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LOOKING FOR MINIMAL GRADED BETTI NUMBERS

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Abstract. We consider O-sequences that occur for arithmetically Cohen-Macaulay (ACM) schemes X of codimension three in $\mathbb{P}^n$. These are Hilbert functions φ of Artinian algebras that are quotients of the coordinate ring of X by a linear system of parameters. Using suitable decompositions of φ, we determine the minimal number of generators possible in some degree c for the defining ideal of any such ACM scheme having the given O-sequence. We apply this result to construct Artinian Gorenstein O-sequences φ of codimension 3 such that the poset of all graded Betti sequences of the Artinian Gorenstein algebras with Hilbert function φ admits more than one minimal element. Finally, for all 3-codimensional complete intersection O-sequences we obtain conditions under which the corresponding poset of graded Betti sequences has more than one minimal element.

Introduction

In the last few years several authors have investigated the postulation Hilbert scheme $\mathbb{H}\mathrm{ilb}^H(\mathbb{P}^n)$, the locally closed subscheme of $\mathbb{H}\mathrm{ilb}^d(\mathbb{P}^n)$ parametrizing the subschemes of $\mathbb{P}^n$ having Hilbert function H. Here $\mathbb{P}^n$ is the n-dimensional projective space over an algebraically closed field k, and $\mathbb{H}\mathrm{ilb}^d(\mathbb{P}^n)$ is the punctual Hilbert scheme parametrizing degree-d punctual subschemes of $\mathbb{P}^n$. One goal was to understand when such a postulation Hilbert scheme is reducible or irreducible (see [Go1], [Ma], [Pa], and [IK] for some work on this subject). In a previous paper [RZ] it was shown that when H can admit more than one minimal graded Betti sequence then the postulation Hilbert scheme $\mathbb{H}\mathrm{ilb}^H(\mathbb{P}^n)$ must be reducible. This motivates us to find conditions on the Hilbert functions H of 0-dimensional schemes such that the partially ordered set $\mathcal{B}_H$ of graded Betti numbers that occur for $\mathbb{H}\mathrm{ilb}^H(\mathbb{P}^n)$ have more than one minimal element. This work is part of a more complete study of determining the sets $\mathcal{B}_H$, which is still one of the most important open problems in this field.

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For $n = 2$ the poset $\mathcal{B}_H$ has only one minimal element for each H (see, for instance, [Ca]). For $n = 3$, using the affirmative answer to the minimal resolution conjecture, we conclude that the maximal Hilbert functions (i.e., maximal among the Hilbert functions of schemes with the same degree) have only one minimum for the graded Betti numbers. On the other hand, Richert [Ri], generalizing an example of Charalambous and Evans [CE], and Sabourin [Sa] produced an infinite class of Hilbert functions H for which $\mathcal{B}_H$ has no minimum element. Richert [Ri] gave some examples of Hilbert functions H of complete intersections of type (a, b, c), with $c = a + b$, for which again $\mathcal{B}_H$ has more than one minimal element. In the present paper, as a consequence of our main results, we determine all complete intersection Hilbert functions $H = H_{CI(a,b,c)}$ for which $\mathcal{B}_H$ has more than one minimal element (except when $c = ab + 1$) (see Proposition 2.3).

In this paper we focus on the number of minimal generators in a fixed degree c for the defining ideal of 0-dimensional schemes in $\mathbb{P}^3$ with a given Hilbert function H. In particular, we look for the smallest number of degree-c generators compatible with H.

To this aim we make use of particular linear decompositions of the Hilbert function which will permit us to construct special schemes which realize the required minimal number of generators in the fixed degree.

Since we are interested in Hilbert functions of 0-dimensional schemes X, respectively ACM schemes of higher dimension s, we will work with those O-sequences φ that are the first, respectively $(s+1)$st, differences of the Hilbert functions H_X. Thus φ is the Hilbert function of an Artinian quotient of the coordinate ring of X by a linear system of parameters.

We will restrict our work to the codimension three case in which the computations are more manageable, but we believe that these ideas can be generalized to higher codimension. We will apply our results, in particular, to Gorenstein sequences and complete intersection O-sequences.

1. Decompositions of O-sequences

We begin by stating the terminology and our general approach for looking for possible graded Betti numbers compatible with a fixed Hilbert function.

Let $R = k[x_1, \dots, x_r]$ and let $A = R/I = \oplus A_i$ be a standard graded Artinian k-algebra quotient of R. We will say that A has codimension r when $\dim_k A_1 = r$. We denote by $\nu_{A,i}(n)$ the number of i-th syzygies of A in degree n, with $n > 0$ and $0 \leq i \leq r - 1$, in the minimal graded free resolution of A as $k[x_1, \dots, x_r]$-module. (Of course, $\nu_{A,0}(n)$ will denote the number of generators of A in degree n; to simplify the notation we will often just write $\nu_A(n)$.)

The Hilbert function of a standard graded Artinian k-algebra will be called an Artinian O-sequence. If φ is an Artinian O-sequence, we define

$$\sigma(\varphi) = \max\{n \in \mathbb{N} \mid \varphi(n) > 0\}.$$

We set

$$\mathcal{S}eq_r = \{r\text{-codimensional Artinian } O\text{-sequences}\},$$

$$\mathcal{A}rt_r = \{\text{graded Artinian standard } k\text{-algebras quotients of } k[x_1, \dots, x_r]\},$$

and for every $\varphi \in \mathcal{S}eq_r$ we define

$$\nu_{\varphi,i}^{\min}(n) = \min\{\nu_{A,i}(n) \mid A \in \mathcal{A}rt_r \text{ and } H_A = \varphi\},$$
$$\nu_{\varphi,i}^{\max}(n) = \max\{\nu_{A,i}(n) \mid A \in \mathcal{A}rt_r \text{ and } H_A = \varphi\}.$$

Despite the fact that the function $\nu_{\varphi,i}^{\max}$ is completely determined by works of Bigatti [Bi], Hulett [Hu], and Pardue [Pa], little is known about the function $\nu_{\varphi,i}^{\min}$. We will write $\nu_{\varphi}^{\max}$ and $\nu_{\varphi}^{\min}$ instead of $\nu_{\varphi,0}^{\max}$ and $\nu_{\varphi,0}^{\min}$, respectively. Our main goal in this paper is to determine $\nu_{\varphi}^{\min}$ when $\varphi \in \mathcal{S}eq_3$ (in codimension $r = 2$ everything is known).

Let $\varphi = (1\ 3\ 6\ \dots\ 0\ \rightarrow)$ be a 3-codimensional Artinian O-sequence, i.e., φ is the $\underline{h}$-vector of an Artinian algebra; in other words, $\varphi = \Delta H_X$ is the first difference of the Hilbert function of a 0-dimensional scheme $X \subset \mathbb{P}_k^3$.

If α and i are positive integers, we recall that the *i-binomial expansion* of α is the unique expression

$$\alpha = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j},$$

where $m_i > m_{i-1} > \cdots > m_j \geq j \geq 1$.

If α has i-binomial expansion as above, then we set

$$\alpha^{\langle i \rangle} = \binom{m_i + 1}{i+1} + \binom{m_{i-1}+1}{i} + \cdots + \binom{m_j + 1}{j+1}.$$

Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a function, $\varphi(0) = 1$. The theorem of Macaulay (see [Mac] and [St]) says that φ is the Hilbert function of a standard graded k-algebra iff $\varphi(n+1) \leq \varphi(n)^{\langle n \rangle}$, for all $n \geq 1$. Notice that, for Artinian O-sequences φ of codimension two, if $\varphi(n-1) \geq \varphi(n)$, then $\varphi(j) \geq \varphi(j+i)$ for all $j \geq n$. This implies that

$$\varphi(n-1) < \varphi(n) \Rightarrow \varphi(n) = n + 1. \tag{1}$$

Moreover, we recall the following Gotzmann Persistence Theorem.

THEOREM 1.1 ([Go2], [BH]). *Let R/I be a standard graded k-algebra with Hilbert function φ. Let $d > 0$ be an integer such that $\varphi(d+1) = \varphi(d)^{\langle d \rangle}$. Let J be the ideal generated by the components of I of degree $\leq d$ and let ψ be the Hilbert function of R/J (so that $\psi(n) = \varphi(n)$ for $n \leq d$). Then $\psi(n+1) = \psi(n)^{\langle n \rangle}$, for all $n \geq d$.*

Let $c \geq 3$ be an integer. Set $\alpha := \varphi(c-2)$ and $h := \alpha^{\langle c-2 \rangle} - \alpha$. Note that $0 \leq \alpha \leq \binom{c}{2}$.

If $\alpha = \binom{c}{2} = \binom{c}{c-2}$, then $\alpha^{\langle c-2 \rangle} = \binom{c+1}{c-1} = \binom{c+1}{2}$ and hence $h = c$.

If $0 \leq \alpha < \binom{c}{2}$, then

$$\alpha = (c-1) + (c-2) + \cdots + (c-h) + \alpha_0$$

with $0 \leq \alpha_0 < c - h - 1$; now, from $\alpha < \binom{c}{2}$ we get $c - h > 1$ and hence $0 \leq h \leq c-2$. We set $\overline{h} = h$ if $\alpha < \binom{c}{2}$, and $\overline{h} = h - 1$ if $\alpha = \binom{c}{2}$.

Next we set

$$u := \varphi(c-1) - \alpha,\ v := \varphi(c) - \alpha - u.$$

Since φ is an O-sequence we have $u \leq h$. The integer $2u - v - \overline{h}$ (which depends on c) will be crucial in the computation of $\nu_\varphi^{\min}(c)$; we therefore set $G = G(c) = 2u - v - \overline{h}$.

We will henceforth deal with O-sequences which satisfy a certain decreasing condition for every $i \geq c$.

Definition 1.2. Let $c \geq 3$ be an integer. An O-sequence φ is said to be of $D^*(c)$-type if it satisfies the following condition:

- If either $v \geq 0$ or $v < 0$ and $2u - h \geq 0$, then $\varphi(i) \geq \varphi(i+1)$ for $i \geq c$.
- If $v < 0$ and $2u - h < 0$, then
  - $\varphi(i) \geq \varphi(i+1) - v$ for all $i \geq c$ such that $\varphi(i) > 2\sum_{j=0}^{u-1}(c - 2j - 1) + \sum_{j=1}^{h-2u+v}(c - j - 2u)$;
  - $\varphi(i) \geq \varphi(i+1)$ for the other values of i, $i \geq c$.

Note that many important O-sequences (e.g., complete intersection sequences, Gorenstein sequences, and generic O-sequences) satisfy this condition for appropriate values of c (cf. Proposition 2.2 and Proposition 2.3).

Proposition 1.3. *Let φ be any 3-codimensional Artinian O-sequence. With the above notation we have*

$$\nu_\varphi^{\min}(c) \geq \max\{0, G\}.$$

Proof. Set $x := \varphi(c-3)$. Then $\Delta^3\varphi(c) = \alpha - 2u + v - x$. Now, for every Artinian algebra A with Hilbert function φ, we know that

$$-\nu_A(c) + \nu_{A,1}(c) - \nu_{A,2}(c) = \Delta^3\varphi(c) = \alpha - 2u + v - x,$$

so we get $\nu_A(c) = \nu_{A,1}(c) - \nu_{A,2}(c) - \alpha + 2u - v + x$. Next, if $\alpha = \binom{c}{2}$, then $h = c$ and $\nu_{A,2}(c) = 0$. Therefore $\nu_A(c) \geq -\alpha + 2u - v + x = 2u - v - h + 1$. If $\alpha < \binom{c}{2}$, then by the above mentioned result on maximal graded Betti numbers, we have $\nu_{A,2}(c) \leq -\Delta\varphi(c-2) + h = x - \alpha + h$. Therefore $\nu_A(c) \geq -\alpha + 2u - v + x - x + \alpha - h = 2u - v - h$. Hence we have $\nu_\varphi^{\min}(c) \geq \max\{0, 2u - v - \overline{h}\}$. $\square$

Indeed, we will show in Corollary 1.23 that, except for a single case, the value of $\nu_\varphi^{\min}(c)$ is exactly $\max\{0, 2u - v - \overline{h}\}$.

In order to determine $\nu_\varphi^{\min}(c)$ we start with the following observation.

Remark 1.4. Suppose that $u = h$, i.e., that φ has maximal growth in $c-1$, and set $k := \varphi(c-1)^{\langle c-1\rangle} - \varphi(c-1)$. We see that if $\alpha = \varphi(c-2) = \binom{c}{2}$, then $u = h = c$ and $\varphi(c-1) = \binom{c+1}{2}$. Therefore $\varphi(c-1)^{\langle c-1\rangle} = \binom{c+2}{2}$, and hence $k = c + 1 = h + 1$. If $\alpha = \varphi(c-2) < \binom{c}{2}$, then $h \leq c - 2$ and $\varphi(c-1) = c + (c-1) + (c-2) + \cdots + (c-h+1) + \alpha_0$, and hence $k = h$. In this situation, by the Persistence Theorem of Gotzmann (Theorem 1.1) we have

$$\nu_\varphi^{\min}(c) = \nu_\varphi^{\max}(c) = \varphi(c-1)^{\langle c-1\rangle} - \varphi(c) = k - v = \max\{0, 2u - v - \overline{h}\}.$$

Moreover, if $\alpha = \varphi(c-2) < c-1$, then $h = 0$, $u = 0$ and $v \leq 0$. Again by the Persistence Theorem (Theorem 1.1) we have

$$\nu_\varphi^{\min}(c) = \nu_\varphi^{\max}(c) = \alpha - \varphi(c) = -v = \max\{0, 2u - v - \overline{h}\}.$$

By the above remark, we can deal with the case when in degree $c-1$ we have a growth which is not maximal, i.e., $0 \leq u < h$ (in particular, $\alpha \geq c-1$).

We will now give certain preferred decompositions (which we will call *c-canonical*) of 3-codimensional O-sequences φ of $D^*(c)$-type that will be used to construct 0-dimensional subschemes of $\mathbb{P}^3$ having $\nu_\varphi^{\min}$ minimal generators in degree c, i.e., the minimum compatible with the O-sequence φ. These decompositions were introduced in [RZ1] and they are a generalization of the maximal decompositions used in [GHS].

Definition 1.5. Let φ be an O-sequence with finite support, $\varphi(1) = r \geq 2$. A *linear decomposition* of φ is a succession of O-sequences of embedding dimension $< r$, $(\varphi_1, \varphi_2, \ldots, \varphi_d)$, such that

(1) $\varphi_1 \geq \varphi_2 \geq \cdots \geq \varphi_d$ (using termwise order),

(2) $\varphi(n) = \sum_{j=1}^{d} \varphi_j(n - j + 1)$ for every $n \in \mathbb{N}_0$ (we use the convention $\varphi(n) = 0$ for all $n < 0$).

A linear decomposition of φ, $(\varphi_1, \varphi_2, \ldots, \varphi_d)$ is called *principal* if

$$d = \min\left\{ n \in \mathbb{N} \;\mid\; \varphi(n) < \binom{r+n-1}{n} \right\}.$$

Thus, if φ is a 3-codimensional O-sequence of $D^*(c)$-type, we will define a principal linear decomposition of it, which we will call *c-canonical*, in the following way. We first define suitable functions φ_i, then show in Proposition 1.9 that they form a linear decomposition of φ. For the sake of simplicity we assume that $\varphi(c-2) < \binom{c}{2}$.

Construction 1.6. For $1 \leq j \leq d$ we define inductively, for every $n \leq c - j - 1$,

$$\varphi_j(n) = \min \left\{ n + 1, \varphi(n + j - 1) - \sum_{s=1}^{j-1} \varphi_s(n + j - s) \right\}.$$

To complete the definition of the decompositions we need to define the values of each φ_j for $n \geq c - j$. To do that we distinguish several cases.

Case 1. $v \geq 0$.
We define

$$\varphi_j(n) = \min \left\{ \varphi_j(c - j + 1),\ \varphi(n + j - 1) - \sum_{s=1}^{j-1} \varphi_s(n + j - s) \right\}$$

for $n \geq c - j + 2$.

Moreover, we define for every $1 \leq j \leq v$ the integers

$$\varphi_j(c - j) = c - j + 1, \quad \varphi_j(c - j + 1) = c - j + 2.$$

(Of course, when $v = 0$, this part is empty.)

For the other values of j we distinguish two cases:

Case 1a. $v \geq u$.
In this case for $v < j \leq d$ we define

$$\varphi_j(c - j) = \min \left\{ \varphi(c - j - 1), \varphi(c - 1) - \sum_{s=1}^{j-1} \varphi_s(c - s) \right\},$$

$$\varphi_j(c - j + 1) = \min \left\{ \varphi(c - j - 1), \varphi(c) - \sum_{s=1}^{j-1} \varphi_s(c + 1 - s) \right\}.$$

Case 1b. $0 \leq v < u$.
We set $m = \min\{h - u, u - v\}$ and define for $v + 1 \leq j \leq v + 2m$

$$\varphi_j(c - j) = \varphi_j(c - j + 1) = \begin{cases} \varphi_j(c - j - 1) & \text{if } j - v \text{ is odd,} \\ \varphi_j(c - j - 1) + 1 & \text{if } j - v \text{ is even,} \end{cases}$$

and for $v + 2m < j \le d$

$$\begin{aligned}\varphi_j(c-j) &= \varphi_j(c-j+1)\\ &= \begin{cases}\min\left\{\varphi_j(c-j-1), \varphi(c-1) - \sum_{s=1}^{j-1}\varphi_s(c-s)\right\} & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } m = u - v, & \\ \min\left\{\varphi_j(c-j-1)+1, \varphi(c-1) - \sum_{s=1}^{j-1}\varphi_s(c-s)\right\} & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } m = h - u. & \end{cases}\end{aligned}$$

Case 2. $v < 0$.

In this case we set $t = \min\{u, h-u\}$ and define for $1 \le j \le 2t$

$$\varphi_j(c-j) = \begin{cases}\varphi_j(c-j-1) & \text{if } j \text{ is odd,}\\ \varphi_j(c-j-1)+1 & \text{if } j \text{ is even,}\end{cases}$$

and

$$\varphi_j(n) = \min\left\{\varphi_j(c-j), \varphi(n+j-1) - \sum_{s=1}^{j-1}\varphi_s(n+j-s)\right\}$$

for $n \ge c - j + 1$.

Case 2a. $h \le 2u$, i.e., $t = h - u$.

In this case we define for $2(h-u)+1 \le j \le d$

$$\varphi_j(c-j) = \min\left\{\varphi_j(c-j-1)+1, \varphi(c-1) - \sum_{s=1}^{j-1}\varphi_s(c-s)\right\}$$

and

$$\varphi_j(n) = \min\left\{\varphi_j(c-j), \varphi(n+j-1) - \sum_{s=1}^{j-1}\varphi_s(n+j-s)\right\}$$

for $n \ge c - j + 1$.

Case 2b. $h > 2u$, i.e., $t = u$.

In this case we define

$$\varphi_j(c-j) = \varphi_j(c-j-1)$$

for every $j \ge 2u + 1$.

Now, if either $2u < h \le 2u - v$, or $h > 2u - v$ and $\alpha_0 \ne 0$, we define for $j = 2u + 1$

$$\varphi_j(n) = \min\left\{\varphi_j(c-j), \varphi_j(n+j-1) - \sum_{s=1}^{j-1}\varphi_s(n+j-s)\right\}$$

for $n \geq c - j + 1$.

Also, for $2u + 1 < j \leq h + v + 1$ we define for $n \geq c - j + 1$

$$\varphi_j(n) = \min\left\{\varphi_j(c-j), \varphi(n+j-1) - \sum_{s=1}^{j-1} \varphi_s(n+j-s)\right\}.$$

(Of course this part is empty if $h \leq 2u - v$.)

Finally, for $\max\{2u+2, h+v+2\} \leq j \leq d$ we define for $n \geq c - j + 1$

$$\varphi_j(n) = \min\left\{\varphi_j(n-1) - 1, \varphi(n+j-1) - \sum_{s=1}^{j-1} \varphi_s(n+j-s)\right\}.$$

If $h > 2u - v$ and $\alpha_0 = 0$, then, we modify this definition slightly as follows: For $2u + 1 < j \leq h + v$ we define for $n \geq c - j + 1$

$$\varphi_j(n) = \min\left\{\varphi_j(c-j), \varphi(n+j-1) - \sum_{s=1}^{j-1} \varphi_s(n+j-s)\right\},$$

and for $h + v + 1 \leq j \leq d$ we define for $n \geq c - j + 1$

$$\varphi_j(n) = \min\left\{\varphi_j(n-1) - 1, \varphi(n+j-1) - \sum_{s=1}^{j-1} \varphi_s(n+j-s)\right\}.$$

We will show that the tuple $(\varphi_1, \varphi_2, \ldots, \varphi_d)$ thus defined provides a principal linear decomposition of φ.

Remark 1.7. Observe that for every n there is only one value of j_n for which

$$\varphi_{j_n}(n - j_n + 1) = \varphi(n) - \sum_{s=1}^{j_n - 1} \varphi_s(n - s + 1) > 0.$$

Moreover, for every $j > j_n$ we have $\varphi_j(n - j + 1) = 0$.

Remark 1.8. Note that in any case the values of $\varphi_j(n)$ are among the following:

(1) $n + 1$;
(2) $\varphi(n) - \sum_{s=1}^{j-1} \varphi_s(n - s + j)$;
(3) $\varphi_j(n-1)$;
(4) $\varphi_j(n-1) - 1$.

Proposition 1.9. *In the above notation, let φ be an O-sequence of $D^*(c)$-type as in Definition 1.2. Then the sequences $\{\varphi_i\}_{i=1,\ldots,d}$ defined above give a principal linear decomposition of φ.*

Proof. First note that, in our definitions, $\varphi_j(n) < \varphi_j(n+1)$ implies $\varphi_j(n) = n+1 = \varphi_j(n+1) - 1$; this is equivalent, by relation (1), to each φ_j being an O-sequence (of codimension two). Moreover, by Remark 1.7, the $\{\varphi_j\}$ are O-sequences which satisfy condition (2) of Definition 1.5 of the "principal decomposition" of φ. Thus, to finish the proof it is enough to show that for every $j = 1, \ldots, d$ and for every n we have $\varphi_j(n) \geq \varphi_{j+1}(n)$. According to Remark 1.8 we have four possibilities for the value of $\varphi_j(n)$. If $\varphi_j(n) = n+1$ the conclusion follows since a 2-codimensional O-sequence can assume at most the value $n+1$ at n. If $\varphi_j(n) = \varphi(n+j-1) - \sum_{s=1}^{j-1} \varphi_s(n+j-s)$, we must have $\varphi_{j+1}(n-1) = 0$ and hence $\varphi_{j+1}(n) = 0$ and we are done. Therefore we can assume that $\varphi_j(n)$ is not of these two forms. When $\varphi_j(n) = \varphi_j(n-1)$ the only non-trivial cases can occur for $n = c-j$ and $n = c-j+1$. The conclusion follows by a straightforward analysis of each case in which the above equality holds. The only remaining case is $\varphi_j(n) = \varphi_j(n-1) - 1$, but this can only occur in Case 2b of the definition for $n \geq c - j + 1$. In this situation we have $\varphi_j(n) = \varphi_j(n-1) - 1 = \varphi_j(c-j-1) - n + c - j = 2c - 2j - n$ and $\varphi_{j+1}(n) = \varphi_{j+1}(n-1) - 1 = \varphi_{j+1}(c-j-2) - n + c - j - 1 = 2c - 2j - 2 - n$. $\square$

Definition 1.10. If φ is an O-sequence of $D^*(c)$-type, the c-canonical decomposition of φ will be denoted by $\lambda(\varphi, c)$.

Remark 1.11. Notice that the above decomposition of the O-sequence φ can be extended to the case when $\varphi(c-2) = \binom{c}{2}$ by replacing h by $\overline{h} = h-1 = c-1$ at every occurrence.

Remark 1.12. We will see that the c-canonical decompositions of φ provide schemes X, $\Delta H_X = \varphi$, with the smallest number of minimal generators in degree c. Indeed, the c-canonical decompositions are maximal among the decompositions with this property.

We present two examples to clarify the c-canonical decomposition of an O-sequence.

Example 1.13. The following table shows our decomposition $\lambda(\varphi, 21)$ for the O-sequence

$$\varphi = \{1\ 3\ 6\ 10\ 15\ 21\ 28\ 36\ 45\ 55\ 66\ 78\ 91\ 105$$
$$120\ 136\ 153\ 171\ 190\ 199\ 209\ 212\ 178\ 177\ 35\}$$

(The value of φ can be read off the second column; the other columns represent the 2-codimensional O-sequences which give the decomposition.) Here $c = 21$, $h = 15$, $u = 10$, and $v = 3$; hence we are in Case 1b.

n	φ	φ_1	φ_2	φ_3	$\dots$															
0	1	1																		
1	3	2	1																	
2	6	3	2	1																
3	10	4	3	2	1															
4	15	5	4	3	2	1														
5	21	6	5	4	3	2	1													
6	28	7	6	5	4	3	2	1												
7	36	8	7	6	5	4	3	2	1											
8	45	9	8	7	6	5	4	3	2	1										
9	55	10	9	8	7	6	5	4	3	2	1									
10	66	11	10	9	8	7	6	5	4	3	2	1								
11	78	12	11	10	9	8	7	6	5	4	3	2	1							
12	91	13	12	11	10	9	8	7	6	5	4	3	2	1						
13	105	14	13	12	11	10	9	8	7	6	5	4	3	2	1					
14	120	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1				
15	136	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1			
16	153	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1		
17	171	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	
18	190	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
19	199	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	4			
20	209	21	20	19	17	17	15	15	13	13	11	11	9	9	8	7	4			
21	212	22	21	20	17	17	15	15	13	13	11	11	9	9	8	7	4			
22	178	22	21	20	17	17	15	15	13	13	11	11	3							
23	177	22	21	20	17	17	15	15	13	13	11	11	2							
24	35	22	13																	

Example 1.14. The following table shows our decomposition $\lambda(\varphi, 21)$ for the O-sequence

$$\varphi = \{1\ 3\ 6\ 10\ 15\ 21\ 28\ 36\ 45\ 55\ 66\ 78\ 91\ 105$$
$$120\ 136\ 153\ 171\ 190\ 199\ 205\ 200\ 195\ 189\ 184\}$$

Here $c = 21$, $h = 15$, $u = 6$, and $v = -5$; hence we are now in Case 2b.

n	φ	φ_1	φ_2	φ_3	$\dots$															
0	1	1																		
1	3	2	1																	
2	6	3	2	1																
3	10	4	3	2	1															
4	15	5	4	3	2	1														
5	21	6	5	4	3	2	1													
6	28	7	6	5	4	3	2	1												
7	36	8	7	6	5	4	3	2	1											
8	45	9	8	7	6	5	4	3	2	1										
9	55	10	9	8	7	6	5	4	3	2	1									
10	66	11	10	9	8	7	6	5	4	3	2	1								
11	78	12	11	10	9	8	7	6	5	4	3	2	1							
12	91	13	12	11	10	9	8	7	6	5	4	3	2	1						
13	105	14	13	12	11	10	9	8	7	6	5	4	3	2	1					
14	120	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1				
15	136	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1			
16	153	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1		
17	171	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	
18	190	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
19	199	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	4			
20	205	20	20	18	18	16	16	14	14	12	12	10	10	8	7	6	4			
21	200	20	20	18	18	16	16	14	14	12	12	10	10	8	6	5	1			
22	195	20	20	18	18	16	16	14	14	12	12	10	10	8	5	2				
23	189	20	20	18	18	16	16	14	14	12	12	10	10	8	1					
24	184	20	20	18	18	16	16	14	14	12	12	10	10	4						

Before proving the main results we restate here, in our situation, some definitions and facts from [RZ1].

Definition 1.15. A subset $\mathcal{A}$ of a poset $(\mathcal{P}, \leq)$ is called a *left segment* if for every $H \in \mathcal{A}$ and $K \in \mathcal{P}$, with $K \leq H$, we have $K \in \mathcal{A}$. In particular, when $\mathcal{P} = \mathbb{N}^r$ with the order induced by the natural order on $\mathbb{N}$, a finite left segment will be called an *r-left segment*.

Definition 1.16. If $\mathcal{A}$ is an r-left segment, we denote by $G(\mathcal{A})$ the set of maximal elements of $\mathcal{A}$ and by $F(\mathcal{A})$ the set of minimal elements of $\mathbb{N}^r \setminus \mathcal{A}$.

Fix an r-left segment $\mathcal{A}$ and consider r families of hyperplanes of $\mathbb{P}^n$, $r \leq n$,

$$\{A_{ij}\}_{i,j}, \quad 1 \leq i \leq r, \quad 1 \leq j \leq a_i,$$

that are sufficiently generic in the sense that $\{A_{1j_1} \cap A_{2j_2} \cap \cdots \cap A_{rj_r}\}$ consists of $a_1 \cdot \ldots \cdot a_r$ distinct linear varieties of codimension r in $\mathbb{P}^n$.

For every $H = (j_1, \ldots, j_r) \in \mathcal{A}$ we define

$$L_H = A_{1j_1} \cap \cdots \cap A_{rj_r}.$$

With this notation we make the following definition:

Definition 1.17. The subscheme of $\mathbb{P}^n$

$$V = \bigcup_{H \in \mathcal{A}} L_H$$

will be called an *r-partial intersection* with respect to the hyperplanes $\{A_{ij}\}$ and with support on the r-left segment $\mathcal{A}$.

It was shown in [RZ1, Theorem 1.9] that every partial intersection is a reduced ACM subscheme of $\mathbb{P}^n$.

Let $V \subset \mathbb{P}^n$ be an r-partial intersection with respect to the hyperplanes $\{A_{ij}\}$, $1 \leq i \leq r$ and $1 \leq j \leq a_i$, and with support on the r-left segment $\mathcal{A}$ of size $T = (a_1, \ldots, a_r)$. Now, set $I_{A_{ij}} = (f_{ij})$, where $f_{ij} \in S_1$ for all i, j, where S is the homogeneous coordinate ring of $\mathbb{P}^n$. Finally, to every $H = (m_1, \ldots, m_r) \leq (a_1 + 1, \ldots, a_r + 1)$ we associate the form

$$P_H = \prod_{i=1}^{r} \prod_{j=1}^{m_i - 1} f_{ij}.$$

Theorem 1.18. *Let $V \subset \mathbb{P}^n$ be an r-partial intersection with support $\mathcal{A}$. Then a minimal set of generators for I_V is*

$$\{P_H \mid H \in F(\mathcal{A})\}.$$

Proof. See [RZ1, Theorem 3.1]. □

Finally we recall how we can associate a partial intersection to each linear decomposition of an Artinian O-sequence φ of codimension r.

To every such an O-sequence we can, in a canonical way, associate an r-left segment $\mathcal{A}^\varphi$ such that, if X is a partial intersection with support on $\mathcal{A}^\varphi$ in $\mathbb{P}^n$, then $\Delta^{n-r+1} H_X = \varphi$ and its graded Betti numbers are maximal with respect to H_X (in the sense of Hulett, Bigatti, Pardue). We call such an X a *maximal partial intersection associated to* φ (see [RZ1, Proposition 4.3]).

Theorem 1.19. *If $\sigma = (\varphi_1, \ldots, \varphi_d)$ is a principal linear decomposition of an O-sequence φ, then there exist $(r-1)$-partial intersections in $\mathbb{P}^n$, $V_1 \supseteq \cdots \supseteq V_d$, where each V_j is a maximal partial intersection associated to the*

O-sequence φ_j. Moreover, if $\{\pi_j\}$ are generic hyperplanes, $1 \leq j \leq d$, and $X_j = V_j \cap \pi_j$, then $X = X_1 \cup \cdots \cup X_d$ is an r-partial intersection with $\Delta^{n-r+1} H_X = \varphi$.

Proof. This is essentially Theorem 4.4 in [RZ1]. □

We call a partial intersection constructed as in Theorem 1.19 *maximal with respect to the decomposition σ.*

Now we are ready to prove our results.

THEOREM 1.20. *Let φ be a 3-codimensional Artinian O-sequence, c any integer, and let h, u and v be as defined above; suppose that φ is not increasing for $i \geq c$. If $u \geq 0$ and either $v \geq 0$ or $v < 0$ and $h \leq 2u$, then there exists a reduced 0-dimensional subscheme X of $\mathbb{P}^3$ with $\Delta H_X = \varphi$ and $\max\{0, G = 2u - v - \overline{h}\}$ generators in degree c.*

Proof. According to Theorem 1.19, if $\lambda = (\varphi_1, \ldots, \varphi_d)$ is a linear decomposition of φ, there exists a reduced subscheme (indeed a partial intersection) $X = X_1 \cup \cdots \cup X_d$ of $\mathbb{P}^3$, where each X_j is a subscheme of $\mathbb{P}^2$, such that $\Delta H_X = \varphi$ and $\Delta H_{X_j} = \varphi_j$ for every $j = 1, \ldots, d$. Moreover, when this is the case,

$$I_X = I_{X_1} + L_1 I_{X_2} + \cdots + (L_1, \ldots, L_{d-1}) I_{X_d} + (L_1, \ldots, L_d)$$

for suitable linear forms L_j. Hence, for every integer c, $\nu_X(c) \leq \sum_{j=1}^{d} \nu_{X_j}(c - j + 1)$. Now we distinguish three cases.

Case 1a. $v \geq u$. In this case, for every partial intersection $X = X_1 \cup \cdots \cup X_d$ associated to $\lambda(\varphi, c)$, since φ_j has maximal growth in $c - j + 1$, we have $\nu_{X_j}(c - j + 1) = 0$. Hence $\nu_X(c) = 0 = \max\{0, G\}$.

Case 1b. $0 \leq v < u$. Let us consider the partial intersection X with respect to the families of hyperplanes $\{x_q\}$, $\{y_r\}$ and $\{z_s\}$, which is maximal with respect to the decomposition $\lambda(\varphi, c)$, i.e., $X = X_1 \cup \cdots \cup X_d$, where each X_j is the maximal partial intersection associated to the O-sequence φ_j. In order to compute $\nu_X(c)$ observe that, if $1 \leq j \leq v$, then φ_j forces $\nu_{X_j}(c - j + 1) = 0$; if $v + 1 \leq j \leq v + 2m$ and $j - v$ is odd, then φ_j forces $\nu_{X_j}(c - j + 1) = 0$ and $\nu_{X_j}(c - j) = 1$; if $v + 1 \leq j \leq v + 2m$ and $j - v$ is even, then φ_j forces $\nu_{X_j}(c - j + 1) = 1$. On the other hand, by the construction of each X_j, we see that the generator of I_{X_j}, for $j - v$ even, in degree $c - j + 1$ is $z_1 \cdot \cdots \cdot z_{c-j+1}$, which is the same as the generator of $I_{X_{j-1}}$. Hence such a generator does not produce a minimal generator for I_X in degree c. For $j > v + 2m$ we observe that φ_j has maximal growth in $c - j$. Hence we have one generator in degree $c - j + 1$ for every $v + 2m < j \leq h$ and no generator in this degree when

$j > h$. Therefore $\nu_X(c) = h - v - 2m$ and an easy computation shows that $h - v - 2m = \max\{0, G\}$.

Case 2a. $v < 0$ and $h \leq 2u$. As in Case 1b let us consider the partial intersection $X = X_1 \cup \cdots \cup X_d$ which is maximal with respect to the decomposition $\lambda(\varphi, c)$. Therefore, since each X_j has maximal graded Betti numbers with respect to its Hilbert function, it has exactly $\nu_{X_j}(c-j+1) = \varphi_j(c-j)^{\langle c-j\rangle} - \varphi_j(c-j+1)$ generators in degree $c-j+1$. For $1 \leq j \leq 2(h-u)$, we have $\nu_{X_j}(c-j+1) = 0$ when j is odd, and $\nu_{X_j}(c-j+1) = 1$ when j is even. But the generator of I_{X_j}, for j even, in degree $c-j+1$ is the same as the generator of $I_{X_{j-1}}$, and hence it does not produce a minimal generator for X in degree c. So

$$\begin{aligned}
\nu_X(c) &= \sum_{j=2(h-u)+1}^{d} \nu_{X_j}(c-j+1) \\
&= \sum_{j=2(h-u)+1}^{d} \big(\varphi_j(c-j)^{\langle c-j\rangle} - \varphi_j(c-j+1)\big) \\
&= \sum_{j=2(h-u)+1}^{h} \big(\varphi_j(c-j)^{\langle c-j\rangle} - \varphi_j(c-j+1)\big) \\
&\qquad + \sum_{j=h+1}^{d} \big(\varphi_j(c-j)^{\langle c-j\rangle} - \varphi_j(c-j+1)\big) \\
&= \sum_{j=2(h-u)+1}^{h} \big(\varphi_j(c-j) - \varphi_j(c-j+1) + 1\big) \\
&\qquad + \sum_{j=h+1}^{d} \big(\varphi_j(c-j) - \varphi_j(c-j+1)\big) \\
&= 2u - h + \sum_{j=2(h-u)+1}^{d} \big(\varphi_j(c-j) - \varphi_j(c-j+1)\big) \\
&= 2u - h - v = G = \max\{0, G\}. \qquad \square
\end{aligned}$$

In order to complete our analysis of $\nu_X(c)$ in the last possible case we need the following technical result.

LEMMA 1.21. *Let $\varphi_1, \ldots, \varphi_w$ be 2-codimensional Artinian O-sequences and $p > w$ an integer such that $\varphi_j(n) = n+1$ for $n \leq p-j-1$, $\varphi_j(p-j) = p-j$, $\varphi_j(n) > \varphi_j(n+1)$ for $p-j \leq n \leq \sigma(\varphi_j)$ and $1 \leq j \leq w$, and $\varphi_j(n) \geq \varphi_{j+1}(n) + 2$ for $p-j \leq n \leq \sigma(\varphi_{j+1})$ and $1 \leq j < w$. Then there* *exist* 2*-partial intersections* $V_1 \supset V_2 \supset \cdots \supset V_w$ *in* $\mathbb{P}^3$*, with* $\Delta^2 H_{V_j} = \varphi_j$*, such that the minimal generator of* I_{V_j} *of degree* $p-j$ *coincides with a minimal generator of* $I_{V_{j+1}}$ *for* $1 \le j < w$*.*

Proof. For every $n \in \mathbb{N}$ let $D_n = \{(x,y) \in \mathbb{N}^2 \mid x+y = n+2\}$, and for $1 \le j \le w$ set

$$R_j = \begin{cases} \{(x,y) \mid (x,y) \le (p-j, p-j+1)\} \text{ if } j \text{ is odd,} \\ \{(x,y) \mid (x,y) \le (p-j+1, p-j)\} \text{ if } j \text{ is even.} \end{cases}$$

We define

$$x_{nj} = \min\{\text{first component of } K \mid K \in D_n \cap R_j\}$$

and

$$P_{nj} = \{(x_{nj}+k, n+2-x_{nj}-k) \mid 0 \le k \le \varphi_j(n)-1\}.$$

Note that $R_1 \supset R_2 \supset \cdots \supset R_w$ and $x_{nj} = x_{n,j+1} = 1$ if $0 \le n \le p-j-2$, $x_{p-j-1,j} = x_{p-j-1,j+1} - 1 = 1$, $x_{n,j} = x_{n,j+1} - 2$ if $p-j \le n \le \sigma(\varphi_{j+1})$. We set $\mathcal{A}_j = \bigcup_n P_{nj}$. Then $\mathcal{A}_j$ is a left segment since φ_j is decreasing for $p-j \le n \le \sigma(\varphi_j)$. Moreover, $\mathcal{A}_{j+1} \subset \mathcal{A}_j$. In fact, if $K = (x,y) \in \mathcal{A}_{j+1}$, then $K \in D_{x+y-2} \cap R_{j+1}$ and $x_{n,j+1} \le x \le x_{n,j+1} + \varphi_{j+1}(n) - 1$. Hence $K \in D_{x+y-2} \cap R_j$ and $x_{nj} \le x \le x_{nj} + \varphi_j(n) - 1$ since $x_{nj} \le x_{n,j+1}$ and, by assumption, $\varphi_j(n) \ge \varphi_{j+1}(n) + 2$ for $p-j \le n \le \sigma(\varphi_{j+1})$.

Now, if we denote by $\{y_r\}$ and $\{z_s\}$ the families of linear forms which we use for defining the partial intersection associated to the left segment $\mathcal{A}_j$, for every j, we see that I_{V_j} has as minimal generators the products $y_1 \cdots y_{p-j}$ and $z_1 \cdots z_{p-j+1}$ if j is odd, and $y_1 \cdots y_{p-j+1}$ and $z_1 \cdots z_{p-j}$ if j is even. Therefore, the minimal generators of degree $p-j$ for I_{V_j} and $I_{V_{j+1}}$ are the same. $\square$

THEOREM 1.22. *Let* φ *be a* 3*-codimensional Artinian O-sequence, let* c *be any integer and let* h*,* u *and* v *be as defined at the beginning of Section 1. Let us suppose that* $u \ge 0$*,* $v < 0$ *and* $h > 2u$ *and that* φ *is of* $D^*(c)$*-type. Then there exists a reduced* 0*-dimensional subscheme* X *of* $\mathbb{P}^3$ *with* $\Delta H_X = \varphi$ *and* $\max\{0, G = 2u - v - \overline{h}\}$ *minimal generators of degree* c*, except when* $2u < h \le 2u - v$ *and* $\alpha_0 = 0$*, in which case there exists such a subscheme with* $G+1$ *minimal generators of degree* c*.*

Proof. Since we are under the hypotheses of Case 2b of Construction 1.6 we consider the corresponding linear decomposition $\lambda(\varphi, c)$ of φ. Define

$$z+1 = \begin{cases} \max\{2u+2, h+v+2\} & \text{if } 2u < h \le 2u-v \text{ or} \\ & h > 2u-v \text{ and } \alpha_0 \ne 0\ , \\ h+v+1 & \text{if } h > 2u-v \text{ and } \alpha_0 = 0. \end{cases}$$

By Theorem 1.19, there exist $V_1 \supseteq \cdots \supseteq V_z$, where each V_j is a maximal 2-partial intersection associated to φ_j. Then one can easily verify that the O-sequences $\{\varphi_{z+1}, \ldots, \varphi_d\}$ satisfy the conditions of Lemma 1.21. Therefore there exist 2-partial intersections $V_{z+1} \supset \cdots \supset V_d$ which have the property required by the lemma. If we prove that the left segment $\mathcal{A}_z$, the support of V_z, contains $\mathcal{A}_{z+1}$, the support of V_{z+1}, then, since all V_j's can be constructed using the same families of hyperplanes, we can assert that V_z contains V_{z+1}. Thus, if we set $X_j = V_j \cap \pi_j$ with π_j, $1 \leq j \leq d$, generic hyperplanes of $\mathbb{P}^3$, then $X = \bigcup_{j=1}^d X_j$ will be a partial intersection associated to φ (cf. Theorem 1.19). Now recall that $\varphi_z(n) = c - z$ for $c - z - 1 \leq n \leq s$, for some s and $\varphi_z(n) < c-z$ for $n > s$. Note that in such a situation $\varphi_{z+1}(n) = 0$ for $n \geq s-1$. Therefore we need to check that if $L = (x, y) \in \mathcal{A}_{z+1}$ and $x + y < s + 2$ then $L \in \mathcal{A}_z$. Now, since $\mathcal{A}_z$ is maximal with respect to φ_z, it contains all pairs $K = (x', y')$, where $c - z + 1 \leq x' + y' < s + 2$ and $1 \leq x' \leq c - z$. Hence it contains all $L \in \mathcal{A}_{z+1}$. Next we compute $\nu_X(c)$. To do that we first observe that $S = \sum_j \nu_{X_j}(c-j+1) = u-v$ since $\nu_{X_j}(c-j+1) = \varphi_j(c-j)-\varphi_j(c-j+1)$ when $\varphi_j(c-j) = \varphi_j(c-j-1)$ and $\nu_{X_j} = \varphi_j(c-j) + 1 - \varphi_j(c-j+1)$ when $\varphi_j(c-j) = \varphi_j(c-j-1) + 1$. Therefore we have S generators of degree c for I_X. To calculate $\nu_X(c)$ it is enough to subtract from S the number of such generators which are redundant, i.e., those generators of degree $c - j + 1$ for I_{X_j} which are also minimal generators for $I_{X_{j-1}}$.

First we observe that, as in Case 2a of Theorem 1.20, for $1 \leq j \leq 2u$ we have u redundant generators. For the remaining cases we distinguish four subcases.

Case 1. $2u < h \leq 2u - v$ and $\alpha_0 > 0$.

In this case $z = 2u + 1$. Now, by construction and by Lemma 1.21, we see that we have a redundant generator of degree $c-j+1$ for each $z+1 \leq j \leq h+1$, $h + 1 - z = h - 2u$. Thus, in this case we obtain $\nu_X(c) \leq S - u - (h - 2u) = 2u - h - v = G$ and hence $\nu_X(c) = G$.

Case 2. $h > 2u - v$ and $\alpha_0 > 0$.

In this case $z = h + v + 1$. Now, again, by construction and by the Lemma 1.21, we see that we have a redundant generator of degree $c - j + 1$ for each $z + 1 \leq j \leq h + 1$, $h + 1 - z = -v$. Thus, in this case we obtain $\nu_X(c) \leq S - u + v = 0$ and hence $\nu_X(c) = 0$.

Case 3. $h > 2u - v$ and $\alpha_0 = 0$.

In this case $z = h + v$. Now, by construction and by Lemma 1.21, we see that we have a redundant generator of degree $c - j + 1$ for each $z + 1 \leq j \leq h$, $h - z = -v$ (note that, since $\alpha_0 = 0$, $\nu_{X_{h+1}}(c - h) = 0$). Thus, again, we obtain $\nu_X(c) \leq S - u + v = 0$, and hence , $\nu_X(c) = 0$.

Case 4 (Exceptional case). $2u < h \leq 2u - v$ and $\alpha_0 = 0$.

In this case $z = 2u + 1$. Now, by construction and by Lemma 1.21, we see that we have a redundant generator of degree $c - j + 1$ for each $z + 1 \leq j \leq h$, $h - z = h - 2u - 1$. Thus, in this case we obtain $\nu_X(c) \leq S - u - (h - 2u - 1) = 2u - h - v + 1 = G + 1$, and, indeed, one can see that $\nu_X(c) = G + 1$. □

Corollary 1.23. *Let φ be a 3-codimensional Artinian O-sequence, c any integer and h, u and v as defined before. If φ is of $D^*(c)$-type, then we have $\nu_\varphi^{\min}(c) = \max\{0, G = 2u - v - \overline{h}\}$ except when $2u < h \leq 2u - v$ and $\alpha_0 = 0$, in which case $G \leq \nu_\varphi^{\min}(c) \leq G + 1$.*

Proof. The result is an easy consequence of Theorems 1.20 and 1.22. □

In the next example we will show that there are O-sequences φ and integers c for which $\nu_\varphi^{\min}(c) = G + 1$.

Example 1.24. Let $\varphi = (1\ 3\ 3\ 3\ 2\ 0\ \ \rightarrow)$ and $c = 4$. In this case $h = 1$, $u = 0$ and $v = -1$. Therefore $G = 0$. We want to show that for every Artinian algebra $A = k[x, y, z]/I = R/I$ with $H_A = \varphi$, I needs at least one generator in degree 4, i.e., $\nu_\varphi^{\min}(4) = 1$. In fact, if A were an Artinian algebra with $H_A = \varphi$ and no generator in degree 4, from $\nu_A(4) - \nu_{A,1}(4) + \nu_{A,2}(4) = -\Delta^3\varphi(4) = 1$ one sees that such an algebra would have $\nu_{A,2}(4) \geq 1$ (indeed 1) and consequently $\nu_{A,1}(3) \geq 3$ (indeed 3) and therefore $\nu_A(3) = 1$. Now, since $\nu_A^{\max}(3) = 1$, (I_2) has maximal growth in degree 2, and so, by the Gotzmann Persistence Theorem (Theorem 1.1), the 3 generators in degree 2, Q_1, Q_2, Q_3, would have a common linear factor L. Hence, if F is the generator of I in degree 3, then $\widetilde{I} = (Q_1, Q_2, Q_3, F) \subset J = (L, F)$. Thus, $H_{R/J}(4) = 3$ and $H_{R/\widetilde{I}}(4) \geq 3$; since I has no generator in degree 4, $H_{R/I}(4) = H_{R/\widetilde{I}}(4) \geq 3$, a contradiction.

2. Applications

In this section we apply the main results of the previous section to particular O-sequences.

In the rest of this section we will denote by $\mathcal{B}_\varphi$ the poset of all graded Betti sequences of Artinian algebras with Hilbert function φ. $CI(a, b)$ and $CI(a, b, c)$ will denote an algebra $k[x, y, z]/I$, where I is an ideal generated by a regular sequence of degrees, respectively, a, b or a, b, c.

The following computational lemma will be useful in obtaining information on Gorenstein O-sequences.

Lemma 2.1. *Let $1 < a \leq b$ and $b + 3 \leq c \leq ab + 1$ be three integers and φ a 3-codimensional Artinian O-sequence such that $\varphi(n) = H_{CI(a,b)}(n)$ for $n \leq c - 1$ and $\varphi(c) < H_{CI(a,b)}(c)$. Then*

$$G = 2\Delta\varphi(c-1) - \Delta\varphi(c) - \varphi(c-2)^{\langle c-2\rangle} + \varphi(c-2) < -\Delta^3\varphi(c).$$

Proof. We use the usual notation $u := \Delta\varphi(c-1)$, $v := \Delta\varphi(c)$, $h := \varphi(c-2)^{\langle c-2\rangle} - \varphi(c-2)$, so $G = 2u - v - h = 2\Delta\varphi(c-1) - \Delta\varphi(c) - h$. Then our conclusion follows if we can prove that $h > 2\Delta\varphi(c-1) - \Delta\varphi(c) + \Delta^3\varphi(c) = \Delta\varphi(c-2)$.

If $c > a+b$ we have $\Delta\varphi(c-2) = 0$, but $h > 0$ since $c \leq ab+1$. If $b+3 \leq c \leq a+b$, then, since $\varphi(c-2) = \sum_{n=0}^{c-2} \Delta\varphi(n)$, we have

$$\varphi(c-2) = \sum_{n=1}^{\Delta\varphi(c-2)} (c-n) + \sum_{n=1}^{a-\Delta\varphi(c-2)} (c - \Delta\varphi(c-2) - 2n).$$

Now $a - \Delta\varphi(c-2) = c-b-1 \geq 2$, and therefore $\varphi(c-2) \geq \sum_{n=1}^{\Delta\varphi(c-2)+1}(c-n)$, i.e., $h > \Delta\varphi(c-2)$. □

We want to apply the above lemma to the 3-codimensional Artinian Gorenstein O-sequences. We recall that such a sequence φ determines $2m+1$ integers $d_0 \leq d_1 \leq \cdots \leq d_{2m}$ which represent the degrees of a minimal set of generators of an Artinian Gorenstein algebra with Hilbert function φ and with the minimum number of generators among all Gorenstein algebras with the same Hilbert function φ (see, for instance, [Di]). Because of this we will call the sequence $((d_0, d_1, \ldots, d_{2m})); (\vartheta - d_{2m}, \ldots, \vartheta - d_0); (\vartheta))$, where $\vartheta = (1/m)\sum_{i=0}^{2m} d_i$, the *minimal Gorenstein Betti sequence associated to* φ. Recall that the following Gaeta conditions hold: $\vartheta - d_i > d_{2m+1-i}$ for $i = 1, \ldots, m$. Note that, from the above conditions, we have

$$\begin{aligned}
\sum_{i=0}^{2m} d_i = m\vartheta \Rightarrow \sum_{i=0}^{m} d_i &= \sum_{i=m+1}^{2m} (\vartheta - d_i) \\
\Rightarrow d_0 + d_1 + \sum_{i=2}^{m} d_i &= \sum_{i=1}^{m} (\vartheta - d_{2m+1-i}) \\
\Rightarrow d_0 + d_1 &= \vartheta - d_{2m} + \sum_{i=2}^{m} (\vartheta - d_i - d_{2m+1-i}) \geq \vartheta - d_{2m}.
\end{aligned}$$

(Indeed, $d_0 + d_1 = \vartheta - d_{2m}$ only if $m = 1$, i.e., φ is the Hilbert function of a $CI(d_0, d_1, d_2)$.)

Proposition 2.2. *Let φ be a 3-codimensional Artinian Gorenstein O-sequence, $\mu = \big((d_0, d_1, \ldots, d_{2m})); (\vartheta - d_{2m}, \ldots, \vartheta - d_0); (\vartheta)\big)$, $2 \leq d_0 \leq d_1 \leq \cdots \leq d_{2m}$, the minimal Gorenstein Betti sequence associated to φ. If $d_1 + 3 \leq d_2 \leq \vartheta - d_{2m}$, then $\mathcal{B}_\varphi$ has more than one minimal element, except for $\varphi = H_{CI(4,4,7)}$.*

Proof. First we observe that μ is a minimal element for $\mathcal{B}_\varphi$. On the other hand, since $\vartheta - d_{2m}$ is the smallest degree for a first syzygy in μ, we have $\varphi(n) = H_{CI(d_0,d_1)}(n)$ for $n \leq d_2 - 1$ and $\varphi(d_2) < H_{CI(d_0,d_1)}(d_2)$. Moreover,

$d_2 \leq \vartheta - d_{2m} \leq d_0 + d_1 \leq d_0 d_1$. Hence, if we set $u := \Delta\varphi(d_2 - 1)$, $v := \Delta\varphi(d_2)$, $h := \varphi(d_2 - 2)^{\langle d_2 - 2\rangle} - \varphi(d_2 - 2)$ and $G := 2u - h - v$, applying the previous lemma we get $G < -\Delta^3\varphi(d_2)$. Now, $-\Delta^3\varphi(d_2)$ is less than or equal to the multiplicity g of d_2 as generators in μ. Thus, applying Corollary 1.23 we get $\nu_\varphi^{\min}(d_2) = \max\{0, G\} < g$ in all cases except when $v < 0$, $2u < h \leq 2u - v$ and $\alpha_0 = 0$ (with the same notation as in Theorem 1.22). Now, under our hypotheses, $v < 0$ implies $u = -v > 0$. In fact, the case $u = 0$ and $v = -1$ only occurs if $d_2 \geq d_0 + d_1$, i.e., $d_2 = d_0 + d_1$. In this case, from $0 < h \leq 1$, i.e., $h = 1$, and $\alpha_0 = 0$ we have $\varphi(d_0 + d_1 - 2) = d_0 + d_1 - 1$. But $\varphi(d_0 + d_1 - 2) = H_{CI(d_0,d_1)}(d_0 + d_1 - 2) = d_0 d_1$, and so $d_0 d_1 = d_0 + d_1 - 1$, which is a contradiction. Therefore, again by Corollary 1.23, $\nu_\varphi^{\min}(d_2) \leq G + 1 = 3u - h + 1$, while, on the other hand, $-\Delta^3\varphi(d_2) = 2u - 1 = g$. Thus, we get $\nu_\varphi^{\min}(d_2) < g$ if we can prove that $3u - h + 1 < 2u - 1$, i.e., $h > u + 2$. Since $h > 2u$, we are done if $2u \geq u + 2$, i.e., if $u \geq 2$. It remains to consider the case $u = -v = 1$. But in this case, $d_2 = d_0 + d_1 - 1$, $h = 3$, and consequently (using $\alpha_0 = 0$) $\varphi(d_0 + d_1 - 3) = (d_0 + d_1 - 2) + (d_0 + d_1 - 3) + (d_0 + d_1 - 4) = 3d_0 + 3d_1 - 9$. But $\varphi(d_0 + d_1 - 3) = H_{CI(d_0,d_1)}(d_0 + d_1 - 3) = d_0 d_1 - 1$, and therefore $d_0 d_1 = 3d_0 + 3d_1 - 8$. Now, a simple computation shows that this happens if and only if $d_0 = d_1 = 2$ (and $d_2 = 3$ which, however, does not fall under our hypotheses) or $d_0 = d_1 = 4$ (and $d_2 = 7$, which implies that φ is the Hilbert function of a complete intersection). This shows that under our hypotheses $\nu_\varphi^{\min}(d_2) < g$. Hence there exists an Artinian algebra A with Hilbert function φ which needs in degree d_2 less than g generators. Since our Betti sequence μ was the minimal Gorenstein Betti sequence associated to φ, this implies that the Betti sequence of A cannot be Gorenstein, and therefore cannot be comparable with μ. We conclude that $\mathcal{B}_\varphi$ has more than one minimal element. □

The above proposition, in particular, applies to O-sequences of complete intersections of type (a, b, c) when $c \leq a + b$. In [Ri] Richert gave some examples of O-sequences φ for which $\mathcal{B}_\varphi$ has more than one minimal element. The next proposition gives a more complete picture of such Betti sequences $\mathcal{B}_\varphi$ for all complete intersection Hilbert functions $\varphi = H_{CI(a,b,c)}$.

Proposition 2.3. *Let $\varphi = H_{CI(a,b,c)}$ be a 3-codimensional Artinian Complete Intersection O-sequence, $3 \leq a \leq b \leq c$.*

1. *If $c \leq b + 2$ or $c \geq ab + 2$, then $\mathcal{B}_\varphi$ has only one minimal element.*
2. *If $b + 3 \leq c \leq ab$ and $(a, b, c) \neq (4, 4, 7)$, then $\mathcal{B}_\varphi$ has more than one minimal element.*

Proof. Observe first that in either case the Betti sequence of the complete intersection Artinian algebra of type (a, b, c) is a minimal element for $\mathcal{B}_\varphi$. In order to have another minimal Betti sequence for φ we would need to find an Artinian algebra, with Hilbert function φ and no minimal generator in degree

c, and thus, when $c \neq a+b$, with at least one second syzygy of degree c. Now, if $c \leq b+2$, such a second syzygy of degree c cannot exist since there is at most one first syzygy in degree $< b+2$ (in fact, in degree $b+1$). If $c \geq ab+2$, we have maximal growth for φ in degree $c-1 > ab$. Hence, by the Gotzmann Persistence Theorem (Theorem 1.1), since $\varphi(c) = ab-1 < \varphi(c-1)$, we would have a minimal generator in degree c. This completes the proof of item (1).

If $b+3 \leq c \leq a+b$, the conclusion of (2) follows from Proposition 2.2. Finally, if $a+b+1 \leq c \leq ab$, since $u = \varphi(c-1) - \varphi(c-2) = ab - ab = 0$, $v = \varphi(c) - \varphi(c-1) = ab - 1 - ab = -1$, $h = (ab)^{\langle c-2 \rangle} - ab \geq 1$ and when $h = 1$ we have $\alpha_0 = ab - (c-1) > 0$, applying Corollary 1.23 we get $\nu_\varphi^{\min}(c) = \max\{0, 2u - h - v\} = 0$. □

Remark 2.4. The case when $\varphi = H_{CI(2,b,c)}$ is similar to the above case; more precisely, if $c \leq b+1$ or $c \geq 2b+2$, then $\mathcal{B}_\varphi$ has only one minimal element, and if $b+2 \leq c \leq 2b$, then $\mathcal{B}_\varphi$ has more than one minimal element.

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