MINIMAL HYPERSURFACES WITH ZERO GAUSS-KRONECKER CURVATURE

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ABSTRACT. We investigate complete minimal hypersurfaces in the Euclidean space R^4, with Gauss-Kronecker curvature identically zero. We prove that, if $f : M^3 \to R^4$ is a complete minimal hypersurface with Gauss-Kronecker curvature identically zero, nowhere vanishing second fundamental form and scalar curvature bounded from below, then $f(M^3)$ splits as a Euclidean product $L^2 \times R$, where L^2 is a complete minimal surface in R^3 with Gaussian curvature bounded from below.

1. Introduction

A classical result of Beez-Killing states that a hypersurface M^n in the Euclidean space R^{n+1} is rigid if the rank of the Gauss map is at least 3. Dajczer and Gromoll in [3] developed a powerful method, the so called "*Gauss parametrization*", which has interesting applications in the study of rigidity problems in the case where the rank is at least 2. The essential point of this method is that it provides a parametrization for every hypersurface with second fundamental form of constant nullity by inverting the Gauss map. The local rigidity of minimal hypersurfaces with nullity $n - 2$ is well understood. In fact, Dajczer and Gromoll in [3] showed that such hypersurfaces allow locally an one-parameter family of isometric deformations, the so-called associated family. Hence, the rigidity of minimal hypersurfaces makes sense under global assumptions. Among other results, Dajczer and Gromoll [3] proved the following global rigidity result:

THEOREM (Dajczer-Gromoll [3]). *If M^n ($n \geq 4$) is a complete Riemannian manifold which does not have R^{n-3} as a factor, then every minimal immersion $f : M^n \to R^{n+1}$ is rigid as a minimal submanifold of R^{n+p} for any $p \geq 1$.*

This is proved by establishing an interesting criterion for a complete hypersurface M^n in R^{n+1} to split as a Euclidean product $M^n = L^3 \times R^{n-3}$. However, the case of dimension $n = 3$ is left open.

The aim of the present paper is to fill in the gap for $n = 3$, by studying complete minimal hypersurfaces in R^4 with Gauss-Kronecker curvature identically zero. One can easily construct complete minimal hypersurfaces with Gauss-Kronecker curvature identically zero in R^4 just by erecting cylinders over complete minimal surfaces in R^3. In fact, let $h : M^2 \to R^3 \hookrightarrow R^4$ be a complete minimal surface. Denote by λ, $-\lambda$ its principal curvatures and by a a unit vector in R^4 normal to R^3. Then the cylinder $f : M^3 = M^2 \times R \to R^4$, $f(p, t) = h(p) + ta$, is a complete minimal hypersurface in R^4 with principal curvatures $k_1 = \lambda$, $k_2 = 0$, $k_3 = -\lambda$. The scalar curvature of M^3 is twice the Gaussian curvature of M^2. Observe that the cylinder over the helicoid in R^3 gives a non-totally geodesic complete minimal hypersurface in R^4, with Gauss-Kronecker curvature identically zero, three distinct principal curvatures and scalar curvature bounded from below. Also, we note that there are complete minimal surfaces in R^3 having their Gaussian curvature not bounded from below [4]. So, erecting the cylinder over those surfaces, one can produce complete minimal hypersurfaces in R^4 with Gauss-Kronecker curvature identically zero and scalar curvature not bounded from below. However, to the best of our knowledge, the cylinders are the only known examples of complete minimal hypersurfaces with Gauss-Kronecker identically zero in R^4. This led us to the following question:

QUESTION. *It is true that any complete minimal hypersurface with vanishing Gauss-Kronecker curvature in R^4 is a cylinder over a minimal surface in R^3?*

We shall give here a partial answer to this question under some additional assumptions on the scalar curvature. In particular, we prove the following theorem.

THEOREM 1. *Let M^3 be an oriented, 3-dimensional, complete Riemannian manifold and $f : M^3 \to R^4$ a minimal isometric immersion with Gauss-Kronecker curvature identically zero and nowhere vanishing second fundamental form. If the scalar curvature is bounded from below, then $f(M^3)$ splits as a Euclidean product $L^2 \times R$, where L^2 is a complete minimal surface in R^3 with Gaussian curvature bounded from below.*

REMARK. It is clear that the cylinders in Theorem 1 are not rigid.

Recently Cheng in [1] proved that complete minimal hypersurfaces in R^4 with scalar curvature bounded from below and constant Gauss-Kronecker curvature have identically zero Gauss-Kronecker curvature. He also gave an example of a minimal hypersurface in R^4 with Gauss-Kronecker curvature identically zero which is in fact a cone shaped hypersurface over the Clifford torus in S^3 and certainly is not complete. Motivated by these facts, we consider complete minimal hypersurfaces in R^4 with constant Gauss-Kronecker curvature and prove that this constant must be zero without any assumption on the scalar curvature. The key of the proof is the powerful Principal Curvature Theorem proved by Smyth and Xavier in [6]. More precisely, we prove the following theorem.

THEOREM 2. *Let M^3 be an oriented, 3-dimensional, complete Riemannian manifold and $f : M^3 \to R^4$ a minimal isometric immersion with constant Gauss-Kronecker curvature. Then the Gauss-Kronecker curvature is identically zero.*

2. Preliminaries

Let $f : M^3 \to R^4$ be an oriented minimal hypersurface equipped with the induced metric and unit normal vector field ξ along f. Denote by A the shape operator associated with ξ and by $k_1 \geq k_2 \geq k_3$ the principal curvatures. The Gauss-Kronecker curvature K and the scalar curvature τ are given by

$$K = \det A = k_1 k_2 k_3, \ \tau = 2(k_1 k_2 + k_1 k_3 + k_2 k_3).$$

Assume now that the Gauss-Kronecker curvature is identically zero and the second fundamental form is nowhere vanishing. Then the principal curvatures satisfy the relation $k_1 = \lambda > k_2 = 0 > k_3 = -\lambda$, where λ is a smooth positive function on M^3. We can choose locally an orthonormal frame field $\{e_1, e_2, e_3\}$ of principal directions corresponding to $\lambda, 0, -\lambda$. Let $\{\omega_1, \omega_2, \omega_3\}$ and $\{\omega_{ij}\}$, $i, j \in \{1, 2, 3\}$, be the corresponding dual and the connection forms. Throughout this paper we make the convention that the indices satisfy

$$1 \leq i, j, k, \ldots \leq 3,$$

and we adopt the method of moving frames. The structure equations are

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_l \omega_{il} \wedge \omega_{lj} - k_i k_j \omega_i \wedge \omega_j, \ i \neq j.$$

Consider the functions

$$u := \omega_{12}(e_3), \quad v := e_2(\log \lambda),$$

which will play a crucial role in the proof of Theorem 1. From the structural equations, and the Codazzi equations,

$$e_i(k_j) = (k_i - k_j)\,\omega_{ij}(e_j)\,,\ i \neq j,$$
$$(k_1 - k_2)\,\omega_{12}(e_3) = (k_2 - k_3)\,\omega_{23}(e_1) = (k_1 - k_3)\,\omega_{13}(e_2)\,,$$

we easily get

$$
\begin{array}{lll}
\omega_{12}(e_1) = v, & \omega_{13}(e_1) = \frac{1}{2}e_3(\log\lambda)\,, & \omega_{23}(e_1) = u, \\
\omega_{12}(e_2) = 0, & \omega_{13}(e_2) = \frac{1}{2}u, & \omega_{23}(e_2) = 0, \\
\omega_{12}(e_3) = u, & \omega_{13}(e_3) = -\frac{1}{2}e_1(\log\lambda)\,, & \omega_{23}(e_3) = -v,
\end{array}
$$

and

(1)
$$
\begin{cases}
e_2(v) = v^2 - u^2, \\
e_1(u) = e_3(v)\,, \\
e_2(u) = 2uv, \\
e_3(u) = -e_1(v)\,, \\
e_1 e_1(\log\lambda) + e_3 e_3(\log\lambda) - \frac{1}{2}\left(e_1(\log\lambda)\right)^2 \\
\qquad -\frac{1}{2}\left(e_3(\log\lambda)\right)^2 - 2v^2 - 4u^2 + 2\lambda^2 = 0.
\end{cases}
$$

Furthermore, the above equations yield

(2)
$$[e_1, e_2] = -ve_1 + \frac{1}{2}ue_3,$$

$$[e_1, e_3] = -\frac{1}{2}e_3(\log\lambda)\,e_1 - 2ue_2 + \frac{1}{2}e_1(\log\lambda)\,e_3,$$

$$[e_2, e_3] = \frac{1}{2}ue_1 + ve_3.$$

3. Proofs

We shall use in the proof of Theorem 1 a result due to S.Y. Cheng and S.T. Yau [2] that we recall in the following lemma. For the reader's convenience we shall include a brief proof, following Nishikawa [5].

LEMMA 3. *Let M^n be an n-dimensional, $n \geq 2$, complete Riemannian manifold with Ricci curvature $Ric \geq -(n-1)k^2$, where k is a positive constant. Suppose that g is a smooth non-negative function on M^n satisfying*

(3)
$$\Delta g \geq cg^2,$$

where c is a positive constant and Δ stands for the Laplacian operator. Then g vanishes identically.

Proof. Take a point $x_0 \in M^n$. If g attains its maximum at x_0, then $g(x_0) = 0$ and the result is true. We shall prove that $g(x) \leq g(x_0)$, for any point $x \in M^3$. Suppose, to the contrary, that there exists a point x_1 such that $g(x_0) < g(x_1)$. We set $\beta = d(x_0, x_1)$ and denote by $B_a(x_0)$ the geodesic ball of radius a centered at x_0, where d is the distance on M^3. We consider

the function $G : B_a(x_0) \to R$, $G(x) = \left(a^2 - \rho^2(x)\right)^2 g(x)$, where $a > \beta$ and $\rho(x) = d(x, x_0)$. Since $G\big|_{\partial B_a(x_0)} = 0$ and G is non-negative we deduce that G attains its maximum at some point $x_2 \in B_a(x_0)$. First let us assume that x_2 is not a cut point of x_0. Then G is smooth near x_2 and by the maximum principle we have

$$(4) \qquad \nabla G(x_2) = 0, \quad \Delta G(x_2) \le 0.$$

It is well known that $\rho \Delta \rho(x_2) \le (n-1)(1 + k\rho(x_2))$ since the Ricci curvature is bounded from below by $-(n-1)k^2$. Then from (4) we get

$$(5) \qquad G(x_2) \le \frac{a^2}{c}(28 + 4k(n-1)a).$$

Assume now that x_2 is a cut point of x_0. Let σ be a unit speed segment from x_0 to x_2 and $\overline{x}_0 \in \sigma$ such that $d(x_0, \overline{x}_0) = \varepsilon$, for sufficiently small $\varepsilon > 0$. Then the function $\overline{\rho}(x) = d(x, \overline{x}_0)$ is smooth near x_2 and the function

$$G_\varepsilon(x) = \left(a^2 - (\overline{\rho}(x) + \varepsilon)^2\right) g(x)$$

is a "support function" of G, i.e., $G_\varepsilon(x) \le G(x)$, $G_\varepsilon(x_2) = G(x_2)$. Thus G_ε attains its maximum at x_2. Proceeding as before and passing to the limit we get the same estimate as in (5). Since x_2 is the maximum point of G in $B_a(x_0)$ we have $G(x_1) \le G(x_2)$ and thus

$$(6) \qquad g(x_1) \le \frac{a^2(28 + 4k(n-1)a)}{c(a^2 - \rho^2(x_1))^2}.$$

Since M^n is complete, letting $a \to \infty$, from (6) we get $g(x_1) = 0$, which is a contradiction. $\qquad\square$

Proof of Theorem 1. Without loss of generality we may assume that M^3 is simply connected, after passing to the universal covering space. Since M^3 is simply connected, the standard monodromy argument allows us to define a global orthonormal frame field $\{e_1, e_2, e_3\}$ of principal directions. The assumptions of Theorem 1 imply that M^3 has three distinct principal curvatures $\lambda > 0 > -\lambda$, and Ricci curvature bounded from below. The functions u and v are well defined on entire M^3. Moreover, we claim that u and v are harmonic

functions. Indeed,

$$\Delta v = e_1 e_1 (v) + e_2 e_2 (v) + e_3 e_3 (v)$$
$$- (\omega_{31} (e_3) + \omega_{21} (e_2)) e_1 (v)$$
$$- (\omega_{12} (e_1) + \omega_{32} (e_3)) e_2 (v)$$
$$- (\omega_{13} (e_1) + \omega_{23} (e_2)) e_3 (v)$$
$$= e_1 e_1 (v) + e_2 e_2 (v) + e_3 e_3 (v)$$
$$- \frac{1}{2} e_1 (\log \lambda) e_1 (v) - 2v e_2 (v)$$
$$- \frac{1}{2} e_3 (\log \lambda) e_3 (v) .$$

Making use of (1), we get

$$e_1 e_1 (v) = -e_1 e_3 (u) ,$$
$$e_2 e_2 (v) = 2v e_2 (v) - 2u e_2 (u)$$
$$= 2v^3 - 6vu^2 ,$$
$$e_3 e_3 (v) = e_3 e_1 (u) .$$

Therefore, taking (2) into account, we obtain

$$\Delta v = -e_1 e_3 (u) + e_3 e_1 (u) + 2v^3 - 6vu^2$$
$$- \frac{1}{2} e_1 (\log \lambda) e_1 (v) - 2v e_2 (v) - \frac{1}{2} e_3 (\log \lambda) e_3 (v)$$
$$= \frac{1}{2} e_3 (\log \lambda) e_1 (u) + 2u e_2 (u) - \frac{1}{2} e_1 (\log \lambda) e_3 (u) + 2v^3 - 6vu^2$$
$$- \frac{1}{2} e_1 (\log \lambda) e_1 (v) - 2v e_2 (v) - \frac{1}{2} e_3 (\log \lambda) e_3 (v)$$
$$= 0.$$

In a similar way, we verify that $\Delta u = 0$.

Using (1) and the fact that $\Delta v = \Delta u = 0$ we get

$$\frac{1}{2} \Delta \left(u^2 + v^2 \right) = \| \nabla u \|^2 + \| \nabla v \|^2$$
$$\geq \left(e_2 (u) \right)^2 + \left(e_2 (v) \right)^2$$
$$= 4u^2 v^2 + \left(v^2 - u^2 \right)^2 .$$

Thus,

$$\Delta \left(u^2 + v^2 \right) \geq 2 \left(u^2 + v^2 \right)^2 .$$

Appealing to the Lemma 3, we infer that $u^2 + v^2$ is identically zero. Thus $u = v \equiv 0$ and λ is constant along the integral curves of e_2. Consider the 2-dimensional distribution V which is spanned by e_1 and e_3. Because $u \equiv 0$, from (2) we see that V is involutive. Let L_x^2 be a maximal integral submanifold

of V passing through a point x of M^3, and denote by $i : L_x^2 \to M^3$ its inclusion map. Then $\widetilde{f} = f \circ i : L_x^2 \to R^4$ defines an immersion. Let A_1 and A_2 be the shape operators of $\widetilde{f}$ associated with $df(e_2)$ and ξ, respectively. A direct calculation shows that $A_1 = 0$ and

$$A_2 \sim \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$$

with respect to the basis $\{e_1, e_3\}$. Thus, $\widetilde{f} : L_x^2 \to R^4$ is a minimal surface with bounded Gaussian curvature which lies in R^3 and $df(e_2)$ is constant along $\widetilde{f}$. Hence $f(M^3)$ splits in the desired form, and this completes the proof of theorem. $\qquad \square$

Proof of Theorem 2. We can choose an orientation such that $k_1 > 0 > k_2 \geq k_3$. According to the Principal Curvature Theorem due to Smyth and Xavier [6], we have $\inf(\Lambda \cap R^+) = 0$, where Λ is the set of values assumed by the non-zero principal curvatures. Hence there exists a sequence of points $\{x_n\}$ such that $k_1(x_n) \to 0$. From the minimality we get $k_1 \geq |k_i|$, $i = 2, 3$, and consequently the Gauss-Kronecker curvature satisfies $K(x_n) \to 0$. Thus K is zero, and this completes the proof. $\qquad \square$

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