to closed sets from Σ. In Section 6 we show that μ is “an image” of some regularly monocompact measure; this result is based on a theorem from Section 5 on measures defined on uncountable products of Polish spaces. Regular monocompactness is a slightly weaker property than countable compactness (and it is not clear if it is preserved under inverse–measure–preserving functions). Finally we mention the infinite game introduced by Fremlin [5], which is related to regularity properties of measures; we give an alternative proof of one of his results.

We would like to thank Mirna Džamonja and Boban Veličković for conversations about games in Boolean algebras and related topics. In particular, our Proposition 3.2 below is related to some ideas presented in [17]. We also thank D.H. Fremlin for several useful comments, and the referee for a very careful reading of an earlier version of this paper.

2. Preliminaries

We consider only finite measures; concerning regularity properties of measures we follow the terminology of Fremlin [5]. (Note that some properties have different names in other sources!) If $\mathcal{K}$ is a family of sets, then we say that $\mathcal{K}$ is

> *countably compact* if every sequence $\langle A_n \rangle_{n\in\omega}$ from $\mathcal{K}$ with the finite intersection property satisfies $\bigcap_{n\in\omega} A_n \neq \emptyset$;
> *monocompact* if $\bigcap_{n\in\omega} A_n \neq \emptyset$ whenever $\langle A_n \rangle_{n\in\omega}$ is a decreasing sequence of nonempty elements from $\mathcal{K}$.

If (X, Σ, μ) is a measure space and $\mathcal{K} \subseteq \Sigma$, then μ is said to be *inner regular with respect to* $\mathcal{K}$ if

$$\mu(A) = \sup\{\mu(K) : \ K \subseteq A, K \in \mathcal{K}\},$$

for every $A \in \Sigma$. A measure μ is *countably compact* (respectively *regularly monocompact*) if it is inner regular with respect to some family $\mathcal{K} \subseteq \Sigma$ which is countably compact (respectively monocompact).

It is a nontrivial result due to Pachl [14] that a countably compact measure μ defined on some Σ remains countably compact when restricted to any sub–σ–algebra $\Sigma_0 \subseteq \Sigma$; see also Fremlin [4]. It is worth recalling that both proofs of Pachl's result use some external characterizations of countable compactness—it is not clear how to explicitly define a suitable countably compact family inside Σ_0. It is not known if regular monocompactness is also preserved under taking restrictions. As was remarked by the referee, the completion of a regularly monocompact measure has the same property; it is unclear, however, if monocompactness of the completion implies monocompactness of the original measure.

If (X, Σ, μ) and $(Y, \mathcal{A}, \nu)$ are measure spaces and $f : X \to Y$ is a measurable function, then we say that f is *inverse–measure–preserving* if $\nu(A) = \mu(f^{-1}[A])$ for $A \in \mathcal{A}$. It can be derived from Pachl's results (e.g., see the lemma below) that if there is a such function and μ is countably compact, then so is ν.

Consider now a measure space (Y, Σ, μ) and a function $f : X \to Y$ with $f[X] = Y$. The algebra Σ induces a σ–algebra $\Sigma' = \{f^{-1}(E) : \ E \in \Sigma\}$ on X, and we can also define on Σ' a measure μ' by $\mu'(f^{-1}[E]) = \mu(E)$, which will be called the preimage of μ. It will be useful to note the following simple fact.

Lemma 2.1. *Let μ' be the preimage of μ (as described above).*

(a) *If μ is inner regular with respect to some $\mathcal{K}$, then μ' is inner regular with respect to $\mathcal{C} = \{f^{-1}[K] : \ K \in \mathcal{K}\}$.*
(b) *If μ' is inner regular with respect to some $\mathcal{C} \subseteq \Sigma'$, then μ is inner regular with respect to $\mathcal{K} = \{E \in \Sigma \colon f^{-1}(E) \in \mathcal{C}\}$.*

(c) *The measure μ' is countably compact (respectively regularly monocompact) if and only if μ is countably compact (respectively regularly monocompact).*

Proof. (a) For a given set $f^{-1}(E) \in \Sigma'$ and $\varepsilon > 0$ we can find $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu(E \setminus K) < \varepsilon$. Since $f^{-1}(E) \setminus f^{-1}(K) \subseteq f^{-1}(E \setminus K)$, we have

$$\varepsilon > \mu(E \setminus K) = \mu'(f^{-1}[E \setminus K]) \geq \mu'(f^{-1}[E] \setminus f^{-1}[K]).$$

(b) Let $E \in \Sigma$ and $\varepsilon > 0$. We can find a set $C \in \mathcal{C}$ such that $C \subseteq f^{-1}(E)$ and $\mu'(C) > \mu'(E) - \varepsilon$. Then the set $K \in \mathcal{K}$ such that $C = f^{-1}(K)$ is a subset of E and we have $\mu(K) = \mu'(C) > \mu'(f^{-1}(E)) - \varepsilon = \mu(E) - \varepsilon$. This shows that μ is inner regular with respect to $\mathcal{K}$.

(c) It is easy to check that $\mathcal{K}$ is countably compact or monocompact if and only if $\mathcal{C}$ has an analogous property. □

If (X, Σ, μ) is a measure space, we denote by μ^* the corresponding outer measure. We repeatedly use the fact that μ^* is upward continuous, i.e., $\mu^*(\bigcup_n Z_n) = \lim \mu^*(Z_n)$ for an arbitrary sequence $Z_1 \subseteq Z_2 \subseteq \ldots \subseteq X$. It will be convenient to single out the following simple observation.

Lemma 2.2. *Let (X, Σ, μ) be a measure space and let $\langle Z_n \rangle_n$ be an increasing sequence of arbitrary subsets of X with union Z. For every $E \in \Sigma$ and $\varepsilon > 0$ there is a set $F \in \Sigma$ with $\mu(E \setminus F) < \varepsilon$, and a number $m \in \omega$ such that if $A \in \Sigma$ and $A \subseteq F$, then $\mu^*(A \cap Z_m) = \mu^*(A \cap Z)$.*

Proof. Let $E \in \Sigma$ and $\varepsilon > 0$. Since the outer measure is upward continuous we can find a number m such that $\mu^*(Z_m) > \mu^*(Z) - \varepsilon$. Let $F_1 \subseteq E$ be a measurable hull of the set $E \cap Z_m$ and F_2 a measurable kernel of $E \cap Z^c$. Then for $F = F_1 \cup F_2$ we have $\mu(E \setminus F) < \varepsilon$ and $\mu^*(A \cap Z_m) = \mu(A \cap F_1) = \mu^*(A \cap Z)$ for every measurable $A \subseteq F$. □

Given any measure space (X, Σ, μ), we say that a sequence $\langle E_n \rangle_{n \in \omega}$ of measurable sets is *μ–centred* if $\mu(\bigcap_{k<n} E_k) > 0$ for every n.

3. Countably compact measures

We present in this section two auxiliary results on countably compact measures. The first lemma is used directly in the proof of Theorem 4.1 below, while the second lemma is related to game–theoretic properties of measures that are mentioned in Section 7.

Lemma 3.1. *Let (X, Σ, μ) be a measure space and suppose that $\mathcal{C} \subseteq \Sigma$ is a family such that the intersection of every μ–centred sequence $\langle F_n \rangle_{n \in \omega}$ from $\mathcal{C}$ is not empty.*

If μ is inner regular with respect to $\mathcal{C}$, then μ is countably compact.

Proof. Let $\widehat{\Sigma}$ be the completion of Σ with respect to μ, and denote by $\mathfrak{A}$ the measure algebra of μ. For $A \in \Sigma$ we write $A^\bullet$ for the corresponding element of $\mathfrak{A}$. Let $\rho\colon \mathfrak{A} \to \widehat{\Sigma}$ be a lifting; i.e., ρ is a Boolean homomorphism such that $\rho(a)^\bullet = a$ for every $a \in \mathfrak{A}$ (see Fremlin's survey [2] for details).

We shall consider the family $\mathcal{C}'$ defined by

$$\mathcal{C}' = \left\{ \bigcap_{k\in\omega} F^k : \ F^k \in \mathcal{C},\ F^{k+1} \subseteq F^k \cap \rho(F^{k\bullet}) \text{ for every } k \right\}.$$

Let us check that μ is inner regular with respect to $\mathcal{C}'$. Take any set $F \in \mathcal{C}$ and $\varepsilon > 0$. We define a sequence of sets F^k from $\mathcal{C}$ in the following way. Put $F^1 = F$; if F^k is given, choose $F^{k+1} \in \mathcal{C}$ so that

$$F^{k+1} \subseteq F^k \cap \rho(F^{k\bullet}) \ \text{ and } \ \mu((F^k \cap \rho(F^{k\bullet})) \setminus F^{k+1}) < \frac{\varepsilon}{2^k}.$$

Then the set $H = \bigcap_{k\in\omega} F^k$ is in $\mathcal{C}'$ and we have $\mu(F \setminus H) \le \varepsilon$. As μ is inner regular with respect to $\mathcal{C}$, it is also inner regular with respect to $\mathcal{C}'$.

Now it remains to check that $\mathcal{C}'$ is countably compact. Consider any centred sequence $\langle C_n\rangle_{n\in\omega}$ of sets from $\mathcal{C}'$. Every set C_n can be written as $C_n = \bigcap_{k\in\omega}^{\infty} F_n^k$, where the sets $F_n^k \in \mathcal{C}$ are as in the definition of $\mathcal{C}'$. Then

$$\bigcap_{n\in\omega} C_n = \bigcap_{n\ge 1} \bigcap_{k,m<n} F_m^k.$$

Observe that for every n

$$\bigcap_{k,m<n} \rho(F_m^{k\bullet}) \supseteq \bigcap_{k,m<n} F_m^k \cap \rho(F_m^{k\bullet}) \supseteq \bigcap_{k,m<n} F_m^{k+1} \supseteq \bigcap_{m<n} C_m \ne \emptyset.$$

Hence

$$\rho\left(\left(\bigcap_{k,m<n} F_m^k \right)^\bullet \right) = \bigcap_{k,m<n} \rho(F_m^{k\bullet}) \ne \emptyset,$$

which means that $\mu(\bigcap_{k,m<n} F_m^k) > 0$. As the family of all F_m^k is μ–centred, by our assumption on $\mathcal{C}$ we get $\bigcap_{n\in\omega} C_n \ne \emptyset$. This completes the proof. $\square$

Corollary 3.2. *Let (X, Σ, μ) be any measure space and let $\Sigma^+ = \{A \in \Sigma : \mu(A) > 0\}$. Suppose that there is a function $\tau : \Sigma^+ \to \Sigma^+$ such that*

(i) *$\tau(A) \subseteq A$ for every $A \in \Sigma^+$;*
(ii) *if $A_n \in \Sigma^+$ and the sequence $\langle \tau(A_n)\rangle_{n\in\omega}$ is μ–centred, then $\bigcap_{n\in\omega} A_n \ne \emptyset$.*

Then the measure μ is countably compact.

Proof. For any $E \in \Sigma^+$ we let $\mathcal{T}(E)$ be the family of all finite unions of sets from $\{\tau(A) : A \in \Sigma^+, A \subseteq E\}$. Moreover we put

$$\mathcal{C} = \left\{ \bigcap_{k\in\omega} B^k : \; B^{k+1} \in \mathcal{T}(B^k) \text{ for every } k \right\}.$$

Claim 1. *μ is inner regular with respect to $\mathcal{C}$.*

Note first that $\mu(E) = \sup\{\mu(B) : B \in \mathcal{T}(E)\}$ for every $E \in \Sigma^+$. Indeed, by (i) E is a countable union, modulo a null set, of sets of the form $\tau(A)$, so $\mu(E)$ is approximated by $\mu(B)$ for $B \in \mathcal{T}(E)$. This implies in a standard way that μ is inner regular with respect to $\mathcal{C}$.

Claim 2. *If $B_n \in \mathcal{T}(E_n)$ and the sequence $\langle B_n \rangle_{n\in\omega}$ is μ–centred, then $\bigcap_{n\in\omega} E_n \neq \emptyset$.*

This is so since if we write $B_n = \tau(A_{n,1}) \cup \tau(A_{n,2}) \cup \cdots \cup \tau(A_{n,k_n})$ for every n then there is a function φ satisfying $\varphi(n) \le k_n$ such that the sequence of sets $\tau(A_{n,\varphi(n)})$ is μ–centred, and the claim follows from (ii).

Now take a μ–centred sequence $\langle B_n \rangle_{n\in\omega}$ from $\mathcal{C}$. Write $B_n = \bigcap_{k\in\omega} B_n^k$ as in the definition of $\mathcal{C}$. Then all sets B_n^k, where $n \in \omega$, $k \ge 1$, are μ–centred, and by Claim 2 $\bigcap_{n\in\omega} B_n \neq \emptyset$. By Claim 1 and Lemma 3.1 μ is a countably compact measure. □

4. Closed-regular measures

We denote by $\mathcal{N}$ the Baire space ω^ω. Recall that for every Polish space X and every $B \in \mathrm{Borel}(X)$, B is analytic, i.e., is a continuous image of $\mathcal{N}$ (or is empty); see, e.g., Kechris [10].

Theorem 4.1. *If Σ is any σ–algebra of subsets of $\mathcal{N}$ and a measure μ defined on Σ is inner regular with respect to closed subsets from Σ, then μ is countably compact.*

Proof. For any $n \in \omega$ and $\psi \in \omega^n$ define

$$V(\psi) = \{x \in \mathcal{N} : \; x(k) \le \psi(k) \text{ for all } k < n\}.$$

Consider the family $\mathcal{C}$ of those closed sets F belonging to Σ for which there is a function $\phi : \omega \to \omega$ such that for every n

$$\mu^*(V(\phi_{\restriction n}) \cap F) = \mu(F).$$

We shall prove that $\mathcal{C}$ μ–approximates Σ and that every μ–centred sequence from $\mathcal{C}$ has a nonempty intersection; in view of Lemma 3.1 this will imply that μ is countably compact.

Take any $E \in \Sigma$ and $\varepsilon > 0$. We construct inductively a function $\phi \in \mathcal{N}$ such that for every n

$$\mu^*(V(\phi_{\restriction n}) \cap E) > \mu(E) - \frac{\varepsilon}{2}.$$

If ϕ is defined on n, then from the fact that the outer measure is upward continuous and that the sequence $V(\phi^\frown m) \cap E$ converges to $V(\phi) \cap E$ as m goes to infinity we deduce that there exists an integer m such that

$$\mu^*(V(\phi^\frown m) \cap E) > \mu(E) - \frac{\varepsilon}{2},$$

and so we can set $\phi(n) = m$.

For every n we can choose a measurable hull $M_n \in \Sigma$ of $V(\phi_{\restriction n}) \cap E$, so that $E \supseteq M_1 \supseteq \ldots$ It follows that for $M = \bigcap_{n\in\omega} M_n$ we have $\mu(E \setminus M) \leq \varepsilon/2$. Now take any closed set $F \in \Sigma$ such that $F \subseteq M$ and $\mu(M \setminus F) < \varepsilon/2$. Then $\mu(E \setminus F) < \varepsilon$; for any n we have $F \subseteq M_n$, so $\mu(F) = \mu^*(F \cap V(\phi_{\restriction n}))$, which means that F is in our class $\mathcal{C}$.

Now consider any μ–centered sequence $(F_n)_{n\in\omega}$ from $\mathcal{C}$. Denote by ϕ a function $\omega \to \omega$ witnessing that $F_0 \in \mathcal{C}$. For every n, $\mu(\bigcap_{k\leq n} F_k) > 0$, so

$$\mu^*\left(\bigcap_{k\leq n} F_k \cap V(\phi_{\restriction n})\right) > 0.$$

Thus we can choose $x_n \in \bigcap_{k\leq n} F_k$ such that $x_n(k) \leq \phi(k)$ for every $k < n$. It follows that the sequence x_n contains a subsequence converging to some $x \in \mathcal{N}$. Every F_k is closed and contains almost all x_n's, so $x \in F_k$ and therefore $\bigcap_{k\in\omega} F_k \neq \emptyset$. □

Corollary 4.2. *If Σ is any σ–algebra of subsets of a Polish space X and the measure μ defined on Σ is inner regular with respect to closed subsets from Σ, then μ is countably compact.*

Proof. Take a continuous surjection $g : \mathcal{N} \to X$, and consider the σ–algebra $\Sigma' = \{g^{-1}(E) : E \in \Sigma\}$. It follows from Lemma 2.1 that the measure μ' on Σ' given by $\mu'(g^{-1}[E]) = \mu(E)$ is inner regular with respect to closed sets from Σ'. By the above theorem μ' is countably compact, and hence μ is countably compact by 2.1. □

D.H. Fremlin remarked that the above result in fact follows from the extension theorem due to Aldaz and Render [1]; see also Fremlin [8, 432D]. Namely, if μ is a measure as in Corollary 4.2, then μ admits an extension to a Borel measure $\widehat{\mu}$ (which is countably compact), so in particular μ is countably compact as the restriction of $\widehat{\mu}$. Our proof of 4.2 has the advantage that it also gives a description of a countably compact family which approximates the measure in question. We shall see in the following sections that using the same idea one can obtain a common generalization of Corollary 4.2 and Theorem 1.1.

5. Measures on $\mathcal{N}^\kappa$

Let κ be any cardinal number. In the product space $\mathcal{N}^\kappa$ the family of all closed sets depending on countably many coordinates will be denoted by $\mathrm{Zero}(\mathcal{N}^\kappa)$; such sets are often called zero sets. Recall that a set $A \subseteq \mathcal{N}^\kappa$ depends on coordinates in $I \subseteq \kappa$ if, for every $x \in A$ and $y \in \mathcal{N}^\kappa$, $x(\alpha) = y(\alpha)$ for all $\alpha \in I$ implies $y \in A$. We shall write $A \sim I$ to indicate that A depends on coordinates in I. Recall that the σ–algebra $\mathrm{Baire}(\mathcal{N}^\kappa)$ generated by $\mathrm{Zero}(\mathcal{N}^\kappa)$, which is called the σ–algebra of Baire sets, is equal to the product of Borel σ–algebras on $\mathcal{N}$. Similar remarks apply to uncountable products of arbitrary Polish spaces; see Wheeler [18] for general background on measures on topological spaces, and Fremlin [6] for applications of sets depending on few coordinates to measure theory.

If μ is a measure on $\mathrm{Baire}(\mathcal{N}^\kappa)$, then, using the fact that every measure on a Polish space is inner regular with respect to compact sets, one can check that μ is countably compact. The following theorem gives a partial generalization of this result.

Theorem 5.1. *Let κ be any cardinal number and Σ any σ–algebra of subsets of $\mathcal{N}^\kappa$. If a measure μ defined on Σ is inner regular with respect to zero subsets from Σ, then μ is regularly monocompact.*

Proof. We shall identify the space $\mathcal{N}^\kappa$ with ω^κ and consider below partial functions from κ into ω. By saying that ϕ is a partial function on κ we mean that the domain of ϕ is a finite subset of κ and the values of ϕ are natural numbers. For every partial function ϕ on κ define

$$V(\phi) = \{x \in \omega^\kappa : \lambda \in \mathrm{Dom}(\phi) \Longrightarrow x(\lambda) \le \phi(\lambda)\}.$$

Moreover, for any $\alpha < \kappa$ and $m \in \omega$ put

$$C_\alpha(m) = V(\langle \alpha, m\rangle) = \{x \in \omega^\kappa : \ x(\alpha) \le m\}.$$

For an arbitrary set $Y \subseteq \omega^\kappa$ and any $E \in \Sigma$, we introduce the following definitions.

(a) We call a partial function ϕ Y*–thick* if $\mu^*(Y \cap V(\phi)) = \mu^*(Y)$.
(b) We call a countable set $I \subseteq \kappa$ *good for* E if for every partial function ϕ on I and $\alpha \in I$ there is an extension of ϕ to an $E \cap V(\phi)$–thick partial function on $\mathrm{dom}(\phi) \cup \{\alpha\}$.

We shall consider the family $\mathcal{K}$ of sets F with the following properties:

(i) $F \in \mathrm{Zero}(\omega^\kappa) \cap \Sigma$;
(ii) $\mu(F) > 0$;
(iii) there is a countable set $I \subseteq \kappa$ such that $F \sim I$ and I is good for F.

We first show that μ is inner regular with respect to $\mathcal{K}$ using the following claim:

Claim 1. *Let $E \in \Sigma$ depend on coordinates in a countable set $I \subseteq \kappa$. For every $\varepsilon > 0$ there is a set $F \in \Sigma \cap \mathrm{Zero}(\omega^\kappa)$ with $F \subseteq E$, $\mu(E \setminus F) < \varepsilon$ such that for every function ϕ defined on a finite set $J \subseteq I$ and every $\alpha \in I$,*

$$(*) \qquad \text{there is an } m \text{ such that} \quad \mu^*(F \cap V(\phi) \cap C_\alpha(m)) = \mu^*(F \cap V(\phi)).$$

To prove this claim note that, for a fixed partial function ϕ on I and any $\alpha \in I$,

$$V(\phi) \cap C_\alpha(m) \nearrow V(\phi) \quad \text{as } m \to \infty,$$

so by Lemma 2.2 there is $F \subseteq E$ with $\mu(E \setminus F) < \varepsilon$ (which can be taken to be a zero set), such that $(*)$ is satisfied. We have countably many pairs (ϕ, α) to consider, so repeating this argument we see that there is F such that $(*)$ holds for every partial function on I and every $\alpha \in I$. This proves the claim.

Let $A \in \Sigma$ and $\varepsilon > 0$ be given. We first find a measurable zero set F_0 and a countable set $I_0 \subseteq \kappa$ such that $F_0 \sim I_0$, $F_0 \subseteq A$, and $\mu(A \setminus F_0) < \varepsilon/2$. We next apply the Claim to $E = F_0$, $I = I_0$ (and $\varepsilon/4$ in place of ε) to obtain a measurable zero set $F_1 \subseteq F_0$ and a countable set $I_1 \supseteq I_0$ such that $F_1 \sim I_1$, $\mu(F_0 \setminus F_1) < \varepsilon/4$, and $(*)$ holds for $F = F_1$ and any partial function ϕ on I_0 and $\alpha \in I_0$.

Continuing in the same manner, we get a decreasing sequence of zero sets $F_n \in \Sigma$ and an increasing sequence I_n of countable sets such that $\mu(F_{n-1} \setminus F_n) < \varepsilon/2^{n+1}$, $F_n \sim I_n$, and $(*)$ holds whenever ϕ is a partial function on I_{n-1} and $\alpha \in I_{n-1}$.

Finally, we put $F = \bigcap_{n\in\omega} F_n$ and $I = \bigcup_{n\in\omega} I_n$. Then $\mu(A \setminus F) \leq \varepsilon$ and $F \sim I$. Moreover, I is good for F: If $J \subseteq I$ is finite, $\phi : J \to \omega$, $\alpha \in I$, then $J \cup \{\alpha\} \subseteq I_n$ for some n, so there is an m such that

$$\mu^*(F_{n+1} \cap V(\phi) \cap C_\alpha(m)) = \mu^*(F_{n+1} \cap V(\phi)),$$

and hence

$$\mu^*(F \cap V(\phi) \cap C_\alpha(m)) = \mu^*(F \cap V(\phi)).$$

In particular, we can extend any partial function ϕ to an $F \cap V(\phi)$–thick function by letting $\phi(\alpha) = m$. This shows that μ is regular with respect to $\mathcal{K}$.

Now it remains to verify that $\mathcal{K}$ is a monocompact class. Let $(F_n)_{n\in\omega}$ be a decreasing sequence of sets from $\mathcal{K}$. Then for every n there is a countable set $I_n \subseteq \kappa$ such that $F_n \sim I_n$ and I_n is good for F_n. Enumerate the elements of $I = \bigcup_{n\in\omega} I_n$ as $I = \{\alpha_k : k \in \omega\}$ and write $T_k = \{\alpha_j : j < k\}$ for every k.

Claim 2. *There is a function $\tau : I \to \omega$ such that for every n and every k its restriction $\tau|(T_k \cap I_n)$ is F_n–thick.*

We define values of τ by induction. Suppose that τ is defined on T_k so that $\tau|(T_k \cap I_n)$ is F_n–thick for every n. There is a natural number p such that for every $n > p$ there is $j \le p$ such that $T_{k+1} \cap I_n \subseteq T_{k+1} \cap I_j$.

For a given $j \le p$ such that $\alpha_k \in I_j$ there is an m_j such that the F_j–thick function $\tau|(T_k \cap I_j)$ can be extended to an F_j–thick function assuming the value m_j at α_k. We let $\tau(\alpha_k)$ be the maximum of these numbers m_j (where $j \le p$).

In this way we have extended τ to T_{k+1} so that $\tau|(T_{k+1} \cap I_j)$ is F_j–thick for every $j \le p$. For any $n > p$ we have $T_{k+1} \cap I_n \subseteq T_{k+1} \cap I_j$, where $j \le p$. It follows that $\tau|(T_{k+1} \cap I_n)$ is F_j–thick (as the restriction of a thick function is thick). Therefore $\tau|(T_{k+1} \cap I_n)$ is also F_n–thick (since $F_n \subseteq F_j$). This verifies the claim.

Using Claim 2 we can check that $\bigcap_{n\in\omega} F_n \neq \emptyset$. For every n the function $\tau|(T_n \cap I_n)$ is F_n–thick. Since $\mu(F_n) > 0$ there is $x_n \in F_n$ such that $x_n(\alpha) \le \tau(\alpha)$ for $\alpha \in T_n \cap I_n$. We can moreover assume that

$$x_n(\alpha) = 0 \text{ for } \alpha \in (T_n \setminus I_n) \cup (\kappa \setminus I),$$

since F_n is determined by $I_n \subseteq I$. Now the sequence of x_n (which is dominated by τ) has a subsequence converging to some $x \in \omega^\kappa$. We have $x_n \in F_k$ for all $n \ge k$, so $x \in F_k$ (as F_k is closed). Finally, $x \in \bigcap_{n\in\omega} F_n$, and the proof is complete. □

Let us remark that if we could refine this argument to prove that the measure in question is countably compact, then we would get the following result: *If a countable set I_j is good for F_j, $j = 1, 2$, then $I_1 \cup I_2$ is good for $F_1 \cap F_2$.* This can, in fact, be done in the case $\kappa = \omega_1$.

THEOREM 5.2. *If Σ is any σ–algebra of subsets of $\mathcal{N}^{\omega_1}$, then every measure μ defined on Σ which is inner regular with respect to zero subsets from Σ is countably compact.*

Proof. We modify the argument from the previous proof as follows. Consider the family $\mathcal{K}$ of sets F with the following properties:

(i) $F \in \mathrm{Zero}(\omega^\kappa) \cap \Sigma$;
(ii) $\mu(F) > 0$;
(iii) there is an *initial segment* I of ω_1 such that $F \sim I$ and I is good for F.

Since every initial segment of ω_1 is countable, we can in a similar way verify that μ is again inner regular with respect to $\mathcal{K}$. The main difference is contained in the following claim.

CLAIM. *If $F, H \in \mathcal{K}$ and $\mu(F \cap H) > 0$, then $F \cap H \in \mathcal{K}$.*

Indeed, let I and J be good for F and H, respectively. We can assume that $I \subseteq J$, but in this case J is good for $F \cap H$, so $F \cap H \in \mathcal{K}$.

Now for any μ–centred sequence $(F_n)_{n\in\omega}$ of sets from $\mathcal{K}$ we have a decreasing sequence $H_n = F_1 \cap F_2 \cap \cdots \cap F_n \in \mathcal{K}$, so by the previous argument $\bigcap_{n\in\omega} H_n \neq \emptyset$, and we are done. □

Corollary 5.3. *Let $X = \prod_{\alpha<\kappa} X_\alpha$, where every X_α is a Polish space. If Σ is a σ–algebra of subsets of X and μ is inner regular with respect to zero sets from Σ, then μ is regularly monocompact. If, moreover, $\kappa = \omega_1$, then μ is countably compact.*

Proof. For every α choose a continuous surjection $g_\alpha \colon \mathcal{N} \to X_\alpha$, and let

$$g = \prod_{\alpha<\kappa} g_\alpha : \mathcal{N}^\kappa \to X.$$

Then for every $Z \in \mathrm{Zero}(X)$ we have $g^{-1}[Z] \in \mathrm{Zero}(\mathcal{N}^\kappa)$, so we can argue as in Corollary 4.2. □

6. Application to measures on Polish spaces

Our motivation for considering measures on uncountable products of Polish spaces came from the following result.

Lemma 6.1. *Let μ be a measure on a σ–algebra $\Sigma \subseteq \mathrm{Borel}(X)$, where X is a Polish space. Suppose that $\{B_\alpha : 1 \le \alpha < \kappa\}$ is a family of analytic subsets of X, and let $\mathcal{F}$ be a family of those sets $E \in \Sigma$ for which there is $\alpha < \kappa$ such that $E \subseteq B_\alpha$ is closed in B_α.*

If μ is inner regular with respect to $\mathcal{F}$, then there is a measure $\widehat{\mu}$ defined on some σ–algebra $\widehat{\Sigma}$ of subsets of $\mathcal{N}^\kappa$ which is inner regular with respect to $\mathrm{Zero}(\mathcal{N}^\kappa) \cap \widehat{\Sigma}$ and an inverse–measure–preserving function $(\mathcal{N}^\kappa, \widehat{\Sigma}, \widehat{\mu}) \longrightarrow (X, \Sigma, \mu)$.

Proof. We can assume that $X = \mathcal{N}$. Every B_α is an analytic subset of $\mathcal{N}$, so there is a closed set $F_\alpha \subseteq \mathcal{N}\times\mathcal{N}$ such that $p[F_\alpha] = B_\alpha$, where $p : \mathcal{N}\times\mathcal{N} \to \mathcal{N}$ is the projection onto the first coordinate.

Let $\pi_\alpha : \mathcal{N}^\kappa \to \mathcal{N}$ be the projection onto the α's axis; we consider $\Delta \subseteq \mathcal{N}^\kappa$, where

$$\Delta = \{x \in \mathcal{N}^\kappa : \text{ for every } \alpha \ge 1, \text{ if } \pi_0(x) \in B_\alpha, \text{ then } (\pi_0(x), \pi_\alpha(x)) \in F_\alpha\}.$$

Let $g : \Delta \to \mathcal{N}$ be π_0 restricted to Δ. We endow Δ with the σ–algebra $\Sigma' = \{g^{-1}(E) \colon E \in \Sigma\}$ and the measure μ' on Σ' given by $\mu' g^{-1}(E) = \mu(E)$.

With every $E \in \mathcal{F}$ we can associate $Z(E) \in \mathrm{Zero}(\mathcal{N}^\kappa)$ as follows. Choose $\alpha < \kappa$ such that $E \subseteq B_\alpha$ is closed; then $p^{-1}[E] \cap F_\alpha$ is a closed subset of $\mathcal{N} \times \mathcal{N}$. Now let

$$Z(E) = \{x \in \mathcal{N}^\kappa \colon (\pi_0(x), \pi_\alpha(x)) \in p^{-1}[E] \cap F_\alpha\}.$$

Note that

(i) $g^{-1}[E] = Z(E) \cap \Delta$ for $E \in \mathcal{F}$;
(ii) if $E_1, E_2 \in \mathcal{F}$ are disjoint, then $Z(E_1) \cap Z(E_2) = \emptyset$.

Let Σ'' be the σ–algebra of subsets of $\mathcal{N}^\kappa$ generated by the family

$$Z(\mathcal{F}) = \{Z(E) :\ E \in \mathcal{F}\},$$

and let $\mu''(C) = \mu'(C \cap \Delta)$ for $C \in \Sigma''$. Then for $E \in \mathcal{F}$ we have $\pi_0^{-1}[E] \supseteq Z(E)$ and

(iii) $\mu''(Z(E)) = \mu'(Z(E) \cap \Delta) = \mu'(g^{-1}[E]) = \mu(E)$.

Observe that, by (ii), (iii), and the $\mathcal{F}$–regularity of μ, for $E \in \mathcal{F}$ we have

$$\mu''(\mathcal{N}^\kappa \setminus Z(E)) = \sup\{\mu''(Z(F)) :\ F \in \mathcal{F}, Z(F) \cap Z(E) = \emptyset\}.$$

This implies that μ'' is inner regular with respect to the closure of the family $Z(\mathcal{F})$ with respect to finite unions and countable intersections. In particular, μ'' is regular with respect to zero sets lying inside Σ''.

We finally let $(\mathcal{N}^\kappa, \widehat{\Sigma}, \widehat{\mu})$ be the completion of $(\mathcal{N}^\kappa, \Sigma'', \mu'')$. Since μ'' is regularly monocompact by Theorem 5.1, so is the measure $\widehat{\mu}$.

By (iii) and the $\mathcal{F}$–regularity of μ, $\pi_0 : \mathcal{N}^\kappa \to \mathcal{N}$ is a measure–preserving function, and the proof is complete. ☐

The above lemma together with the result from Section 5 (and the fact that countable compactness is preserved under images) gives the following corollary.

COROLLARY 6.2. *Let μ be a measure on a σ–algebra $\Sigma \subseteq$ Borel(X), where X is a Polish space.*

(a) *There is a regularly monocompact measure space $(\widehat{X}, \widehat{\Sigma}, \widehat{\mu})$ and an inverse–measure–preserving function $(\widehat{X}, \widehat{\Sigma}, \widehat{\mu}) \longrightarrow (X, \Sigma, \mu)$.*
(b) *The measure μ is countably compact provided there is a family $\{B_\alpha : 1 \le \alpha < \omega_1\}$ of analytic subsets of X such that μ is regular with respect to the family $\mathcal{F}$ of those $E \in \Sigma$ for which there is an $\alpha < \omega_1$ such that $E \subseteq B_\alpha$ is closed in B_α.*

Unfortunately, it is not known if regular monocompactness is preserved under inverse–measure–preserving mappings (see Fremlin [5]), so one cannot write in 6.2(a) that μ is simply regularly monocompact. Note that Theorem 1.1 follows from 6.2(b).

7. Measures and games

Let (X, Σ, μ) be any measure space and write $\Sigma^+ = \{E \in \Sigma :\ \mu(E) > 0\}$. In [5] Fremlin introduced the following Banach–Mazur game associated to μ.

The game $\Gamma(\mu)$ has two players, I and II, who choose sets $A_n, B_n \in \Sigma^+$, respectively, so that $A_1 \supseteq B_1 \supseteq A_2 \supseteq B_2 \supseteq \ldots$. Player II wins if $\bigcap_{n\in\omega} A_n \neq \emptyset$.

Fremlin [5] calls the measure μ *weakly α–favourable* if Player II has a winning strategy in $\Gamma(\mu)$, and *α–favourable* if II has a winning tactic in this game, where a tactic is a function $\tau : \Sigma^+ \to \Sigma^+$ such that II wins by playing $B_n = \tau(A_n)$ at each step. For such classes of measure spaces we have the following implications:

$$\begin{aligned}\text{regularly monocompact} \implies{}& \alpha\text{–favourable}\\ &\implies \text{weakly } \alpha\text{–favourable} \implies \text{perfect}\ .\end{aligned}$$

For instance, if μ is inner regular with respect to a monocompact class $\mathcal{K}$, then II wins simply by choosing elements from $\mathcal{K} \cap \Sigma^+$. Fremlin [5] showed that the class of weakly α–favourable measures is properly contained in the class of perfect measures, and he posed the question whether any of the first two implications can be reversed.

Note that we could consider a less restrictive game $\Gamma'(\mu)$, in which the players form a sequence of sets which is μ–centred rather than decreasing. Then our Proposition 3.2 says that Player II has a winning tactic in $\Gamma'(\mu)$ if and only if μ is countably compact.

Fremlin showed in [5] that every weakly α–favourable measure defined on a σ–algebra Σ generated by ω_1 sets is countably compact, and in [7] he proved that μ is weakly α–favourable whenever μ is defined on some $\Sigma \subseteq \mathrm{Borel}(X)$, where X is a Polish space. We show below how one can apply some of the above ideas to prove the latter result; in fact, in the case of $X = [0, 1]$ we are able to explicitly construct a winning strategy for the second player.

THEOREM 7.1 (Fremlin). *If $\Sigma \subseteq \mathrm{Borel}[0, 1]$, then every measure on Σ is weakly α–favourable.*

Proof. As in the proof of Theorem 5.1 we write

$$V(\psi) = \{x \in \mathcal{N} : \ x(k) \le \psi(k) \text{ for all } k < n\},$$

for any $n \in \omega$ and $\psi \in \omega^n$. We shall work in the space $[0, 1] \times \mathcal{N}$. Given $V(\psi)$ as above, we let $G(\psi) = [0, 1] \times V(\psi)$. We denote by $\pi : [0, 1] \times \mathcal{N} \to [0, 1]$ the projection onto the first coordinate.

Every move A_n of the first player is a Borel set, so we can find a closed set $F_n \subseteq [0, 1] \times \mathcal{N}$ such that $\pi[F_n] = A_n$. The second player defines inductively functions $\varphi_n : \omega \to \omega$ such that for every n the set

$$Y_n = \bigcap_{i\le n} \pi[F_i \cap G(\varphi_i|n)]$$

satisfies $\mu^*(Y_n) > 0$, and for the n–th move chooses a set B_n which is a measurable hull of Y_n. Player I is obliged to choose $A_{n+1} \subseteq B_n$, so

$\mu^*(\pi[F_{n+1}] \cap Y_n) = \mu(A_{n+1}) > 0$, and it is easily seen that one can define $\varphi_{n+1}|(n+1)$ and $\varphi_i(n)$ for $i \le n$ in such a way that Y_{n+1} will be a set of positive outer measure.

By following this strategy Player II wins: For every n choose $t_n \in Y_n$. Then the sequence $t_n \in [0,1]$ has a subsequence converging to some t. Fix k. For every $n > k$ there is y_n such that $y_n \in V(\varphi_k|n)$ and $(t_n, y_n) \in F_k$. The sequence of y_n in turn has a subsequence that converges to some y. It follows that $(t,y) \in F_k$ since F_k is closed and $t = \pi(t,y) \in \pi[F_k] = A_k$. Finally, $t \in \bigcap_{k\in\omega} A_k$. This finishes the proof. □

References

[1] J. M. Aldaz and H. Render, *Borel measure extensions of measures defined on sub-σ-algebras*, Adv. Math. **150** (2000), 233–263. MR 1749252 (2001f:28018)

[2] D. H. Fremlin, *Measure algebras*, Handbook of Boolean algebras, Vol. 3, North-Holland, Amsterdam, 1989, pp. 877–980. MR 991611

[3] ______, *Measure-additive coverings and measurable selectors*, Dissertationes Math. **260** (1987). MR 928693 (89e:28012)

[4] ______, *Compact measure spaces*, Mathematika **46** (1999), 331–336. MR 1832624 (2002f:28004)

[5] ______, *Weakly α-favourable measure spaces*, Fund. Math. **165** (2000), 67–94. MR 1805001 (2002c:28021)

[6] ______, *Sets determined by few coordinates*, Atti Sem. Mat. Fis. Univ. Modena **50** (2002), 23–36. MR 1910776 (2003d:28002)

[7] ______, *Problem FN*, note of September 9, 2002.

[8] ______, *Measure Theory, Vol. 4: Topological measure spaces*, Torres Fremlin, Colchester 2003.

[9] ______, *Problem list*, available at `http://www.essex.ac.uk/maths/staff/fremlin/mt.htm`.

[10] A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR 1321597 (96e:03057)

[11] E. Marczewski, *Ensembles indépendant et measures non séparables*, C. R. Acad. Sci. Paris **207** (1938), 768–770.

[12] ______, *On compact measures*, Fund. Math. **40** (1953), 113–124. MR 0059994 (15,610a)

[13] K. Musiał, *Inheritness of compactness and perfectness of measures by thick subsets*, Measure theory (Proc. Conf. Oberwolfach, 1975), Lecture Notes in Math, Vol. 541, Springer, Berlin, 1976, pp. 31–42. MR 0442181 (56 #567)

[14] J. K. Pachl, *Two classes of measures*, Colloq. Math. **42** (1979), 331–340. MR 567571 (82b:28012)

[15] G. Plebanek, *A separable perfect noncompact measure*, Bull. Polish Acad. Sci. Math. **49** (2001), 285–289. MR 1863268 (2002i:28003)

[16] C. Ryll-Nardzewski, *On quasi-compact measures*, Fund. Math. **40** (1953), 125–130. MR 0059997 (15,610d)

[17] B. Veličković, *Playful Boolean algebras*, Trans. Amer. Math. Soc. **296** (1986), 727–740. MR 846604 (88a:06017)

[18] R. F. Wheeler, *A survey of Baire measures and strict topologies*, Exposition. Math. **1** (1983), 97–190. MR 710569 (85b:46035)

Institute of Mathematics, University of Wrocław, 50–384 Wrocław, Poland
E-mail address, P. Borodulin: `pborod@math.uni.wroc.pl`
E-mail address, G. Plebanek: `grzes@math.uni.wroc.pl`