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GROMOV-HAUSDORFF LIMITS IN DEFINABLE FAMILIES

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Dedicated to Ludwig Bröcker on the occasion of his 65th anniversary

ABSTRACT. The notion of piecewise definable metric space is introduced. It is shown that the set of all Gromov-Hausdorff limits of piecewise definable spaces belonging to a fixed bounded definable family is again a definable family. The Gromov-Hausdorff limit is taken with respect to the geodesic metric and the word *definable* means *definable in some o-minimal structure over* $\mathbb{R}$.

1. Introduction and statement of results

The aim of this note is to study Gromov-Hausdorff limits of piecewise definable spaces belonging to a fixed definable family. Here the word *definable* means *definable in some o-minimal structure over the reals*. See [12] and [14] for o-minimal structures.

A piecewise definable space is a space X obtained by gluing finitely many compact, connected and definable sets $X^1, \ldots, X^k \subset \mathbb{R}^n$ along compact, definable subsets $X^{ij} = X^{ji} \subset X^i \cap X^j, i, j = 1, \ldots, k$. X is equipped with a natural length metric d_X induced by the Euclidean metrics on the pieces. The details of this construction can be found in Section 2.

Let $A \subset \mathbb{R}^{m+n}$ be a bounded piecewise definable space. Let $A' := \pi_m(A)$, where $\pi_m : \mathbb{R}^{m+n} \to \mathbb{R}^m$ is the projection onto the first m coordinates. Each fiber of π_m over a point $a \in A'$ can be considered as a piecewise definable space in $\mathbb{R}^n$, which we suppose to be compact and which we denote by A_a. Let $F(A) := \{(A_a, d_{A_a}) : a \in A'\}$ denote the set of geodesic metric spaces in the family A.

We refer to [3] for the definition of Gromov-Hausdorff distance between compact metric spaces. For any family F of compact metric spaces, we denote by $cl(F)$ the closure of F in the Gromov-Hausdorff topology, i.e., the family of all compact metric spaces which are Gromov-Hausdorff limits of sequences

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in F. We say that F is definable if there exists a bounded definable family A of compact piecewise definable spaces as above with $F = F(A)$.

THEOREM 1. *Let F be a definable family of compact metric spaces. Then $cl(F)$ is also a definable family of compact metric spaces.*

COROLLARY 1.1. *Let X be the Gromov-Hausdorff limit of a sequence $X_1, X_2, \ldots$ of compact piecewise definable spaces belonging to a fixed bounded definable family. Then X is piecewise definable. Suppose that also the Hausdorff limit Y of this sequence exists. Then there exists a finite-to-one map $\pi : X \to Y$ which preserves lengths of curves. The number of points in each fiber is bounded by a constant which depends only on the family.*

As an example, consider a family of ellipsoids in $\mathbb{R}^3$ getting flatter and flatter. As Gromov-Hausdorff limit, we obtain a double disk, which is clearly piecewise definable.

A similar statement as that of Theorem 1, but with Gromov-Hausdorff limit replaced by Hausdorff limit, and geodesic metric replaced by Euclidean metric, is well-known. In the semialgebraic setting, this goes back to Bröcker [4] and was later extended, using model theory, to o-minimal structures by Marker-Steinhorn [9], Pillay [11] and van den Dries [13]. Lion-Speissegger [8] gave a geometric proof of the same fact, and their version will be used in the proof of our main theorem. Gromov-Hausdorff limits, but still with respect to Euclidean metric, were considered by van den Dries [13].

For geometric and practical applications, the geodesic metric is more interesting and more natural than the Euclidean one. However, it is much less understood. One obstacle when dealing with the geodesic metric is that, in general, it is not a definable function. In [1] it is shown that the Gromov-Hausdorff limit of a definable 1-parameter family exists, and this fact was used to study the local geometry of definable sets. Our main theorem extends this result in two directions: first we allow arbitrary definable families and secondly we describe all limit spaces as piecewise definable spaces.

2. Piecewise definable metric spaces

We first collect some facts about the geodesic distance on definable sets.

Let $X \subset \mathbb{R}^n$ be a connected, compact definable set. The Euclidean metric induces a length metric d_X, called geodesic metric on X. For $x, y \in X$, $d_X(x, y)$ is defined to be the minimal length of a curve in X between x and y. It is not known for which o-minimal structures $d_X : X \times X \to \mathbb{R}$ is again a definable function. However, by a result of Kurdyka and Orro [7] there exists for each $\epsilon > 0$ a continuous definable function $d_{\text{def}} : X \times X \to \mathbb{R}$ with $d_X \leq d_{\text{def}} \leq (1 + \epsilon) d_X$.

One version of the Łojasiewicz inequality in o-minimal structures [14] states that if f, g are continuous non-negative definable functions on a compact definable set with $f^{-1}(0) \subset g^{-1}(0)$, then there exists a continuous, monotone, definable function $\phi : [0, \infty) \to [0, \infty)$ with $g \leq \phi \circ f$ and $\phi(0) = 0$.

By applying this to $g := d_{\mathrm{def}} : X \times X \to [0, \infty)$ and f equal to the Euclidean distance $X \times X \to [0, \infty)$, we obtain that

$$\|x - y\| \leq d_X(x, y) \leq \phi(\|x - y\|).$$

It follows that d_X and the Euclidean metric induce the same topology on X. In particular, (X, d_X) is compact, and hence complete. The theorem of Hopf-Rinow [3, I.3.7] implies that two points $x, y \in X$ can be joined by a geodesic, i.e., a curve $\gamma : [0, d_X(x, y)] \to X$ with $\gamma(0) = x$, $\gamma(d_X(x, y)) = y$ and $d_X(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in [0, d_X(x, y)]$.

Given $x, y \in X$ and $\epsilon > 0$, there exists a continuous, *definable* curve γ between x and y of length bounded by $d(x, y) + \epsilon$. Such a curve is constructed in the proof of Proposition 3 in [7].

Let $X^1, \ldots, X^k \subset \mathbb{R}^n$ be compact, connected, definable sets. We extend the geodesic metrics d_{X^j} on X^j to a metric d on the disjoint union $\sqcup_{j=1}^k X^j$ by setting $d(x, y) = d_{X^j}(x, y)$ if x and y are both contained in some X^j and $d(x, y) = \infty$ otherwise. The canonical map $\sqcup_{j=1}^k X^j \to \bigcup_{j=1}^k X^j \subset \mathbb{R}^n$ is denoted by π.

Let $X^{12}, X^{13}, \ldots, X^{k-1,k} \subset \mathbb{R}^n$ be compact definable subsets with $X^{ij} = X^{ji} \subset X^i \cap X^j$. We consider the graph with vertices $1, \ldots, k$ whose edges are the pairs ij such that $X^{ij} \neq \emptyset$ and assume that this graph is connected.

The X^{ij} generate an equivalence relation on $\sqcup_{j=1}^k X^j$. Two points $x, y \in \sqcup_{j=1}^k X^j$ are equivalent if and only if $\pi(x) = \pi(y)$ and there exists a sequence $j_1, \ldots, j_l$ such that $x \in X^{j_1}$, $y \in X^{j_l}$ and $\pi(x) \in X^{j_i, j_{i+1}}$ for $i = 1, \ldots, l - 1$.

The quotient pseudo metric space is denoted by $X := (\sqcup_{j=1}^k X^j, d_X)$ (see [3] for quotients of metric spaces) and called a *piecewise definable metric space*.

Let us describe d_X more explicitly. For $x, y \in X$ we have

$$d_X(x, y) = \inf \left\{ \sum_{j=1}^N d(x^j, y^j) \right\},$$

where the inf is taken over all finite sequences $x^1 = x, y^1, x^2, y^2, \ldots, x^N, y^N = y$ such that y^j and x^{j+1} are equivalent for $j = 1, \ldots, N - 1$ (there is no bound on N). The assumption made on the sets X^{ij} implies that $d_X(x, y) < \infty$ for all $x, y \in X$.

Lemma 2.1. *(X, d_X) is a geodesic space, i.e., two points of X can be joined by a geodesic.*

Proof. By definition, d_X is a pseudo-metric. We first show that $d_X(x, y) = 0$ implies that x and y are equivalent.

For all $x, y \in X$ we have $d_X(x,y) \geq \|\pi(x) - \pi(y)\|$.

Let r be the minimal Euclidean distance between $\pi(x)$ and one of those X^{ij} which do not contain $\pi(x)$. Denote the canonical embedding of X^j in X by τ_j.

Choose a sequence $x^1 = x, y^1, x^2, y^2, \ldots, x^N, y^N = y$ as above with $\sum_{j=1}^N d(x^j, y^j) < r$. Let i_j be such that $x^j, y^j \in X^{i_j}$. Since y^j and x^{j+1} are equivalent and $\|\pi(y^j) - \pi(x)\| = \|\pi(x^{j+1}) - \pi(x)\| < r$, we get by the definition of r that $\tau_{i_j}(\pi(x)) \sim \tau_{i_{j+1}}(\pi(x))$. This holds for all j and shows that $x = \tau_{i_1}(\pi(x)) \sim y = \tau_{i_N}(\pi(y))$. Therefore $d_X(x,y) = 0$, which shows that d_X is a metric.

We join x^j and y^j by a geodesic in X^{i_j}. Pasting these curves together yields a continuous curve between x and y whose length is $\sum_{j=1}^N d(x^j, y^j)$. This implies that d_X is a length metric.

The embedding $\tau_j : (X^j, d_{X^j}) \to (X, d_X)$ is 1-Lipschitz. Since X is covered by the compact sets $\tau_j(X^j)$, it is compact. By the theorem of Hopf-Rinow, it is a geodesic metric space [3]. $\square$

3. Proof of the main theorem modulo some propositions

In this section, we will state three propositions without proofs and deduce the proof of the main theorem. The proofs of the propositions will be given in later sections.

Definition 3.1. Let a piecewise definable space X be given by sets $X^1, \ldots, X^N$ and gluing sets $X^{12}, \ldots, X^{N-1,N}$. A subdivision of X is a piecewise definable space Y given by sets $Y^1, \ldots, Y^M$ and gluing sets $Y^{12}, \ldots, Y^{M-1,M}$ such that

(a) each X^i is a union of some of the Y^j,
(b) if $Y^{j_1}, Y^{j_2} \subset X^i, j_1 \neq j_2$, then $Y^{j_1 j_2} = Y^{j_1} \cap Y^{j_2}$,
(c) if $Y^{j_1} \subset X^{i_1}, Y^{j_2} \subset X^{i_2}$ with $i_1 \neq i_2$ then $Y^{j_1 j_2} = Y^{j_1} \cap Y^{j_2} \cap X^{i_1 i_2}$.

Lemma 3.2 (Subdivision Lemma). *If Y is a subdivision of X, then $d_Y = d_X$.*

Definition 3.3. A compact definable set $X \subset \mathbb{R}^n$ is called C-normal, where $C > 1$ is a real number, if $d_X(x,y) \leq C\|x - y\|$ for all $x, y \in X$.

If X is C-normal for some $C > 1$, then X is also called normally embedded (cf. [2]).

Proposition 3.4 (Convergence of normal sets). *Let $X_1, X_2, \ldots$ belong to a fixed bounded definable family of compact subsets of $\mathbb{R}^n$. Suppose that each X_i is C-normal for some fixed constant $C > 1$, and that the Hausdorff limit $X := \lim_{i\to\infty} X_i$ exists. Let $x_i, y_i \in X_i$ and suppose $x_i \to x, y_i \to y$ for*

$i \to \infty$. *Then*

$$d_X(x,y) = \lim_{i\to\infty} d_{X_i}(x_i, y_i).$$

In particular, X is also C-normal.

Note that the assumption that X_i is C-normal with C independent of i cannot be dropped, as can be seen from the example of flat ellipsoids in $\mathbb{R}^3$ mentioned in the introduction.

PROPOSITION 3.5 (Convergence of normal families). *Let $X_1, X_2, \ldots$ belong to a fixed bounded definable family of compact piecewise definable spaces, such that X_i is given by sets $X_i^1, \ldots, X_i^k \subset \mathbb{R}^n$ and $X_i^{1,2}, \ldots, X_i^{k-1,k} \subset \mathbb{R}^n$ (with k independent of i). Suppose that $X_i^1, \ldots, X_i^k$ are C-normal for some $C > 1$ independent of i. Suppose furthermore that each Hausdorff limit $X^j := \lim_{i\to\infty} X_i^j, X^{j,l} := \lim_{i\to\infty} X_i^{j,l}$ exists. Then the piecewise definable space X given by the sets $X^1, \ldots, X^k, X^{1,2}, \ldots, X^{k-1,k}$ is the Gromov-Hausdorff limit of the sequence (X_i, d_{X_i}) (and $\pi(X)$ is the Hausdorff limit of this sequence).*

Proof of the Theorem 1. Let $A \subset \mathbb{R}^{m+n}$ be bounded and piecewise definable, $A' = \pi_m(A)$ and $A_a, a \in A'$ (which is supposed to be compact) the piecewise definable space canonically associated to a fiber.

Let us suppose that A is given by the pieces $A^1, \ldots, A^N, A^{12}, \ldots, A^{N-1,N}$. By taking a subdivision if necessary, we can suppose that $A_a^1, \ldots, A_a^N$ are C-normal for some constant $C > 1$ and all $a \in A'$. This follows from the theorem of Kurdyka-Orro [7]. The set $F(A)$ remains the same by Lemma 3.2.

Consider the family $\tilde{A} \subset A' \times \mathbb{R}^{n\left(N+\binom{N}{2}\right)}$ with $\tilde{A}_a = A_a^1 \times \ldots \times A_a^N \times A_a^{12} \times \ldots \times A_a^{N-1,N} \subset \mathbb{R}^{n\left(N+\binom{N}{2}\right)}$. Applying the theorem of Lion-Speissegger [8] to this family and noting that Hausdorff convergence of a product is equivalent to Hausdorff convergence of each of its factors, we get an integer M and a compact, definable family $B \subset \mathbb{R}^{M+n}$ of piecewise definable spaces such that

(a) for every $a \in A'$ there exists $b \in B' = \pi_M(B)$ with $A_a = B_b$;
(b) for every sequence $(b_i)_i$ in B' such that $\lim_{i\to\infty} b_i = b$, each piece of B_{b_i} converges in the Hausdorff topology to the corresponding piece of B_b.

Note that (b) implies, by Propositions 3.4 and 3.5, that $(B_{b_i}, d_{B_{b_i}})$ converge in the Gromov-Hausdorff metric to (B_b, d_{B_b}).

Replacing B' by the closure of the definable set $\{b \in B' : \exists a \in A' : A_a = B_b\}$ and B by $\pi_M^{-1}(B') \cap B$, we get a compact, definable family which satisfies (a) and (b) and moreover the following condition:

(c) The set of $b \in B'$ such that there exists $a \in A'$ with $A_a = B_b$ is dense in B'.

We claim that $cl(F(A)) = F(B)$.

The compactness of B and (b) imply that $cl(F(B)) = F(B)$. Since $F(A) \subset F(B)$, we get $cl(F(A)) \subset F(B)$.

For $b \in B'$, choose a sequence $b_i \in B'$ converging to b such that there exist $a_i \in A'$ with $A_{a_i} = B_{b_i}$. This is possible by (c). The metric spaces $A_{a_i} = B_{b_i}$ converge to B_b by (b). It follows that $B_b \in cl(F(A))$ and thus $F(B) \subset cl(F(A))$.

This finishes the proof of Theorem 1 (modulo the proof of the propositions, which will be given in the next sections). $\square$

Proof of Corollary 1.1. Let A be a bounded definable family of compact piecewise definable spaces such that for each i there exists $a_i \in A'$ with $X_i = A_{a_i}$. Let B be a compact definable family as in the proof of the theorem and $b_i \in B'$ with $X_i = B_{b_i}$. Passing to a subsequence if necessary, we can suppose that $b_i \to b'$ for some $b' \in B'$. Then Property (b) of B and Proposition 3.5 imply that $\pi(X) = \pi(B_b)$ is the Hausdorff limit of the sequence. The cardinality of $\pi^{-1}(x'), x' \in \pi(X)$ is bounded by the number N of pieces in a C-normal subdivision of B.

By the definition of the metric in X, the length of a curve γ in X is the same as the length of the curve $\pi \circ \gamma$ in $\pi(X)$. $\square$

4. Proof of the Subdivision Lemma

Proof. Let Y denote a subdivision of X. As sets, $X = Y$, and we have to show that $d_Y = d_X$.

By definition,

$$d_X(x,y) = \inf \left\{ \sum_{j=1}^{N} d(x^j, y^j) \right\},$$

where the inf is taken over all finite sequences $x^1 = x, y^1, x^2, y^2, \ldots, x^N, y^N = y$ such that x^k, y^k lie in one of the sets X^{i_k} and y^k and x^{k+1} are equivalent with respect to the equivalence relation generated by the X^{ij} (with no bound on N). The distance $d_Y(x,y)$ is defined similarly, but this time the inf is taken over all sequences $x^1 = x, y^1, x^2, y^2, \ldots, x^N, y^N = y$ with $x^k, y^k \in Y^{i_k}$ such that y^k and x^{k+1} are equivalent with respect to the equivalence relation generated by the Y^{ij}.

Given any sequence of points as above for d_Y, we delete all pairs y^j, x^{j+1} which lie in the same X^j. This gives a sequence as in the definition for d_X, whose length is not longer by triangle inequality and by the fact that $d_{Y^j} \geq d_{X^i}|_{Y^j}$ for $Y^j \subset X^i$. We deduce that $d_X \leq d_Y$.

Conversely, given a sequence $x^1, y^1, \ldots, x^N, y^N$ for d_X and $\epsilon > 0$, we can join each pair $x^j, y^j \in X^{\mu_j}$ by a definable curve in X^{μ_j} of length less than $d_{X^{\mu_j}}(x^j, y^j) + \epsilon$ (compare Section 2). Since the curve is definable, we can partition it into finitely many parts which are completely contained in one of the Y^j. Then we take the endpoints of these parts as new points and obtain a sequence as in the definition for d_Y. The length of this new sequence is bounded by the length of the old one plus ϵN. It follows $d_Y \le d_X + \epsilon N$. Since ϵ was arbitrary, we get $d_Y \le d_X$. □

5. Convergence of normal sets

DEFINITION 5.1. Let $X \subset \mathbb{R}^n$ be a connected compact definable set and $x, y \in X$. An ϵ-path c between x and y is a sequence $c = (x_1, x_2, \ldots, x_N)$ of points of X such that $x_1 = x, x_N = y$, $\|x_{i+1} - x_i\| \le \epsilon$. The length of c is given by $l(c) := \sum_{i=1}^{N-1} \|x_{i+1} - x_i\|$.

LEMMA 5.2. *Let $X \subset \mathbb{R}^n$ be a compact definable set and $x, y \in X$. Define*

$$d_X^\epsilon(x, y) := \inf \{l(c) : c \text{ is an } \epsilon\text{-path between } x, y\}.$$

Then $\lim_{\epsilon \to 0} d_X^\epsilon(x, y) = d_X(x, y)$.

Proof. Let γ be a geodesic between x and y. Choosing points on γ at distances $\le \epsilon$, we get that $d_X^\epsilon(x, y) \le d(x, y)$; therefore $\limsup_{\epsilon \to 0} d_X^\epsilon(x, y) \le d(x, y)$.

For the opposite direction, fix $\eta > 1$ and a covering $X = \bigcup_{j=1}^k X^j$ by compact η-normal subsets X^j [7].

Given an ϵ-path $x_1 = x, x_2, \ldots, x_N = y$, we construct a new ϵ-path as follows.

Let $n_0 := 0$ and let $j_1 \in \{1, \ldots, k\}$ be such that $x_1 \in X^{j_1}$. Let n_1 be the largest integer (possibly equal to 1) such that $x_{n_1} \in X^{j_1}$.

If $n_1 < N$, let j_2 be such that $x_{n_1+1} \in X^{j_2}$. Let n_2 be the largest integer with $x_{n_2} \in X^{j_2}$.

We continue in this way. After $k' \le k$ steps, the process terminates and we get finite sequences $j_1, j_2, \ldots, j_{k'}$ and $0 = n_0 < n_1 < \ldots < n_{k'} = N$ such that x_{n_i+1} and $x_{n_{i+1}}$ belong to $X^{j_{i+1}}$ for $i = 0, \ldots, k' - 1$.

Choose for each $i = 0, \ldots, k'-1$ a sequence of points $z_1^i = x_{n_i+1}, z_2^i, \ldots, z_{N_i}^i = x_{n_{i+1}}$ in $X^{j_{i+1}}$ such that $d_{X^{j_{i+1}}}(z_j^i, z_{j+1}^i) \le \epsilon$ and $\sum_{j=1}^{N_i-1} d_{X^{j_{i+1}}}(z_j^i, z_{j+1}^i) = d_{X^{j_{i+1}}}(x_{n_i+1}, x_{n_{i+1}})$. The existence of such points follows as above by subdividing a geodesic joining x_{n_i+1} and $x_{n_{i+1}}$ in $X^{j_{i+1}}$.

The sequence $z_1^0 = x, \ldots, z_{N_0}^0 = x_{n_1}, z_1^1 = x_{n_1+1}, \ldots, z_{N_{k'}}^{k'} = y$ still has the property that consecutive terms are at Euclidean distance at most ϵ. From

$$\begin{aligned}\sum_{j=n_{i+1}}^{n_{i+1}-1} \|x_{j+1} - x_j\| \ge \|x_{n_{i+1}} - x_{n_i+1}\| &\ge \eta^{-1} d_{X^{j_{i+1}}}(x_{n_i+1}, x_{n_{i+1}}) \\ &\ge \eta^{-1} \sum_{j=1}^{N_i-1} \|z_j^i - z_{j+1}^i\|\end{aligned}$$

we see that the length of the new sequence is at most η times the length of the original sequence.

Let $\phi : [0,\infty) \to [0,\infty)$ be a continuous, monotone and definable function such that $d_X(x,y) \leq \phi(\|x-y\|)$ for all $x,y \in X$ (compare Section 2). Then

$$\begin{aligned} d_X(x,y) &\leq \sum_{i=0}^{k'} \sum_{j=1}^{N_i-1} d_{X^{j_{i+1}}}(z_{j+1}^i, z_j^i) + \sum_{i=1}^{k'-1} d_X(x_{n_i}, x_{n_i+1}) \\ &\leq \eta \sum_{i=0}^{k'} \sum_{j=1}^{N_i-1} \|z_{j+1}^i - z_j^i\| + k\phi(\epsilon) \\ &\leq \eta^2 \sum_{i=1}^{N-1} \|x_i - x_{i+1}\| + k\phi(\epsilon). \end{aligned}$$

It follows that $d_X(x,y) \leq \eta^2 d_X^\epsilon(x,y) + k\phi(\epsilon)$. Letting ϵ tend to 0 we obtain $d_X(x,y) \leq \eta^2 \liminf_{\epsilon\to 0} d_X^\epsilon(x,y)$. Since $\eta > 1$ was arbitrary, we even have $d_X(x,y) \leq \liminf_{\epsilon \to 0} d_X^\epsilon(x,y)$ and the lemma is proved. □

Proof of Proposition 3.4. Proceeding as in the previous proof, but with the explicit choice $\phi(t) = Ct$, we get for each i, all $\epsilon > 0$ and $\eta > 1$,

$$d_{X_i}(x_i,y_i) \leq \eta^2 d_{X_i}^\epsilon(x_i,y_i) + C(\eta)\epsilon,$$

where $C(\eta)$ only depends on η, but not on i.

Fix $\epsilon > 0$ and $\eta > 1$. Choose an ϵ-path c in X between x and y of length $l(c) \leq d_X^\epsilon(x,y) + \epsilon$. Since X is the Hausdorff limit of $X_1, X_2, \ldots$, we find a sequence of 2ϵ-paths c_i in X_i between x_i and y_i converging to c. The triangle inequality implies $l(c) = \lim_{i\to\infty} l(c_i)$. On the other hand, $l(c_i) \geq d_{X_i}^{2\epsilon}(x_i,y_i) \geq \eta^{-2}\left(d_{X_i}(x_i,y_i) - 2C(\eta)\epsilon\right)$.

We deduce that

$$d_X^\epsilon(x,y) \geq l(c) - \epsilon \geq \eta^{-2}\left(\limsup_{i\to\infty} d_{X_i}(x_i,y_i) - 2C(\eta)\epsilon\right) - \epsilon.$$

Letting ϵ tend to 0 and afterwards η tend to 1 we obtain $d_X(x,y) \geq \limsup_{i\to\infty} d_{X_i}(x_i,y_i)$.

Now let us prove the other direction.

Since the C-normal sets $X_1, X_2, \ldots$ belong to a fixed bounded definable family, their geodesic diameters are uniformly bounded.

Let γ_i be a geodesic in X_i between x_i and y_i. Given $\epsilon > 0$, we can choose sufficiently many points on γ_i in order to get an ϵ-path between x_i and y_i whose length is not larger than $d_{X_i}(x_i,y_i)$. Actually, since the length of γ_i is uniformly bounded, the number of points needed is bounded from above by some number $N(\epsilon)$ which is independent of i.

Passing to a subsequence if necessary, we can assume that these ϵ-paths converge to an ϵ-path between x and y. The triangle inequality implies that its length is bounded by $\liminf_{i\to\infty} d_{X_i}(x_i, y_i)$. Therefore $d^\epsilon_X(x, y) \le \liminf_{i\to\infty} d_{X_i}(x_i, y_i)$. Taking the limit as ϵ tends to 0 yields $d_X(x, y) \le \liminf_{i\to\infty} d_{X_i}(x_i, y_i)$. □

6. Convergence of normal families

LEMMA 6.1. *In the situation of Proposition 3.5, suppose that $x_i \in X_i^{\mu_x}$ converges to $x \in X^{\mu_x}$ and that $y_i \in X_i^{\mu_y}$ converges to $y \in X^{\mu_y}$ as $i \to \infty$. Then*

$$d_X(x, y) = \lim_{i\to\infty} d_{X_i}(x_i, y_i).$$

Proof. Choose $\epsilon > 0$ and a sequence $x = x^1, y^1, \dots, x^N, y^N = y$ such that x^j, y^j belong to the same X^{μ_j}, y^j and x^{j+1} belong to $X^{\mu_j,\mu_{j+1}}$ and such that

$$\sum_{j=1}^N d_{X^{\mu_j}}(x^j, y^j) \le d_X(x, y) + \epsilon.$$

We can choose a sequence $x_i^1 = x_i, y_i^1, x_i^2, y_i^2, \dots, x_i^N, y_i^N = y_i$ in each X_i such that x_i^j, y_i^j belong to $X_i^{\mu_j}$; y_i^j and x_i^{j+1} belong to $X_i^{\mu_j,\mu_{j+1}}$ and such that $x_i^j \to x^j, y_i^j \to y^j$ for $i \to \infty$.

By Proposition 3.4, the distances $d_{X_i^{\mu_j}}(x_i^j, y_i^j)$ converge to $d_{X^{\mu_j}}(x^j, y^j)$ as $i \to \infty$. Since $\sum_{j=1}^N d_{X_i^{\mu_j}}(x_i^j, y_i^j) \ge d_{X_i}(x_i, y_i)$ we obtain

$$\begin{aligned} d_X(x, y) &\ge \sum_{j=1}^N d_{X^{\mu_j}}(x^j, y^j) - \epsilon = \lim_{i\to\infty} \sum_{j=1}^N d_{X_i^{\mu_j}}(x_i^j, y_i^j) - \epsilon \\ &\ge \limsup_{i\to\infty} d_{X_i}(x_i, y_i) - \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we thus have $d_X(x, y) \ge \limsup_{i\to\infty} d_{X_i}(x_i, y_i)$.

For the other direction, fix $\eta > 1$. By subdividing if necessary, we can assume that each X_i^j is η-normal.

Define

$$\tilde{d}_{X_i}(x, y) = \inf\left\{\sum_{j=1}^N \|x_i^j - y_i^j\|\right\},$$

where the inf is taken over all finite sequences $x_i^1 = x_i, y_i^1, x_i^2, y_i^2, \dots, x_i^N, y_i^N = y$ such that x_i^j and y_i^j lie in the same $X_i^{\mu_j}$ and y_i^j and x_i^{j+1} lie in $X^{\mu_j,\mu_{j+1}}$.

Clearly $\tilde{d}_{X_i} \le d_{X_i} \le \eta\tilde{d}_{X_i}$. Working with $\tilde{d}_{X_i}$ has the advantage that we can use a uniform bound on the number N, namely the number of sets in the description of X_i as piecewise definable space. This follows at once from triangle inequality for the Euclidean distance. We also get that the inf is a minimum.

Choose for each i a minimal sequence $x_i = x_i^1, y_i^1, \ldots, x_i^N, y_i^N = y_i$ as above. By passing to a subsequence, we can assume that $x_i^j \to x^j, y_i^j \to y^j$ for $i \to \infty$. Then $x^1 = x, y^N = y$, $x^j, y^j \in X^{\mu_j}$, $y^j, x^{j+1} \in X^{\mu_j, \mu_{j+1}}$.

By Proposition 3.4 we get that X^{μ_j} is η-normal and therefore

$$\begin{aligned} d_X(x,y) &\leq \sum_{j=1}^N d_{X^{\mu_j}}(x^j, y^j) \leq \eta \sum_{j=1}^N \|x^j - y^j\| = \eta \lim_{i\to\infty} \sum_{j=1}^N \|x_i^j - y_i^j\| \\ &= \eta \lim_{i\to\infty} \tilde{d}_{X_i}(x_i, y_i) \leq \eta \liminf_{i\to\infty} d_{X_i}(x_i, y_i). \end{aligned}$$

This is true for every $\eta > 1$. Hence $d_X(x,y) \leq \liminf_{i\to\infty} d_{X_i}(x_i, y_i)$. $\square$

Proof of Proposition 3.5. Fix $\epsilon > 0$ and a finite ϵ-dense net $\{x^1, \ldots, x^k\}$ in (X, d_X) (this means that each point of X is at distance at most ϵ from one of the points $x^1, \ldots, x^k$). Since X is piecewise definable of bounded geodesic diameter, the existence of such a net is clear.

Each point x^j of this net lies in (at least) one of the sets $X^1, \ldots, X^k$, say X^{μ_j}. We choose a sequence of points $x_i^j \in X_i^{\mu_j}$ converging to x^j.

By Lemma 6.1, $\lim_{i\to\infty} d_{X_i}(x_i^{j_1}, x_i^{j_2}) = d_X(x^{j_1}, x^{j_2})$.

We claim that $\{x_i^1, \ldots, x_i^k\}$ is a 2ϵ-net in X_i for i sufficiently large. If not, we could find a sequence of points $p_i \in X_i$ with distance to these sets at least 2ϵ. By passing to subsequences, we can assume that $p_i \in X_i^\mu$ for some fixed μ and that p_i converges to some $p \in X^\mu$. Then the distance from p to $\{x^1, \ldots, x^k\}$ is at least 2ϵ (by Lemma 6.1), which is a contradiction.

Since the Gromov-Hausdorff distance between a metric space and a finite ϵ-net in it is bounded by ϵ, we get, using the triangle inequality for the Gromov-Hausdorff distance, that (X_i, d_{X_i}) converges in the Gromov-Hausdorff distance to (X, d_X).

The fact that $\pi(X)$ is the Hausdorff limit of the sequence $X_1, X_2, \ldots$ is trivial. $\square$

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