Illinois Journal of Mathematics
Volume 49, Number 2, Summer 2005, Pages 517–522
S 0019-2082

# THE BRAID INDEX OF SURFACE-KNOTS AND QUANDLE COLORINGS

KOKORO TANAKA

*Dedicated to Professor Yukio Matsumoto on the occasion of his 60th birthday*

Abstract. The braid index of a surface-knot $F$ is the minimal number among the degrees of all simple surface braids whose closures are ambient isotopic to $F$. We prove that there exists a surface-knot with braid index $k$ for any positive integer $k$. To prove it, we use colorings of surface-knots by quandles and give lower bounds of the braid index of surface-knots.

A *surface-knot* is a closed, connected, oriented surface embedded locally flatly in $\mathbb{R}^4$. The notion of a *surface braid* was defined by Viro [17] and extensively studied by Kamada [10]. A similar notion was also investigated by Rudolph [14], [15]. A *surface braid of degree m* is a compact oriented surface $S$ embedded properly and locally flatly in $B_1^2 \times B_2^2$, where $B_i^2$ is a 2-disk ($i = 1, 2$), such that

(i) the restriction map $\pi|_S$ of the projection $\pi : B_1^2 \times B_2^2 \to B_2^2$ is a branched covering map of degree $m$, and
(ii) $\partial S = P_m \times \partial B_2^2 \left(\subset B_1^2 \times \partial B_2^2\right)$ for a fixed set $P_m$ of $m$ distinct interior points of $B_1^2$.

A surface braid $S$ is called *simple* if the covering $\pi|_S$ is simple (i.e., the preimage of each branch locus consists of $m-1$ points).

A surface braid $S$ of degree $m$ is extended to a closed surface embedded in $\mathbb{R}^4$, called the *closure* of $S$, by embedding the 4-disk $B_1^2 \times B_2^2$ in $\mathbb{R}^4$ and attaching $m$ sheets of 2-disks along the boundary of $S$ in $\mathbb{R}^4 \setminus \mathrm{int}(B_1^2 \times B_2^2)$ in the obvious way. Surface braids are closely related to surface-knots; as an analogue of Alexander's theorem in classical knot theory, Viro [17] and Kamada [8] proved that any surface-knot is ambient isotopic to the closure of a simple surface braid. We refer to [10], [2] for more details.

The *braid index* of a surface-knot $F$ is defined to be the minimal number among the degrees of all simple surface braids whose closures are ambient

Received November 25, 2004; received in final form January 12, 2005.
2000 *Mathematics Subject Classification.* Primary 57Q45. Secondary 57M25.

©2005 University of Illinois

isotopic to $F$ in $\mathbb{R}^4$. There exist several results on the braid index of a surface-knot; see [7], [9], [11], for example. Surface-knots with braid index less than three are unknotted, and those with braid index three are "ribbon" [7]. The 2-twist spun trefoil, for example, is not ribbon, and hence has braid index four [7]. However, a braid index greater than four has never been obtained for any specific examples of surface-knots. In this paper, we prove:

THEOREM 1. *For any integer $k > 0$ there exists a surface-knot with braid index $k$.*

To prove the theorem, we use colorings of surface-knots by quandles.

A *quandle* [3], [6], [12] is a non-empty set $X$ equipped with a binary operation $(a, b) \mapsto a * b$ such that

(i) $a * a = a$ for any $a \in X$,
(ii) the map $*a : X \to X$ $(x \mapsto x * a)$ is bijective for each $a \in X$, and
(iii) $(a * b) * c = (a * c) * (b * c)$ for any $a, b, c \in X$.

The *dihedral quandle of order* $p$, denoted by $R_p$, is a quandle consisting of the set $\{0, 1, \ldots, p-1\}$ with the binary operation defined by $i * j \equiv 2j - i \pmod{p}$.

A *diagram* of a surface-knot is a generic projection image in $\mathbb{R}^3$, where one of the two sheets near the double point curve is broken depending on the relative height. This convention is similar to classical knot diagrams. A diagram consists of *broken sheets*, which are mutually disjoint compact oriented surfaces in $\mathbb{R}^3$, and the orientations are specified by normal vectors. We refer to [2] for more details.

A *coloring* of a surface-knot diagram by a quandle $X$ is an assignment of an element of $X$ to each broken sheet such that $a * b = c$ holds along each double point curve, where $a$ (resp. $c$) is the color of under-sheet that is behind (resp. in front of) the over-sheet colored $b$ with respect to the normal vector of the over-sheet. We remark that the number of colorings is an invariant of a surface-knot and that the coloring by $R_p$ is the same as the Fox *$p$-coloring* [4], [5].

PROPOSITION 2. *Let $F$ be a surface-knot which is not a trivial $S^2$-knot. If there is a finite quandle $X$ with $n$ elements such that $F$ admits at least $n^s$ colorings by $X$ for integers $n > 1$ and $s > 0$, then the braid index of $F$ is at least $s + 1$.*

*Proof.* Let $m$ be the braid index of $F$. Consider a simple surface braid $S$ of degree $m$ whose closure presents $F$. Regarding $B_1^2$ as $I_1 \times I_2$, where $I_i$ is the unit interval $(i = 1, 2)$, the projection of $B_1^2$ onto the first factor $I_1$ induces $\pi' : B_1^2 \times B_2^2 \to I_1 \times B_2^2$ and we obtain a diagram $D$ of $S$ as the projection of $S$ by $\pi'$. The boundary circles $\partial S = P_m \times \partial B_2^2$ project to embedded circles in $I_1 \times \partial B_2^2$ by $\pi'$. Branch points appear at the end of double point curves and correspond to branch loci of the covering $\pi|_S$. Since $F$ is not a trivial $S^2$-knot, the diagram $D$ has branch points. By definition, each coloring of $D$ by $X$ is determined by a vector $(x_1, x_2, \ldots, x_m) \in X^m$ such that the $i$th boundary circle of $D$ receives the color $x_i$ ($i = 1, 2, \ldots, m$). By [11, Lemma 12], we may assume that the first and second boundary circles belong to the same broken sheet; Figure 1 shows this situation, where a branch point connects the first and second broken sheets near the boundary circles. It follows from $x_1 = x_2$ that the surface-knot $F$ admits at most $n^{m-1}$ colorings by $X$. Thus we obtain $n^s \leq n^{m-1}$, that is, $m \geq s + 1$. □

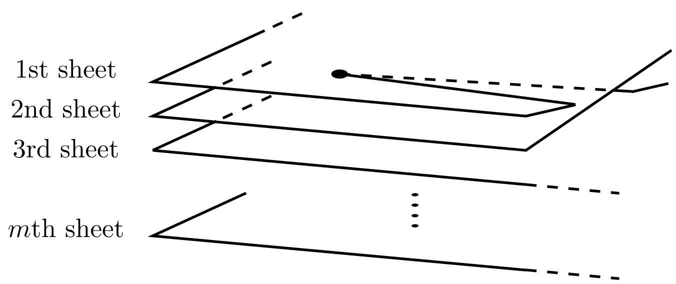


FIGURE 1

For the proof of Theorem 1, since it is known that there exist surface-knots with braid index less than three, it is sufficient to prove:

PROPOSITION 3. *The connected sum of $\ell$ copies of the spun $(2, p)$-torus knot has braid index $\ell + 2$, where $p$ is an odd integer with $p \geq 3$.*

*Proof.* Let $F_p(\ell)$ be the connected sum of $\ell$ copies of the spun $(2, p)$-torus knot. Since the number of colorings of $F_p(1)$ by the dihedral quandle $R_p$ of order $p$ is equal to $p^2$, that of $F_p(\ell)$ is equal to $p^{\ell+1}$ (cf. [13]). Hence the braid index of $F_p(\ell)$ is at least $\ell + 2$ by Proposition 2.

On the other hand, the following was proved by Kamada, Satoh and Takabayashi [11, Theorem 3]: if neither $F_1$ nor $F_2$ is a trivial $S^2$-knot, then the inequality

$$(*) \qquad \mathrm{Braid}(F_1 \# F_2) \leq \mathrm{Braid}(F_1) + \mathrm{Braid}(F_2) - 2$$

holds for the connected sum $F_1 \# F_2$ of two surface-knots $F_1$ and $F_2$, where $\mathrm{Braid}(F)$ is the braid index of a surface-knot $F$. Thus the braid index of $F_p(\ell)$ is at most $\ell + 2$, since that of $F_p(1)$ is three. □

We obtain the following by an argument similar to that in the proof of Proposition 3.

Corollary 4. *The connected sum of $\ell$ copies of the spun $(2,p)$-torus knot and $g$ copies of the trivial $T^2$-knot has the braid index $\ell+2$, where $p$ is an odd integer with $p \geq 3$.*

*Proof.* Let $T$ be the trivial $T^2$-knot, and let $F_p(\ell)\#_g T$ be the connected sum of $F_p(\ell)$ and $g$ copies of $T$. In general, the number of colorings is invariant under the connected sum by a trivial surface-knot. Thus the number of colorings of $F_p(\ell)\#_g T$ by $R_p$ is equal to that of $F_p(\ell)$, that is, it is equal to $p^{\ell+1}$. Hence the braid index of $F_p(\ell)\#_g T$ is at least $\ell+2$ by Proposition 2. On the other hand, since the braid index of $T$ is two, that of $F_p(\ell)\#_g T$ is at most $\ell+2$ by Proposition 3 and inequality $(*)$. □

If $p'$ is less than $p$, then we can show by a direct calculation that the number of colorings of $F_{p'}(\ell)\#_g T$ by $R_p$ is less than $p^{\ell+1}$. Hence the two ribbon surface-knots $F_p(\ell)\#_g T$ and $F_{p'}(\ell)\#_g T$ are not ambient isotopic to each other, and the following is a direct consequence of Proposition 3 and Corollary 4.

Corollary 5. *For each pair of integers $k \geq 3$ and $g \geq 0$ there exists an infinite series of ribbon surface-knots of genus $g$ with braid index $k$.*

Finally, we consider the braid index of a *non-ribbon* surface-knot. Let $G(\ell)$ be the connected sum of the 2-twist spun trefoil and $\ell$ copies of the spun trefoil, where $\ell$ is an integer with $\ell \geq 0$.

Lemma 6. *For each integer $\ell > 0$, the $S^2$-knot $G(\ell)$ is non-ribbon and the braid index of $G(\ell)$ is either $\ell+3$ or $\ell+4$.*

*Proof.* It follows, from the quandle cocycle invariant [1] of $G(\ell)$ associated with a certain 3-cocycle of the dihedral quandle $R_3$ and the coefficient group $\mathbb{Z}_3$, that $G(\ell)$ is non-ribbon and that the number of colorings of $G(\ell)$ by $R_3$ is equal to $3^{\ell+2}$. We refer to [16, proof of Theorem 1.1] for the quandle cocycle invariant of $G(\ell)$. Hence the braid index of $G(\ell)$ is at least $\ell+3$ by Proposition 2. On the other hand, since the braid index of the 2-twist spun trefoil $G(0)$ is four [7], that of $G(\ell)$ is at most $\ell+4$ by Proposition 3 and inequality $(*)$. □

We recall here that the braid index of a non-ribbon surface-knot is greater than three [7]. Using Lemma 6, we prove:

Proposition 7. *For any integer $k > 3$ there exists a non-ribbon surface-knot with braid index $k$.*

*Proof. Case 1: The braid index of $G(k-4)$ is $k-1$.* Then we take the non-ribbon $S^2$-knot $G(k-3)$. Using inequality $(*)$ again for $G(k-3)$, the braid index of $G(k-3)$ is at most $k$ $\big(=(k-1)+3-2\big)$. On the other hand, we have already proved that the braid index of $G(k-3)$ is at least $k$.

*Case 2: The braid index of $G(k-4)$ is $k$.* In this case the non-ribbon $S^2$-knot $G(k-4)$ is what we want. $\square$

PROBLEM 8. *For each integer $\ell > 0$ determine the braid index of $G(\ell)$ exactly. Which is the correct value of this index, $\ell+3$ or $\ell+4$?*

**Acknowledgments.** This research is supported by the 21st century COE program at the Graduate School of Mathematical Sciences of the University of Tokyo. The author would like to express his sincere gratitude to Yukio Matsumoto for encouraging him. He would also like to thank Seiichi Kamada, Masahico Saito and Shin Satoh for helpful comments on the draft, and Isao Hasegawa for stimulating discussions.

## References

[1] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, and M. Saito, *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc. **355** (2003), 3947–3989. MR 1990571 (2005b:57048)

[2] J. S. Carter and M. Saito, *Knotted surfaces and their diagrams*, Mathematical Surveys and Monographs, vol. 55, American Mathematical Society, Providence, RI, 1998. MR 1487374 (98m:57027)

[3] R. Fenn and C. Rourke, *Racks and links in codimension two*, J. Knot Theory Ramifications **1** (1992), 343–406. MR 1194995 (94e:57006)

[4] R. H. Fox, *A quick trip through knot theory*, Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961), Prentice-Hall, Englewood Cliffs, N.J., 1962, pp. 120–167. MR 0140099 (25 #3522)

[5] ———, *Metacyclic invariants of knots and links*, Canad. J. Math. **22** (1970), 193–201. MR 0261584 (41 #6197)

[6] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra **23** (1982), 37–65. MR 638121 (83m:57007)

[7] S. Kamada, *Surfaces in* $\mathbf{R}^4$ *of braid index three are ribbon*, J. Knot Theory Ramifications **1** (1992), 137–160. MR 1164113 (93h:57039)

[8] ———, *A characterization of groups of closed orientable surfaces in* 4*-space*, Topology **33** (1994), 113–122. MR 1259518 (95a:57002)

[9] ———, *Standard forms of* 3*-braid* 2*-knots and their Alexander polynomials*, Michigan Math. J. **45** (1998), 189–205. MR 1617423 (99h:57048)

[10] ———, *Braid and knot theory in dimension four*, Mathematical Surveys and Monographs, vol. 95, American Mathematical Society, Providence, RI, 2002. MR 1900979 (2003d:57050)

[11] S. Kamada, S. Satoh and M. Takabayashi, *The braid index is not additive for the connected sum of* 2*-knots*, Trans. Amer. Math. Soc., to appear.

[12] S. V. Matveev, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.) **119(161)** (1982), 78–88. MR 672410 (84e:57008)

[13] J. H. Przytycki, 3*-coloring and other elementary invariants of knots*, Knot theory (Warsaw, 1995), Banach Center Publ., vol. 42, Polish Acad. Sci., Warsaw, 1998, pp. 275–295. MR 1634462

[14] L. Rudolph, *Braided surfaces and Seifert ribbons for closed braids*, Comment. Math. Helv. **58** (1983), 1–37. MR 699004 (84j:57006)

[15] ______, *Special positions for surfaces bounded by closed braids*, Rev. Mat. Iberoamericana **1** (1985), 93–133. MR 836285 (88a:57018)

[16] S. Satoh and A. Shima, *Triple point numbers and quandle cocycle invariants of knotted surfaces in 4-space*, New Zealand J. Math. **34** (2005), 71–79. MR 2141479

[17] O. Ya. Viro, Lecture given at Osaka City University, September, 1990

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba Meguro, Tokyo 153-8914, Japan

*E-mail address*: k-tanaka@ms.u-tokyo.ac.jp