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SOME RATIONALITY PROPERTIES OF OBSERVABLE GROUPS AND RELATED QUESTIONS

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Abstract. We investigate in this paper some rationality questions related to observable, epimorphic, and Grosshans subgroups of linear algebraic groups over non-algebraically closed fields.

1. Introduction

Let G be a linear algebraic group defined over an algebraically closed field k. Then G acts naturally on its regular function ring $k[G]$ by right translation $(r_g \cdot f)(x) = f(x \cdot g)$, for all $x, g \in G, f \in k[G]$. For H a closed k-subgroup of G, we put

$$H' = k[G]^H := \{f \in k[G] \ : r_h \cdot f = f, \text{ for all } h \in H\}.$$

Then $k[G]^H$ is the k-subalgebra of H-invariant functions of $k[G]$. By convention, we identify the algebraic groups considered here with their points in a fixed algebraically closed field. For a k-subalgebra R of $k[G]$, we put

$$R' = \{g \in G : r_g \cdot f = f \text{ for all } f \in R\}.$$

Then for any closed subgroup $H \subset G$ we have

$$H \subseteq H'' \subseteq G.$$

Motivated by representation theory, Bialynicki-Birula, Hochschild and Mostow (see [1, p. 134]) introduced the concept of "observable subgroup". A closed subgroup H of G is called an *observable subgroup* of G if any finite dimensional rational representation of H can be extended to a finite dimensional rational representation on the whole group G (or, equivalently, if every finite dimensional rational H-module is an H-submodule of a finite dimensional rational G-module). In [1] equivalent conditions for a subgroup to be observable were given. Grosshans (see [6], [7] and the references therein) has added several other conditions. It later turned out that for closed subgroups

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the property of being observable for a subgroup H is equivalent to the equality $H = H''$. There are now several equivalent conditions known for a subgroup to be observable, which are more or less easy to verify; these are gathered in Theorem 1 below.

On the opposite side, a closed subgroup $H \subseteq G$ may satisfy the equality $H'' = G$. If this holds, H is called an *epimorphic subgroup* of G. In fact, under an equivalent condition, this notion was first introduced and studied by Bien and Borel [2], [3] (see also [7] for a recent treatment), which, in turn, is based on a similar notion for Lie algebras given by Bergman (unpublished). Several other equivalent conditions for a closed subgroup to be epimorphic have been given (see Theorem 11 below).

In connection with the solution of Hilbert's 14th Problem, the following well-known problem is of great interest. Assume that X is an affine variety, G is a reductive group acting upon X morphically, H is a closed subgroup of G, and consider the G-action on the regular function ring $k[X]$ by left translation: $(l_g \cdot f)(x) = f(g^{-1} \cdot x)$. It is natural to ask when $k[X]^H$ is a finitely generated k-algebra.

For a closed subgroup $H \subset G$, we have $k[X]^H = k[X]^{H''}$ (see [6], [7]). On the other hand, it is well-known (see, e.g., [6], [7]) that H'' is the smallest observable subgroup of G containing H. So the problem is reduced to the case when H is an observable subgroup. To solve this problem, Grosshans [6], [7] introduced the "codimension 2 condition" for observable subgroups, and subgroups satisfying this condition are now called *Grosshans subgroups* of G (see Section 4).

In this paper, we continue the study initiated in [1]. Namely, we are interested in some questions of rationality related to observable, epimorphic, and Grosshans subgroups. The first rationality results regarding observable (resp. epimorphic) subgroups were obtained in [1], and then in [7], [10] (resp. [2], [3] and [10]), where also some arithmetical applications to ergodic actions were given. In this paper we prove some new results on rationality properties of observable, epimorphic, and Grosshans subgroups (which were stated initially for algebraically closed fields). In a subsequent paper under preparation some arithmetic and geometric applications will be considered.

Throughout, we consider only linear algebraic groups (i.e., absolutely reduced affine group schemes of finite type) defined over some field k, which, in short, are called *k-groups*. For the basic theory of linear algebraic groups over non-algebraically closed fields we refer to [4]. For a k-group G, the notion of a rational k-module V for G is as in [6], [7].

2. Some rationality results for observable groups

First we recall well-known results over algebraically closed fields. For an algebraic group G we denote by G° the identity connected component subgroup of G.

Theorem 1 ([1], [7, Theorems 2.1 and 1.12]). *Let G be a linear algebraic group defined over an algebraically closed field k and let H be a closed k-subgroup of G. Then the following conditions are equivalent:*

(a) $H = H''$.

(b) *There exists a finite dimensional rational representation $\rho : G \to GL(V)$ and a vector $v \in V$, all defined over k, such that*

$$H = G_v = \{g \in G : \rho(g) \cdot v = v\}.$$

(c) *There are finitely many functions $f \in k[G/H]$ which separate the points in G/H.*

(d) *G/H is a quasi-affine k-variety.*

(e) *Every finite dimensional rational k-representation $\rho : H \to GL(V)$ can be extended to a finite dimensional rational k-representation $\rho' : G \to GL(V')$, where $V \hookrightarrow V'$, i.e., every finite dimensional rational H-module is an H-submodule of a finite dimensional rational G-module.*

(f) *There is a finite dimensional rational k-representation $\rho : G \to GL(V)$ and a vector $v \in V$ such that $H = G_v$, the isotropy group of v, and*

$$G/H \cong G \cdot v = \{\rho(g) \cdot v : g \in G\}$$

(as algebraic varieties).

(g) *The quotient field of the ring of $G^\circ \cap H$-invariants in $k[G^\circ]$ is equal to the field of $G^\circ \cap H$-invariants in $k(G^\circ)$.*

(h) *If a 1-dimensional rational H-module M is an H-submodule of a finite dimensional rational G-module, then the H-dual module M^* of M is also an H-submodule of a finite dimensional rational G-module.*

Now let k be any field. If a closed k-subgroup H of a linear algebraic k-group G satisfies condition (b) (resp. (e)) in Theorem 1, where $v \in V(k)$ and the corresponding representation ρ is defined over k, then we say that H is an *isotropy k-subgroup of G* (resp. has the *extension property* over k).

We first recall the following rationality results proved in [1].

Theorem 2 ([1, Theorem 5]). *Let G be a linear algebraic k-group, H a closed k-subgroup of G, and $k \subset K$ an algebraic extension of k. Then H has the extension property over k if and only if it has the extension property over K.*

Theorem 3 ([1, Theorem 8]). *If H is a closed k-subgroup of a linear algebraic k-group G with the extension property over k, then H is an isotropy k-subgroup of G. Conversely, if k is algebraically closed and H is an isotropy k-subgroup, then it has the extension property over k.*

From Theorem 2 and Theorem 3 we derive the following result.

Proposition 4. *Let k be an arbitrary field and H a closed k-subgroup of a k-group G. The following two conditions are equivalent:*

(a) *H is an isotropy subgroup of G over $\overline{k}$.*

(b) *H is an isotropy subgroup of G over k, i.e., there exists a finite dimensional k-rational representation $\rho : G \to GL(V)$ and a vector $v \in V(k)$ such that $H = G_v$.*

Proof. (b)⇒(a): Trivial.

(a)⇒(b): By Theorem 1, since H is an isotropy subgroup over $\overline{k}$, H has the extension property over $\overline{k}$. Therefore, by Theorem 2, H has the extension property over k. By Theorem 3, H is an isotropy k-subgroup of G. □

Remark 1. In [10], another proof of Proposition 4 was given, which is based on some ideas of Grosshans [6], under the condition (which is not essential) that $k = \mathbf{Q}$ and H is connected.

We put

$$H'_k = k[G]^{H(k)} = \{f \in k[G] : r_h \cdot f = f, \forall h \in H(k)\},$$

and

$$(H'_k)' = \{g \in G : r_g \cdot f = f, \forall f \in H'_k\}.$$

Then $k[G]^{H(k)}$ and $k[G]^H := \{f \in k[G] : r_h \cdot f = f, \forall h \in H\}$ are k-subalgebras of $k[G]$. In general we have the following diagram:

$$\begin{array}{ccc} H'_k = k[G]^{H(k)} & \subseteq & \bar{k}[G]^{H(k)} \\ \cup\!| & & \cup\!| \\ k[G]^H & \subseteq & \bar{k}[G]^H = H'. \end{array}$$

Thus we have

$$\begin{array}{ccc} (H'_k)' = (k[G]^{H(k)})' & \supseteq & (\bar{k}[G]^{H(k)})' \\ \cap\!| & & \cap\!| \\ (k[G]^H)' & \supseteq & (\bar{k}[G]^H)' = H''. \end{array}$$

If, moreover, $H(k)$ is Zariski dense in H, then we have

$$H'_k = k[G]^H = \overline{k}[G]^H \cap k[G]\,.$$

We say that H is *relatively observable over* k if $H = (H'_k)'$, and k*-observable* if $(k[G]^H)' = H$. It is clear that if k is algebraically closed, then these notions coincide with the observability. We have the obvious implication

$$H \text{ is } k\text{-observable} \Rightarrow H \text{ is observable.}$$

PROPOSITION 5. *Let k be a field, and let H be a closed k-subgroup of a k-group G. Then:*

(a) $H' = \overline{k}[G]^H = \overline{k} \otimes_k k[G]^H$.
(b) *H is observable if and only if H is k-observable.*
(c) *Assume that $H(k)$ is Zariski dense in H. Then H is observable $\Leftrightarrow$ H is k-observable $\Leftrightarrow$ H is relatively observable over k.*

Proof. (a) We need the following lemma.

LEMMA 6. *Let X be an affine scheme of finite type over k upon which a k-group H acts k-morphically, such that the (good) quotient scheme X/H exists. Then we have*

$$\overline{k}[X]^H = \overline{k} \otimes_k k[X]^H.$$

(Here, by convention, $X = \mathrm{Spec}(\bar{k}[X])$, and $k[X]$ gives the k-structure of $\bar{k}[X]$.)

Proof. Since X/H is defined over k, we have $\overline{k}[X/H] = \overline{k} \otimes_k k[X/H]$. Moreover, the quotient morphism $\pi : X \to X/H$ is also defined over k, so the comorphism π^0 sends $k[X/H]$ into $k[X]$. On the other hand, $\pi^0 : \overline{k}[X/H] \to \overline{k}[X]^H$ is an isomorphism. So $\pi^0|_{k[X/H]} : k[X/H] \to k[X] \cap \overline{k}[X]^H = k[X]^H$ is a monomorphism. Because of the k-linearity of π^0, we have

$$\begin{aligned}
\overline{k}[X]^H &= \pi^0(\overline{k}[X/H]) \\
&= \pi^0(\overline{k} \otimes k[X/H]) \\
&= \overline{k} \otimes \pi^0(k[X/H]) \\
&\subseteq \overline{k} \otimes (k[X]^H) \\
&\subseteq \overline{k}[X]^H.
\end{aligned}$$

Thus the above equalities imply that $\pi^0(k[X/H]) = k[X]^H$. Since π^0 is an isomorphism, we have $\overline{k}[X]^H = \overline{k} \otimes_k k[X]^H$. The lemma is proved. $\square$

Now (a) follows by taking $X = G$ in the lemma.

(b) It suffices to show that if H is observable then it is also k-observable. But this follows directly from (a) .

(c) By part (b) we need only show that

$$H \text{ is relatively observable over } k \Leftrightarrow H \text{ is } k\text{-observable.}$$

"⇒": Since $H(k)$ is Zariski dense in H, we have

$$f \in \bar{k}[G]^{H(k)} \Leftrightarrow f \in \bar{k}[G]^{H}.$$

Therefore

$$H = (k[G]^{H(k)})' \supseteq (\bar{k}[G]^{H(k)})' = (\bar{k}[G]^{H})' \supseteq H,$$

i.e., H is observable, hence also k-observable, by (b).

"⇐": If H is k-observable, then we have

$$H = (k[G]^{H})' \supseteq (k[G]^{H(k)})' \supseteq H,$$

so H is relatively observable over k. □

Proposition 7. *Let H be a k-subgroup of a k-group G. The following are equivalent:*

(a) *There exist finitely many functions in $\overline{k}[G/H]$ which separate the points in G/H.*
(b) *There exist finitely many functions in $k[G/H]$ which separate the points in G/H.*

Proof. The assertion (b)⇒(a) is obvious. To prove (a)⇒(b), notice that since G/H defined over k, we have

$$\overline{k}[G/H] = \overline{k} \otimes k[G/H].$$

Assume that the functions $f_1, \ldots, f_n \in \overline{k}[G/H]$ separate the points in G/H. We have

$$f_i = \sum_j \lambda_{ij}\varphi_{ij},$$

with $\lambda_{ij} \in \overline{k}, \varphi_{ij} \in k[G/H]$. If $xH \neq yH \in G/H$, there exists i such that $f_i(xH) \neq f_i(yH)$. So there exists j such that $\varphi_{ij}(xH) \neq \varphi_{ij}(yH)$. Hence the functions $\{\varphi_{ij}\} \subseteq k[G/H]$ separate the points in G/H. □

Proposition 8. *Let G be a k-group, H a closed k-subgroup of G. Assume that there exists a finite dimensional k-rational representation $\rho : G \to GL(V)$ and $v \in V(k)$ such that $H = G_v$. Then there is a finite dimensional k-rational representation $\rho' : G \to GL(W)$ and $w \in W(k)$ such that $H = G_w$ and $G/H \cong_k G \cdot w$.*

Proof. (Our original proof of this result was lengthy; the following proof is based on communications with F. Grosshans.) By Theorem 1.12 of [7], there exists a vector space V', a representation $\rho' : G \to \mathrm{GL}(V')$, and a vector $v \in V'$ such that $H = G_v$ and there is an isomorphism

$$G/H \simeq G \cdot v.$$

Let $X = \overline{G \cdot v}$ which is a closed subvariety of V', V'^* the dual vector space of V', and $\{\lambda_1, \dots, \lambda_n\}$ a basis of V'^*. Thus, considered as an affine space, we have $\bar{k}[V'] = \bar{k}[\lambda_1, \dots, \lambda_n]$. The morphisms

$$\varphi : G \xrightarrow{\pi} G/H \overset{p}{\hookrightarrow} X \hookrightarrow V', g \mapsto g \cdot v$$

correspond to the comorphisms

$$\varphi^* : \bar{k}[V'] \xrightarrow{r} \bar{k}[X] \xrightarrow{p^*} \bar{k}[G/H] \overset{\pi^*}{\hookrightarrow} \bar{k}[G], \varphi^*(\lambda_i)(g) = \lambda_i(g \cdot v),$$

where r is the restriction. We may identify $\bar{k}[G/H]$ with $\bar{k}[G]^H$, and thus consider it as a subalgebra of $\bar{k}[G]$. It is clear that φ^* is G-equivariant with respect to left translation, and (by the construction) we have

$$\varphi^*(\bar{k}(X)) = \bar{k}(G)^H$$

and

$$l_i := \varphi^*(\lambda_i) = p^*(r(\lambda_i)) \in \bar{k}(G)^H \cap \bar{k}[G] = \bar{k}[G]^H.$$

By Proposition 5 we may write

$$l_i = \sum_j c_{ij} \otimes \mu_{ij}, \quad c_{ij} \in \bar{k}, \mu_{ij} \in k[G]^H, \forall i, j.$$

Since G is defined over k, the G-orbit of μ_{ij} spans a finite dimensional vector subspace of $\bar{k}[G]$, which is defined over k. By adding a finite number of functions (see, e.g., [4, Proposition, p. 54]) we may therefore assume that the functions $\{\mu_{ij}\}$ are k-linearly independent and that the $\bar{k}$-vector space W' with k-basis $\{\mu_{ij}\}$ is defined over k and is G-stable. If we let W be the dual k-vector space of W', this gives to a representation $\rho : G \to \mathrm{GL}(W)$, which is defined over k.

Denote by Y the affine k-variety with $\bar{k}[\mu_{ij}]$ as $\bar{k}$-algebra of functions. By considering the algebra of regular functions $\bar{k}[W]$ on the vector space W defined over k, we have the k-homomorphisms of k-algebras

$$k[W] \to k[\mu_{ij}] \to k[G/H] \to k[G],$$

which corresponds to G-equivariant k-morphisms of k-varieties with G-action

$$G \to G/H \xrightarrow{q} Y \xrightarrow{r} W.$$

One checks that the k-morphism $q : G/H \to Y$ is dominant.

Set $y = q(eH) \in Y$. Then Y is the closure of the G-orbit $G \cdot y$, which is isomorphic to G/H (since it is so over $\bar{k}$). Hence it is a k-isomorphism, and the representation $\rho : G \to \mathrm{GL}(W)$ is the one required. Therefore Proposition 8 is proved. □

From results proved above, we have the following theorem, which is an analog of Theorem 1 for arbitrary fields.

Theorem 9. *Let G be a linear algebraic group defined over a field k and let H be a closed k-subgroup of G. Then the following conditions are equivalent:*

(a) *$H = H''$, i.e., H is observable.*
(a') *$H = (k[G]^H)'$, i.e., H is k-observable.*
(b') *There exists a k-rational representation $\rho : G \longrightarrow GL(V)$ and a vector $v \in V(k)$ such that*

$$H = G_v = \{g \in G : g \cdot v = v\}.$$

(c') *There are finitely many functions $f \in k[G/H]$ which separate the points in G/H.*
(d') *G/H is a quasiaffine variety defined over k.*
(e') *Every k-rational representation $\rho : H \longrightarrow GL(V)$ can be extended to a k-rational representation $\rho' : G \longrightarrow GL(V')$.*
(f') *There is a k-rational representation $\rho : G \longrightarrow GL(V)$ and a vector $v \in V(k)$ such that $H = G_v$ and*

$$G/H \cong_k G \cdot v = \{\rho(g)v : g \in G\}.$$

(g') *The quotient field of the ring of $G^\circ \cap H$-invariants in $k[G^\circ]$ is equal to the field of $G^\circ \cap H$-invariants in $k(G^\circ)$.*

If, moreover, $H(k)$ is Zariski dense in H, then the above conditions are equivalent to the relative observability of H over k.

Proof. First, by Proposition 4, with the conditions labelled as in Theorem 1 we have (b)$\Leftrightarrow$(b'), by Proposition 5 we have (a)$\Leftrightarrow$(a'), and by Proposition 7 we have (c) $\Leftrightarrow$(c'). The fact that (d)$\Leftrightarrow$(d') is trivial, and we have (e)$\Leftrightarrow$(e') by Theorem 2, and the same proof as that of [7, Theorem 1.12] shows that we have (f)$\Leftrightarrow$(f').

To prove the equivalence of (g') with the other conditions, we use the other equivalent conditions. We need the following lemma.

Lemma 10. *With the above assumptions, H is k-observable in G if and only if $H \cap G^\circ$ is k-observable in G°.*

Proof. First observe that since H and G^0 are defined over k, so is $H \cap G^0$. We have (a)$\Leftrightarrow$(a'), so H is observable in G if and only if H is k-observable in G, and $H \cap G^\circ$ is observable in G° if and only if $H \cap G^\circ$ is k-observable in G°. By [7, Corollary 1.3], H is observable in G if and only if $H \cap G^\circ$ is observable in G°. It follows that H is k-observable in G if and only $H \cap G^\circ$ is k-observable in G°. The lemma is proved. □

Now, by [1, Theorem 3], $H \cap G^\circ$ is k-observable in G° if and only if (g') holds. □

3. Rationality properties for epimorphic subgroups

Recall that by a result of Grosshans [7] epimorphic subgroups $H \subseteq G$ are those closed subgroups of G that satisfy the condition $H'' = G$. We have the following characterizations of epimorphic subgroups over an algebraically closed fields.

Theorem 11 ([1, Théorème 1], [7, Lemma 23.7]). *Let H be a closed subgroup of G, all defined over an algebraically closed field k. Then the following conditions are equivalent:*

(a) *H is epimorphic, i.e., $H'' = G$.*
(b) *$k[G/H] = k$.*
(c) *$k[G/H]$ is finite dimensional over k.*
(d) *If V is any rational G-module, then the spaces of fixed points of G and H in V coincide.*
(e) *If V is a rational G-module such that $V = X \oplus Y$, where X, Y are H-invariant, then X, Y are also G-invariant.*
(f) *Morphisms of algebraic groups from G to another one L are defined by their values on H.*

Remark 2. The initial definition of epimorphic subgroups given in [2] only required that condition (f) above hold.

Let the notation be as in Section 2, and let k be an arbitrary field. Then for a k-subgroup H of a k-group G we say that H is *relatively epimorphic over k* if $(H'_k)' = G$, and *k-epimorphic* if $(k[G]^H)' = G$. Recall that we have the following inclusions:

$$\begin{array}{ccc} (H'_k)' = (k[G]^{H(k)})' & \supseteq & (\bar{k}[G]^{H(k)})' \\ \bigcap\mid & & \bigcap\mid \\ (k[G]^H)' & \supseteq & (\bar{k}[G]^H)' = H''. \end{array}$$

Therefore the following implications hold:

$$H \text{ is epimorphic} \Rightarrow H \text{ is } k\text{-epimorphic},$$
$$H \text{ is } k\text{-epimorphic} \Leftarrow H \text{ is relatively epimorphic over } k.$$

In fact, we have the following result:

Proposition 12. *With above notation, if H is either (a) relatively epimorphic over k or (b) k-epimorphic, then it is also epimorphic.*

Proof. We need only check that $H'' \supseteq G$. Assume that (a) holds. Let $g \in G$ be an arbitrary element, and let $f \in H'$. Then $r_h(f) = f$ for all $h \in H$.

By Proposition 5, we have

$$H' = \overline{k}[G]^H = \overline{k} \otimes_k k[G]^H.$$

Therefore we may write $f = \sum_i c_i f_i, c_i \in \bar{k}, f_i \in k[G]^H$. Since $f_i \in k[G]^H \subset k[G]^{H(k)} = H'_k$ and, by assumption, $g \in G = (H'_k)'$, we have $r_g(f_i) = f_i$ for all i. Therefore $r_g(f) = f$, i.e., $g \in H''$. Thus $G = H''$.

Now assume (b) holds. Then by Proposition 5 again, we have

$$\bar{k}[G]^H = k[G]^H \otimes \bar{k},$$

so

$$\begin{aligned}(\bar{k}[G]^H)' &= (k[G]^H \otimes \bar{k})' \\ &= (k[G]^H)' = G.\end{aligned}$$

Thus H is also epimorphic. □

We have the following analog of Theorem 11 over an arbitrary field.

Theorem 13. *Let k be any field and let H be a closed k-subgroup of a k-group G. Then the following conditions are equivalent:*

(a') *H is k-epimorphic, i.e., $(k[G]^H)' = G$.*
(b') *$k[G/H] = k$.*
(c') *$k[G/H]$ is finite dimensional over k.*
(d') *For any rational G-module V defined over k the spaces of fixed points of G and H in V coincide.*
(e') *For any rational G-module V defined over k, if $V = X \oplus Y$, where X, Y are H-invariant, then X, Y are also G-invariant.*
(f') *Morphisms defined over k of algebraic k-groups from G to another one are defined by their values on H.*

Proof. In what follows we refer to Theorem 11 for the properties (a)–(f). By Proposition 12 and the implications before it, we have (a)⇔(a'). Since G/H is defined over k, we have (b)⇔(b') and (c)⇔(c').

The proof of the implication (i)⇒(ii) of [2, Théorème 1] (i.e., (f)⇒(b) above) gives also a proof of the implication (f')⇒(b'). The implication (b')⇒(c') is trivial. We have (c')⇔(c)⇔(d)⇒(d') and the same proof as that of [2, Théorème 1] shows that (d')⇒(e')⇒(f'). Thus we have the equivalence of statements (b'), (c'), (d'), (e'), (f'). Since the statements (a), (b), (c), (d), (e), (f) are equivalent and (a)⇔(a'), the theorem follows. □

Remark 3. It was mentioned in [10, p. 195] that Bien and Borel (unpublished) have also proved that if G is connected, then (d)⇔(d').

4. Some rationality properties for Grosshans subgroups

One of the main results on the finite generation problem mentioned in the Introduction (and hence also on Hilbert's 14th Problem) is the following result of Grosshans (Theorem 15). Before stating this result, we recall another very useful result which reduces the problem to the case of connected groups.

THEOREM 14 ([7, Theorem 4.1]). *Let k be an algebraically closed field. For any closed subgroup H of G, if one of the k-algebras $k[G]^H$, $k[G]^{H^\circ}$, $k[G^\circ]^{H \cap G^\circ}$, $k[G^\circ]^{H^\circ}$ is a finitely generated k-algebra, then the same holds for the other k-algebras.*

THEOREM 15 ([7, Theorem 4.3]). *For an observable subgroup H of a linear algebraic group G, all defined over an algebraically closed field k, the following are equivalent:*

(a) *There is a finite dimensional rational representation $\varphi : G \to GL(V)$ and an element $v \in V$ such that $H = G_v$ and each irreducible component of $\overline{G \cdot v} - G \cdot v$ has codimension ≥ 2 in $\overline{G \cdot v}$.*
(b) *The k-algebra $k[G]^H$ is a finitely generated k-algebra.*

If (b) *holds, let X be an affine variety with $k[X] = k[G]^H$, and with G-action via left translations of G on G/H. There is a point $x \in X$ such that $G \cdot x$ is open in X, $G \cdot x \simeq G/H$ via $gH \mapsto g \cdot x$, and each irreducible component of $X \setminus G \cdot x$ has codimension ≥ 2 in X.*

The observable subgroups which satisfy one of the equivalent conditions in Theorem 15 are called *Grosshans subgroups* (see [7]). There exist nice geometrical characterizations and examples of Grosshans subgroups; see [7] and the references therein.

For a field k, a k-group G and an observable k-subgroup $H \subset G$, we say that H satisfies the *codimension* 2 *condition over* k if H satisfies condition (a) above, where V, φ are all defined over k and $v \in V(k)$.

We call H a *Grosshans subgroup relatively over* k (resp. *k-Grosshans subgroup*) of G if $k[G]^{H(k)}$ (resp. $k[G]^H$) is a finitely generated k-algebra.

We have a result similar to Theorem 15 for k-Grosshans subgroups.

THEOREM 16. *Let k be any perfect field with infinitely many elements and G a connected k-group. Assume that H is an observable k-subgroup of G. Consider the following conditions:*

(a') *H satisfies the codimension 2 condition over k.*
(b') *One of the k-algebras $k[G]^H$, $k[G]^{H^\circ}$, $k[G^\circ]^{H \cap G^\circ}$, $k[G^\circ]^{H^\circ}$ is a finitely generated k-algebra.*
(c') *H is a Grosshans subgroup of G relative to k (i.e., $k[G]^{H(k)}$ is a finitely generated k-algebra).*

Then, together with conditions in Theorem 15, we have the following implications:

$$(a) \Leftrightarrow (a') \Leftrightarrow (b) \Leftrightarrow (b') \Rightarrow (c').$$

If, moreover, $H(k)$ is Zariski dense in H, then all these conditions are equivalent.

Proof. (a)⇔(a'): We have trivially (a')⇒(a). The proof of Proposition 8 and that of Theorem 4.3 of [7] (the computation of the dimension) show that we also have (a)⇒(a'). Thus (a)⇔(a').

(b)⇔(b'): Recall that we have (a)⇔(b) (see Theorem 15 above). By Lemma 6 we have

$$\bar{k}[G]^H \simeq \bar{k} \otimes_k k[G]^H.$$

Therefore it is clear that $\bar{k}[G]^H$ is a finitely generated $\bar{k}$-algebra if and only if so is $k[G]^H$. Similarly, $\bar{k}[G]^{H^\circ} \simeq k[G]^{H^\circ} \otimes_k \bar{k}$ is finitely generated as $\bar{k}$-algebra if and only if $k[G]^{H^\circ}$ is a finitely generated k-algebra. This is also true if H° is replaced by $H \cap G^\circ$, etc. Thus (b)⇔(b').

(b')⇒(c'): If H is connected, then $H(k)$ is Zariski dense in H (see, e.g., [4, 18.3], or [5]), and we have

$$(*) \qquad k[G]^H = k[G]^{H(k)},$$

and the assertion is trivial. Otherwise, assume that $H \neq H^\circ$. Then we can use Theorem 14 above. In fact, $H^\circ(k)$ is a normal subgroup of finite index in $H(k)$, and we see that

$$k[G]^{H(k)} = (k[G]^{H^\circ(k)})^{H(k)/H^\circ(k)}$$

is a finitely generated k-algebra, since from the equivalence (a)⇔(a')⇔(b) ⇔(b') and from Theorem 14 it follows that $k[G]^{H^\circ(k)}$ is a finitely generated k-algebra, and that $H(k)/H^\circ(k)$ is a finite group.

Assume further that $H(k)$ is Zariski dense in H. Then $(*)$ holds, so the theorem is proved. □

Remark 4. It is of interest to find examples for which condition (c') holds, but the other conditions do not. This will, perhaps, ultimately lead to counter-examples to the (generalized) Hilbert's 14th Problem in the case when $\operatorname{char} k > 0$. (Various extensions of classical results in (geometric) invariant theory to the case of characteristic $p > 0$ were discussed at length in [9, Appendices].) It would be more interesting to have examples with G, H connected groups.

A connection to the subalgebra of invariants of a Grosshans subgroup of a reductive group acting rationally upon a finitely generated commutative algebra is established in the following result:

THEOREM 17 ([7, Theorem 9.3]). *Let k be an algebraically closed field. For any closed subgroup H of a reductive group G, all defined over k, the following conditions are equivalent:*

(a) *$k[G]^H$ is a finitely generated k-algebra.*
(b) *For any finitely generated commutative k-algebra A on which G acts rationally, the algebra of invariants A^H is a finitely generated k-algebra.*

We consider the following relative version of this theorem.

THEOREM 18. *Let k be a perfect field with infinitely many elements, and H a closed k-subgroup of a connected reductive k-group G. Consider the following conditions.*

(a') *$k[G]^H$ is a finitely generated k-algebra.*
(b') *For any finitely generated commutative k-algebra A_k on which G acts k-rationally, the algebra of invariants A_k^H is a finitely generated k-algebra.*
(c') *For any finitely generated commutative k-algebra A_k on which G acts k-rationally, the algebra of invariants $A_k^{H(k)}$ is a finitely generated k-algebra.*

Then, with the notations as in Theorem 17, we have

$$(a) \Leftrightarrow (a') \Leftrightarrow (b) \Leftrightarrow (b') \Rightarrow (c').$$

If, moreover, $H(k)$ is Zariski dense in H, then all of the above conditions are equivalent.

Proof. The proof follows the same lines as that of Theorem 16 by using Theorem 17. □

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REFERENCES

[1] A. Białynicki-Birula, G. Hochschild, and G. D. Mostow, *Extensions of representations of algebraic linear groups*, Amer. J. Math. **85** (1963), 131–144. MR 0155938 (27 #5871)
[2] F. Bien and A. Borel, *Sous-groupes épimorphiques de groupes linéaires algébriques. I*, C. R. Acad. Sci. Paris Sér. I Math. **315** (1992), 649–653. MR 1183796 (93i:20048)
[3] ———, *Sous-groupes épimorphiques de groupes linéaires algébriques. II*, C. R. Acad. Sci. Paris Sér. I Math. **315** (1992), 1341–1346. MR 1198999 (93k:20063)

[4] A. Borel, *Linear algebraic groups*, Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991. MR 1102012 (92d:20001)

[5] M. Demazure et A. Grothendieck, *Schémas en groupes. I–III*, Séminaire de Géometrie Algébrique Du Bois Marie (SGA 3), Lecture Notes in Mathematics, vol. 151–153, Springer-Verlag, Berlin, 1962/1964. MR 0274458 (43#223a); MR 0274459 (43#223b); MR 0274460 (43#223c)

[6] F. Grosshans, *Observable groups and Hilbert's fourteenth problem*, Amer. J. Math. **95** (1973), 229–253. MR 0325628 (48 #3975)

[7] ______, *Algebraic homogeneous spaces and invariant theory*, Lecture Notes in Mathematics, vol. 1673, Springer-Verlag, Berlin, 1997. MR 1489234 (99b:13005)

[8] J. E. Humphreys, *Linear algebraic groups*, Springer-Verlag, New York, 1975. MR 0396773 (53 #633)

[9] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (2), vol. 34, Springer-Verlag, Berlin, 1994. MR 1304906 (95m:14012)

[10] B. Weiss, *Finite-dimensional representations and subgroup actions on homogeneous spaces*, Israel J. Math. **106** (1998), 189–207. MR 1656889 (99g:22017)

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