

## MAPS PRESERVING ZERO JORDAN PRODUCTS ON HERMITIAN OPERATORS

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ABSTRACT. Let  $H$  be a separable complex Hilbert space and  $B_s(H)$  the Jordan algebra of all Hermitian operators on  $H$ . Let  $\theta : B_s(H) \rightarrow B_s(H)$  be a surjective  $\mathbb{R}$ -linear map which is continuous in the strong operator topology such that  $\theta(x)\theta(y) + \theta(y)\theta(x) = 0$  for all  $x, y \in B_s(H)$  with  $xy + yx = 0$ . We show that  $\theta(x) = \lambda u x u^*$  for all  $x \in B_s(H)$ , where  $\lambda$  is a nonzero real number and  $u$  is a unitary or anti-unitary operator on  $H$ .

### 1. Introduction

Given any (associative) ring  $R$  we can render it into a Jordan ring by defining, for any two elements  $a, b \in R$ , the Jordan product  $a \circ b = ab + ba$ . An additive map  $\varphi : R \rightarrow R'$  of rings is called a Jordan homomorphism if  $\varphi(a \circ b) = \varphi(a) \circ \varphi(b)$  for all  $a, b \in R$ . In the case that 2 is invertible in  $R'$  in the sense that  $2x = a$  has a unique solution in  $R'$  for every  $a \in R'$ , this condition is equivalent to  $\varphi(a^2) = \varphi(a)^2$  for all  $a \in R$ . Obviously, (associative) homomorphisms and anti-homomorphisms are Jordan homomorphisms. Jordan homomorphisms have been thoroughly investigated in the literature [1], [2], [5], [8], [9], [11], [22]. The results proved usually read that a Jordan homomorphism must be a homomorphism or an anti-homomorphism. For instance, Herstein [5] showed that a Jordan homomorphism of any ring onto a prime ring is either a homomorphism or an anti-homomorphism.

In the case that the ring  $R$  is endowed with an involution  $a \rightarrow a^*$ , the set  $S = S(R) = \{a \in R \mid a^* = a\}$  of all symmetric elements of  $R$  is itself a Jordan ring. In [10] Jacobson and Rickart proved that, given a matrix ring  $R = M_n(A)$ , where  $A$  is a ring and  $n \geq 3$ , with a “canonical” involution  $*$  such that the symmetric elements are trace-valued, any Jordan homomorphism of  $S$  can be lifted to a homomorphism of  $R$  in a unique way. At his 1961 AMS talk [6] Herstein posed the problem of characterizing the Jordan homomorphisms of the symmetric elements of a simple ring with involution. This problem was

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Received October 4, 2004; received in final form December 9, 2004.

2000 *Mathematics Subject Classification*. Primary 47B48. Secondary 16W10, 17C30.

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solved by Martindale [12] in the presence of orthogonal idempotents. Using Zelmanov's brilliant work on prime Jordan algebras [23], an idempotent-free solution was later obtained by McCrimmon [14] and Martindale [13].

Let  $H$  be a complex Hilbert space and  $B(H)$  the algebra of all bounded linear operators on  $H$  with the adjoint map  $a \rightarrow a^*$ . The Jordan subalgebra  $B_s(H) = S(B(H))$  of all Hermitian operators on  $H$  (i.e., the set of all bounded observables) plays an important role in the mathematical description of quantum mechanics. As with any algebraic structure the study of the automorphisms of this algebra is of considerable importance. It is well-known (see, for example, [16, p. 99]) that every Jordan automorphism on  $B_s(H)$  is of the form  $x \rightarrow uxu^*$ , where  $u$  is a unitary or anti-unitary operator on  $H$ . A unified treatment of the Jordan automorphisms is presented in [4].

In many cases maps on  $B_s(H)$  defined by certain local properties or preserving certain properties are exactly Jordan automorphisms [16], [17], [18], [19]. In this note we shall show that Jordan automorphisms can be characterized by their action on zero Jordan products. More precisely, we show that a surjective  $\mathbb{R}$ -linear map  $\theta : B_s(H) \rightarrow B_s(H)$  which is continuous in the strong operator topology and preserves zero Jordan products must be of the form  $x \rightarrow \lambda\varphi(x)$  where  $\lambda$  is a nonzero real number and  $\varphi$  is a Jordan automorphism of  $B_s(H)$ . Moreover,  $\theta$  is a Jordan automorphism if, in addition,  $\theta(1) = 1$ .

## 2. The results

Let  $C$  be a commutative ring with  $\frac{1}{2}$  (i.e.,  $C$  is a unital ring in which 2 is invertible),  $c \rightarrow \bar{c}$  an automorphism of order 1 or 2 on  $C$ , and  $R = M_n(C)$ , the  $n$  by  $n$  matrix algebra over  $C$ . We denote by  $e_{ij}$  the matrix which has 1 in the  $(i, j)$ -position and zeros elsewhere. Note that  $R$  can be equipped with the involution  $*$  defined by  $a^* = \bar{a}$  for  $a \in C$  and  $e_{ij}^* = e_{ji}$ . For simplicity we shall write  $a_{ij} = ae_{ij}$  for  $a \in C$  and put

$$M = \{e_{ii} \mid 1 \leq i \leq n\} \cup \{a_{ij} + \bar{a}_{ji} \mid a \in C, 1 \leq i \neq j \leq n\}.$$

Then the Jordan ring  $S = S(R)$  is the linear span of  $M$  over the subring  $F = \{c \in C \mid \bar{c} = c\}$  of  $C$ .

Let  $J$  be an arbitrary Jordan algebra over  $F$  and  $\theta : S \rightarrow J$  an  $F$ -linear map which preserves zero Jordan products, that is,  $\theta(x) \circ \theta(y) = 0$  whenever  $x, y \in S$  satisfy  $x \circ y = 0$ . To begin with, we investigate the product  $\theta(x) \circ \theta(y)$ . A relation between  $\theta(x \circ y)$  and  $\theta(x) \circ \theta(y)$  will be established for arbitrary  $x, y \in S$ . As a consequence, we see that the map  $\theta$  which preserves zero Jordan products also preserves equal Jordan products, that is,  $\theta(x) \circ \theta(y) = \theta(u) \circ \theta(v)$  for all  $x, y, u, v \in S$  with  $x \circ y = u \circ v$ .

**THEOREM 2.1.** *Let  $S$  be the Jordan algebra defined above,  $J$  a Jordan algebra over  $F$  and  $\theta : S \rightarrow J$  an  $F$ -linear map preserving zero Jordan products.*

Then

$$(2.1) \quad \theta(x) \circ \theta(y) = \frac{1}{2} \theta(1) \circ \theta(x \circ y)$$

for all  $x, y \in S$ . In particular, we have  $\theta(x) \circ \theta(y) = \theta(u) \circ \theta(v)$  for all  $x, y, u, v \in S$  with  $x \circ y = u \circ v$ .

*Proof.* Since  $S$  is spanned by  $M$  over  $F$  and  $\theta$  is  $F$ -linear, it suffices to verify (2.1) for  $x, y \in M$ . Note that (2.1) is obvious if  $x \circ y = 0$ . Thus we need only consider the following cases:

(a)  $x = y = e_{ii}$ .

From  $(e_{ii} - 1) \circ e_{ii} = 0$  it follows that  $\theta(e_{ii} - 1) \circ \theta(e_{ii}) = 0$  and hence

$$(2.2) \quad \theta(e_{ii}) \circ \theta(e_{ii}) = \theta(1) \circ \theta(e_{ii}).$$

Now  $x \circ y = 2e_{ii}$ , so we have

$$\theta(x) \circ \theta(y) = \theta(1) \circ \theta(e_{ii}) = \frac{1}{2} \theta(1) \circ \theta(x \circ y).$$

(b)  $x = e_{ii}, y = a_{ij} + \bar{a}_{ji}$ , where  $i \neq j$  and  $a \in C$ .

From  $(e_{ii} - e_{jj}) \circ (a_{ij} + \bar{a}_{ji}) = 0$  it follows that  $\theta(e_{ii} - e_{jj}) \circ \theta(a_{ij} + \bar{a}_{ji}) = 0$ , or equivalently,

$$(2.3) \quad \theta(e_{ii}) \circ \theta(a_{ij} + \bar{a}_{ji}) = \theta(e_{jj}) \circ \theta(a_{ij} + \bar{a}_{ji}).$$

Since  $\theta(e_{hh}) \circ \theta(a_{ij} + \bar{a}_{ji}) = 0$  for  $h \notin \{i, j\}$ , this together with (2.3) yields

$$\begin{aligned} \theta(1) \circ \theta(a_{ij} + \bar{a}_{ji}) &= \sum_{h=1}^n \theta(e_{hh}) \circ \theta(a_{ij} + \bar{a}_{ji}) \\ &= \theta(e_{ii}) \circ \theta(a_{ij} + \bar{a}_{ji}) + \theta(e_{jj}) \circ \theta(a_{ij} + \bar{a}_{ji}) \\ &= 2\theta(e_{ii}) \circ \theta(a_{ij} + \bar{a}_{ji}), \end{aligned}$$

and hence

$$(2.4) \quad \theta(e_{ii}) \circ \theta(a_{ij} + \bar{a}_{ji}) = \frac{1}{2} \theta(1) \circ \theta(a_{ij} + \bar{a}_{ji}).$$

Now  $x \circ y = a_{ij} + \bar{a}_{ji}$ , so (2.4) means exactly  $\theta(x) \circ \theta(y) = \frac{1}{2} \theta(1) \circ \theta(x \circ y)$ .

(c)  $x = a_{ij} + \bar{a}_{ji}, y = b_{ij} + \bar{b}_{ji}$ , where  $i \neq j$  and  $a, b \in C$ .

Let  $c = \frac{1}{2}(a\bar{b} + \bar{a}b) \in F$ . From

$$(e_{ii} + a_{ij} + \bar{a}_{ji} - e_{jj}) \circ (-c_{ii} + b_{ij} + \bar{b}_{ji} + c_{jj}) = 0$$

it follows that

$$\theta(e_{ii} + a_{ij} + \bar{a}_{ji} - e_{jj}) \circ \theta(-c_{ii} + b_{ij} + \bar{b}_{ji} + c_{jj}) = 0.$$

Expansion of the last identity yields  $A + B = 0$ , where

$$A = \theta(e_{ii} - e_{jj}) \circ \theta(-c_{ii} + c_{jj}) + \theta(a_{ij} + \bar{a}_{ji}) \circ \theta(b_{ij} + \bar{b}_{ji}),$$

and

$$B = \theta(a_{ij} + \bar{a}_{ji}) \circ \theta(-c_{ii} + c_{jj}),$$

since  $\theta(e_{ii} - e_{jj}) \circ \theta(b_{ij} + \bar{b}_{ji}) = 0$  by (2.3). Replacing  $a$  by  $2a$ , we get  $2A + 4B = 0$ . Therefore,  $A = B = 0$ . By (2.2) it follows from  $A = 0$  that

$$\begin{aligned} \theta(a_{ij} + \bar{a}_{ji}) \circ \theta(b_{ij} + \bar{b}_{ji}) &= \theta(e_{ii} - e_{jj}) \circ \theta(c_{ii} - c_{jj}) \\ &= c\theta(e_{ii} - e_{jj}) \circ \theta(e_{ii} - e_{jj}) \\ &= c(\theta(e_{ii}) \circ \theta(e_{ii}) + \theta(e_{jj}) \circ \theta(e_{jj})) \\ &= c\theta(1) \circ (\theta(e_{ii}) + \theta(e_{jj})). \end{aligned}$$

Now  $x \circ y = 2c(e_{ii} + e_{jj})$ , so we have

$$\begin{aligned} \theta(x) \circ \theta(y) &= c\theta(1) \circ (\theta(e_{ii}) + \theta(e_{jj})) \\ &= \frac{1}{2}\theta(1) \circ \theta(2c(e_{ii} + e_{jj})) \\ &= \frac{1}{2}\theta(1) \circ \theta(x \circ y). \end{aligned}$$

(d)  $x = a_{ij} + \bar{a}_{ji}$ ,  $y = b_{jk} + \bar{b}_{kj}$ , where  $i, j, k$  are distinct and  $a, b \in C$ .

From

$$(a_{ij} + [ab]_{ik} + \bar{a}_{ji} + \overline{[ab]}_{ki}) \circ (-[b\bar{b}]_{jj} + b_{jk} + \bar{b}_{kj} - e_{kk}) = 0$$

it follows that

$$\theta(a_{ij} + [ab]_{ik} + \bar{a}_{ji} + \overline{[ab]}_{ki}) \circ \theta(-[b\bar{b}]_{jj} + b_{jk} + \bar{b}_{kj} - e_{kk}) = 0.$$

Expansion of the last identity yields  $A + B = 0$ , where

$$A = \theta(a_{ij} + \bar{a}_{ji}) \circ \theta(b_{jk} + \bar{b}_{kj}) - \theta([ab]_{ik} + \overline{[ab]}_{ki}) \circ \theta(e_{kk}),$$

and

$$B = \theta([ab]_{ik} + \overline{[ab]}_{ki}) \circ \theta(b_{jk} + \bar{b}_{kj}) - \theta(a_{ij} + \bar{a}_{ji}) \circ \theta([b\bar{b}]_{jj}),$$

since  $\theta(a_{ij} + \bar{a}_{ji}) \circ \theta(e_{kk}) = 0$  and  $\theta([ab]_{ik} + \overline{[ab]}_{ki}) \circ \theta([b\bar{b}]_{jj}) = 0$ . Replacing  $b$  by  $2b$ , we get  $2A + 4B = 0$ . Therefore,  $A = B = 0$ . By (2.4) it follows from  $A = 0$  that

$$\begin{aligned} \theta(a_{ij} + \bar{a}_{ji}) \circ \theta(b_{jk} + \bar{b}_{kj}) &= \theta(e_{kk}) \circ \theta([ab]_{ik} + \overline{[ab]}_{ki}) \\ &= \frac{1}{2}\theta(1) \circ \theta([ab]_{ik} + \overline{[ab]}_{ki}). \end{aligned}$$

Now  $x \circ y = [ab]_{ik} + \overline{[ab]}_{ki}$ , so the last identity means exactly  $\theta(x) \circ \theta(y) = \frac{1}{2}\theta(1) \circ \theta(x \circ y)$ .

Thus we have proved that  $\theta(x) \circ \theta(y) = \frac{1}{2}\theta(1) \circ \theta(x \circ y)$  for all  $x, y \in S$ . Therefore, if  $x, y, u, v \in S$  satisfy  $x \circ y = u \circ v$ , then it is obvious that  $\theta(x) \circ \theta(y) = \theta(u) \circ \theta(v)$ .  $\square$

Using an argument similar to that for [3, Theorem 3.2], we may apply the preceding theorem to the case of Hermitian operators.

**COROLLARY 2.2.** *Let  $F_s(H)$  be the Jordan algebra of all Hermitian operators of finite rank on a complex Hilbert space  $H$ ,  $J$  a Jordan algebra over  $\mathbb{R}$  and  $\theta : F_s(H) \rightarrow J$  an  $\mathbb{R}$ -linear map preserving zero Jordan products. Then  $\theta(x) \circ \theta(y) = \theta(u) \circ \theta(v)$  for all  $x, y, u, v \in F_s(H)$  with  $x \circ y = u \circ v$ .*

*Proof.* Fix  $x, y, u, v \in F_s(H)$ . There is a projection  $p \in F_s(H)$  such that  $pxp = x$ ,  $pyy = y$ ,  $pup = u$  and  $pvp = v$ . Let  $\{e_1, e_2, \dots, e_n\}$  be an orthonormal basis of the range of  $p$  and  $A$  the subalgebra of  $B(H)$  consisting of all operators  $a$  on  $H$  defined by  $a(w) = \sum_{i,j=1}^n \alpha_{ij} \langle w | e_j \rangle e_i$  for  $w \in H$ , where  $\alpha_{ij} \in \mathbb{C}$  and  $\langle w | e_j \rangle$  denotes the inner product of  $w$  and  $e_j$ . Note that  $A$  is isomorphic to  $R = M_n(\mathbb{C})$  via the isomorphism  $a \rightarrow \sum_{i,j=1}^n \alpha_{ij} e_{ij}$  and we may assume that  $x, y, u, v \in S = S(R)$ , the Jordan algebra of all Hermitian matrices. Therefore for the restriction of  $\theta$  to  $S$  we may apply Theorem 2.1. Hence  $\theta(x) \circ \theta(y) = \theta(u) \circ \theta(v)$  for all  $x, y, u, v \in F_s(H)$  with  $x \circ y = u \circ v$  and the proof is complete.  $\square$

We are now in a position to prove the main theorem of the present paper using some ideas from the proof of [3, Theorem 3.3].

**THEOREM 2.3.** *Let  $B_s(H)$  be the Jordan algebra of all Hermitian operators on a separable complex Hilbert space  $H$  and  $\theta : B_s(H) \rightarrow B_s(H)$  a surjective  $\mathbb{R}$ -linear map which is continuous in the strong operator topology. Suppose that  $\theta$  preserves zero Jordan products. Then  $\lambda = \theta(1)$  is a nonzero real number and there exists a unitary or anti-unitary operator  $u$  on  $H$  such that*

$$\theta(x) = \lambda x u x^* \quad \text{for all } x \in B_s(H).$$

*Proof.* First we show that

$$(2.5) \quad \frac{1}{2} \theta(1) \circ \theta(x \circ y) = \theta(x) \circ \theta(y)$$

for all  $x, y \in B_s(H)$ .

Let  $x_0, y_0 \in F_s(H)$ . Since  $H$  is separable,  $F_s(H)$  is dense in  $B_s(H)$  in the strong operator topology. In particular, there exists a sequence  $\{p_n\}$  in  $F_s(H)$  converging to 1 in the strong operator topology. We may assume further that  $p_n(x_0 \circ y_0) = (x_0 \circ y_0)p_n = x_0 \circ y_0$  for all  $n$ . According to Corollary 2.2 the restriction of  $\theta$  to  $F_s(H)$  preserves equal Jordan products, so it follows from  $\frac{1}{2} p_n \circ (x_0 \circ y_0) = x_0 \circ y_0$  that

$$\frac{1}{2} \theta(p_n) \circ \theta(x_0 \circ y_0) = \theta(x_0) \circ \theta(y_0)$$

for all  $n$ . Since  $\theta$  is continuous in the strong operator topology, we obtain

$$(2.6) \quad \frac{1}{2} \theta(1) \circ \theta(x_0 \circ y_0) = \theta(x_0) \circ \theta(y_0)$$

by passing to the limit in the previous equation.

Let  $x \in B_s(H)$  and  $\{x_n\}$  a sequence in  $F_s(H)$  converging to  $x$ . It follows from (2.6) that

$$\frac{1}{2} \theta(1) \circ \theta(x_n \circ y_0) = \theta(x_n) \circ \theta(y_0)$$

for all  $n$ . Passing to the limits in this equation we get

$$(2.7) \quad \frac{1}{2} \theta(1) \circ \theta(x \circ y_0) = \theta(x) \circ \theta(y_0).$$

Finally, let  $y \in B_s(H)$  and  $\{y_n\}$  a sequence in  $F_s(H)$  converging to  $y$ . It follows from (2.7) that

$$\frac{1}{2} \theta(1) \circ \theta(x \circ y_n) = \theta(x) \circ \theta(y_n)$$

for all  $n$ . Passing to the limits in this equation we obtain (2.5).

For any projection  $p$  in  $B_s(H)$  we have

$$(2.8) \quad \theta(1) \circ \theta(p) = \theta(p) \circ \theta(p) = 2\theta(p)^2.$$

Thus

$$[\theta(1) \circ \theta(p)]\theta(p) = \theta(p)[\theta(1) \circ \theta(p)],$$

and its expansion yields

$$\theta(1)\theta(p)^2 = \theta(p)^2\theta(1).$$

This together with (2.8) gives

$$\theta(1)[\theta(1) \circ \theta(p)] = [\theta(1) \circ \theta(p)]\theta(1).$$

Since every element of  $B_s(H)$  is an  $\mathbb{R}$ -linear combination of projections [20, Theorem 3], we have

$$\theta(1)[\theta(1) \circ \theta(x)] = [\theta(1) \circ \theta(x)]\theta(1)$$

for all  $x \in B_s(H)$ , and, *a fortiori*,

$$\theta(1)[\theta(1) \circ \theta(x \circ y)] = [\theta(1) \circ \theta(x \circ y)]\theta(1)$$

for all  $x, y \in B_s(H)$ . In view of (2.5), we conclude that  $\theta(1)$  commutes with  $\theta(x) \circ \theta(y)$  for all  $x, y \in B_s(H)$ . Since  $\theta : B_s(H) \rightarrow B_s(H)$  is surjective, this implies that  $\theta(1)$  commutes with  $x \circ y$  for all  $x, y \in B_s(H)$ , and, in particular, commutes with every projection  $p$  in  $B_s(H)$ . In other words,  $\theta(1)$  is a central element in  $B_s(H)$ , that is, a real number. Note that  $\theta(1)$  is nonzero by (2.5). Set  $\lambda = \theta(1)$ . Then (2.5) reduces to

$$\lambda\theta(x \circ y) = \theta(x) \circ \theta(y),$$

and hence  $\varphi = \lambda^{-1}\theta$  is a surjective Jordan homomorphism on  $B_s(H)$  which is also continuous in the strong operator topology. We claim that  $\varphi$  is injective. Suppose on the contrary that  $\varphi(a) = 0$  for some nonzero  $a \in B_s(H)$ . Let  $J$  be the Jordan ideal of  $B_s(H)$  generated by  $a$ . Then the restriction of  $\varphi$  to the Jordan ideal  $J$  is zero. Let  $I$  be a nonzero  $*$ -ideal of  $B(H)$  such that  $J$  contains all the symmetric elements of  $I$ . (For instance, take  $I$  to be the ideal

generated by all the  $x^4$  for  $x \in J$ ; see the proof of [7, Theorem 2.6].) Since the nonzero  $*$ -ideal  $I$  contains  $F(H)$ , the ideal of all finite-rank operators on  $H$ , we have  $F_s(H) = S(F(H)) \subseteq J$  and so  $\varphi$  is zero on  $F_s(H)$ . Recall that  $\varphi$  is continuous and  $F_s(H)$  is dense in  $B_s(H)$  in the strong operator topology, so we conclude that  $\varphi$  is zero on  $B_s(H)$ , a contradiction. Therefore  $\varphi$  is a Jordan automorphism. As we mentioned in the introduction, there exists a unitary or anti-unitary operator  $u$  on  $H$  such that  $\varphi(x) = uxu^*$  for all  $x \in B_s(H)$  and the proof is thereby complete.  $\square$

**Acknowledgement.** We would like to thank the referee for her/his valuable comments.

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