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VECTOR MEASURES AND NUCLEAR OPERATORS

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ABSTRACT. Among other results we prove that for a Banach space X and $1 < p < \infty$, all p-unconditionally Cauchy sequences in X lie inside the range of a Y-valued measure of bounded variation for some Banach space Y containing X if and only if each ℓ_1-valued 2-summing map on X induces a nuclear map on X valued in ℓ_q, q being conjugate to p. We also characterise Banach spaces X with the property that all ℓ_2-valued absolutely summing maps on X are already nuclear as those for which X^* has the (GT) and (GL) properties.

1. Introduction

The study of nuclear operators—which generalises trace class operators to the Banach space setting—has come in as a handy tool in the study of the geometry of Banach spaces. The fact that a Banach space X has the Radon-Nikodym property precisely when X-valued absolutely summing maps on $C[0,1]$ are nuclear is a typical example of this phenomenon. In the theory of vector measures, it is known (see, e.g., [16]) that ℓ_1-valued absolutely summing maps on X coincide with nuclear maps exactly when null sequences in X are included inside the range of an X-valued measure. In the same work, it is also proved that assuming the vector measures involved to have bounded variation results in the class of finite dimensional Banach spaces. In light of these extreme cases, it is natural to address the question of describing those Banach spaces X such that every null sequence in X is included inside the range of a vector measure of bounded variation which takes its values inside a (larger) space containing X isometrically. In [13], Pineiro shows that such spaces are isomorphic copies of Hilbert spaces.

One of our main aims in the present paper is to provide alternative proofs, which are short and more transparent, of these and some other results of Pineiro by exploiting a by-now famous theorem of Johnson, König, Maurey and Retherford [9] on the characterisation of a Hilbert space X in terms of the eigenvalue distribution of nuclear operators on X.

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THEOREM 1.1 ([9]). *A Banach space X is isomorphic to a Hilbert space if and only if nuclear operators on X have absolutely summing eigenvalues.*

For further applications and ramifications of this theorem, see also the survey article [18].

We shall also use this occasion to prove some new results on the characterisation of Banach spaces in terms of ℓ_2-valued nuclear operators on X and use this result to show that in the presence of the Gordon-Lewis property finite dimensional spaces are the only Banach spaces X such that every set in X included inside the range of an X-valued measure is already included inside the range of an X^{**}-valued measure of bounded variation. This is done in Section 4. Section 3 deals with the question of describing those Banach spaces X that have the property that p-unconditionally Cauchy sequences in X are included inside the range of a vector measure of bounded variation taking its values inside a Banach space containing X. This result, which is subsequently used in Section 4, is of independent interest and refines certain results already known in the literature.

2. Notations and definitions

We shall follow [5] for the theory of vector measures and [4] for various concepts pertaining to Banach spaces and the theory surrounding nuclear and absolutely summing maps as used in this paper. In what follows, X shall denote a Banach space with B_X and X^* denoting its closed unit ball and the dual, respectively, whereas μ will stand for an X-valued vector measure $\mu : (\Omega, \Sigma) \to X$ defined on a measure space (Ω, Σ) such that μ is countably additive (c.a.).

DEFINITION 2.1. Given a bounded linear map $T : X \to Y$ between Banach spaces X and Y, we shall say that T is

(a) *nuclear* ($T \in N(X,Y)$) if there exist bounded sequences $\{f_n\}_{n=1}^{\infty} \subset B_{X^*}, \{y_n\}_{n=1}^{\infty} \subset B_Y$ and $\{\lambda_n\}_{n=1}^{\infty} \in \ell_1$ such that

$$T(x) = \sum_{n=1}^{\infty} \lambda_n \langle x, f_n \rangle y_n, \ x \in X.$$

Equivalently, $T : X \to Y$ is nuclear if and only if T can be factorised as $T = T_2 \circ D \circ T_1$, where $T_1 : X \to \ell_\infty, T_2 : \ell_1 \to Y$ are bounded linear maps and $D = D_{\bar{\xi}} : \ell_\infty \to \ell_1$ is a diagonal map induced by $\bar{\xi} = (\xi_n) \in \ell_1 : D_\xi(\bar{\alpha}) = (\alpha_n \xi_n)_{n=1}^{\infty}, \bar{\alpha} = (\alpha_n) \in \ell_\infty$.

(b) *p-absolutely summing* $(1 \leq p < \infty)$ ($T \in \Pi_p(X,Y)$) if there exists $c > 0$ such that

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p \right)^{1/p} \leq c \sup_{f \in B_{X^*}} \left(\sum_{i=1}^{n} |\langle x_i, f \rangle|^p \right)^{1/p}$$

for all $x_i \in X, 1 \le i \le n, n \ge 1$.

The infimum of $\sum_{n=1}^{\infty} |\lambda_n|$ taking over all representations of T as given in (a) shall be denoted by $\nu(T)$, the *nuclear norm* of T, whereas the *p-summing norm* of T, denoted by $\pi_p(T)$, is the infimum of all $c > 0$ appearing in (b). Clearly

$$\|T\| \le \pi_p(T) \le \nu(T),$$

as can be easily checked.

Grothendieck's theorem (see [17, Theorem 5.12]) states that all bounded linear maps from an L_1 space into a Hilbert space are absolutely summing.

We collect below some important properties of these classes of operators which shall be used in the sequel.

PROPOSITION 2.2 ([10, Chapter 2], [13, Chapter 1]). *Let X and Y be Banach spaces and assume that $1 \le p, q < \infty$. Then the following assertions hold:*

(a) $N(X,Y) \subset \Pi_p(X,Y) \ni: \pi_p(T) \le \nu(T),\ \forall\, T \in N(X,Y)$.
(b) $\Pi_p(X,Y) \subset \Pi_q(X,Y) \ni: \pi_q(T) \le \pi_p(T),\ \forall\, T \in \Pi_p(X,Y), (p \le q)$.
(c) $\Pi_2^{(2)}(X,Y) \subset N(X,Y) \ni: \nu(TS) \le \pi_2(T)\pi_2(S),\ \forall\, S \in \Pi_2(X,Z)$, $\forall\, T \in \Pi_2(Z,Y)$.
(d) $\Pi_p(X) \subset \mathcal{E}_q(X), q = \max(p,2)$.
(e) $\Pi_p \circ \Pi_q(X) \subset \mathcal{E}_r(X),\ (\frac{1}{r} = \frac{1}{p} + \frac{1}{q})$.

Here we recall that for operator ideals $\mathcal{A}$ and $\mathcal{B}$ the symbol $\mathcal{A} \circ \mathcal{B}(X,Y)$ has been used for the *component* of $\mathcal{A} \circ \mathcal{B}$ on the pair (X,Y), defined by

$$\mathcal{A} \circ \mathcal{B}(X,Y) = \{T : X \to Y :\ \exists\ Z \text{ and } T_1 \in \mathcal{A}(X,Z), T_2 \in \mathcal{B}(Z,Y) \text{ such that } T = T_2T_1\},$$

where $\mathcal{E}_p(X)$ stands for those operators on X which have p-summable eigenvalues.

DEFINITION 2.3. For $p \ge 1$, $\ell_p[X]$ shall denote the vector space of all sequences $(x_n)_{n=1}^{\infty}$ in X such that $\sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p < \infty$ for all $x^* \in X^*$. This turns into a Banach space when equipped with the norm

$$\epsilon_p((x_n)) = \sup \left\{ \left(\sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p \right)^{1/p} ; x^* \in B_{X^*} \right\}.$$

We shall be mainly concerned with the (closed) subspace $\ell_p(X)$ of $\ell_p[X]$ given by

$$\ell_p(X) = \{\hat{x} = (x_n) \in \ell_p[X] : \epsilon_p((0,\ldots,0,x_n,x_{n+1},\ldots)) \xrightarrow{n} 0\}.$$

The elements of $\ell_p[X]$ shall be called p-weakly summable sequences, whereas those from $\ell_p(X)$ shall be referred to as p-unconditionally Cauchy sequences in X. An easy consequence of the uniform boundedness theorem shows that $\ell_\infty[X]$ coincides with the space of all bounded sequences in X, whereas $\ell_\infty(X) = c_0(X)$, the Banach space of null sequences in X. It is also easy to see that $\ell_\infty[X]$ gets identified with $L(\ell_1, X)$, the space of bounded linear operators, via the map $\ell_\infty[X] \ni \bar{x} = (x_n) \to T_{\bar{x}} \in L(\ell_1, X)$, where

$$T_{\bar{x}}(\bar{\alpha}) = \sum_{n=1}^{\infty} \alpha_n x_n,\ \bar{\alpha} = (\alpha_n) \in \ell_1.$$

This identification can be used to describe a useful relationship between different types of vector measures and absolutely summing maps. To this end, we introduce the following vector-valued sequence spaces in X:

$$\begin{aligned} R(X) &= \{\bar{x} = (x_n) \in \ell_\infty[X] :\ \exists \text{ vector measure } \mu : \Sigma \to X \\ &\qquad \text{such that } (x_n) \subset \mathrm{rg}(\mu)\}, \\ R_{vbv}(X) &= \{\bar{x} = (x_n) \in \ell_\infty[X] :\ \exists\ X_0,\ \text{a Banach space,} \\ &\qquad \text{an isometry } T : X \to X_0 \text{ and an } X_0\text{-valued measure } \mu \\ &\qquad \text{of bounded variation such that } Tx_n \in \mathrm{rg}(\mu), n \geq 1\}, \\ R^*_{bv}(X) &= R_{vbv}(X) \text{ for } X_0 = X^{**}. \end{aligned}$$

Clearly, $R^*_{bv}(X) \subset R_{vbv}(X)$, whereas $R_{vbv}(X) \subset R(X)$ follows from a theorem of Anantharaman and Diestel (see Remark 3.3). Here $\mathrm{rg}(\mu) = \{\mu(A); A \in \Sigma\}$ denotes the range of μ in X_0.

We shall also find it convenient—at the expense of abuse of notation—to write $A \subset R(X)$ to mean that the subset A of X lies inside the range of an X-valued measure. The same shall apply to the other X-valued sequence spaces $R_{vbv}(X)$ and $R^*_{bv}(X)$ introduced above.

3. Main results

We start with the following theorem, giving necessary and sufficient conditions for the containment of sequences in X from $\ell_p(X)$ inside the range of a vector measure of bounded variation taking its values in a large space. In what follows, e_i shall denote the ith unit vector in ℓ_p (resp. ℓ_p^n).

Theorem 3.1. *For $1 < p < \infty$, the following statements are equivalent for a Banach space X:*

(i) $\ell_p(X) \subset R_{vbv}(X)$.
(ii) $\Pi_2(X, \ell_1) \subset N(X, \ell_q), (q = p^*)$.
(iii) $\exists\ c > 0$ *s.t. for* $(x_i)_{i=1}^n \subset X, (x_i^*)_{i=1}^n \subset X^*$,

$$\left| \sum_{i=1}^{n} \langle x_i, x_i^* \rangle \right| \leq c\ \pi_2 \left(\sum_{i=1}^{n} x_i^* \otimes e_i : X \to \ell_1^n \right) \epsilon_p((x_i)_{i=1}^n).$$

Before we proceed to prove the theorem, a few remarks are in order. First of all, the inclusion in (ii) above is meant in the sense that for each $T \in \Pi_2(X, \ell_1)$, the composite map $i_{1q} \circ T$ belongs to $N(X, \ell_q)$, where $i_{1q} : \ell_1 \to \ell_q$ is the inclusion map. Further, it is easy to see that in the definition of $R_{vbv}(X)$ one can assume without loss of generality that $X_0 = \ell_\infty(B_{X^*})$, the Banach space of bounded functions on B_{X^*}, which we shall throughout denote by the same symbol X_0. In view of this, it is possible to define a norm on $R_{vbv}(X)$ by means of the formula

$$\|\bar{x}\|_{vbv} = \inf \left\{ \begin{array}{l} \mathrm{tv}(\mu);\ \exists \text{ a vector measure } \mu : \Sigma \to X_0 \\ \text{of bounded variation s.t. } (x_n) \subseteq \mathrm{rg}(\mu) \end{array} \right\},$$

which makes $R_{vbv}(X)$ into a Banach space. Finally, we recall (see [5, p. 162]) that the vector measure $\mu : \Sigma \to X$ is of bounded variation if and only if the corresponding integration map $T_\mu : C(K) \to X$ is absolutely summing. In this case, $\pi_1(T_\mu) = \mathrm{tv}(\mu)$. Using this, it is not difficult to prove the following characterisation of such vector measures, which will be used in the proof (see [14]).

Fact. *Given a bounded sequence $\bar{x} = (x_n)$ in a Banach space X, we have $\bar{x} \in R_{vbv}(X)$ if and only if $T_{\bar{x}} \in \Pi_1(\ell_1, X)$. In this case,* $\mathrm{tv}(\mu) = \pi_1(T_{\bar{x}})$.

Note that ℓ_1 has cotype 2, so that $\Pi_2(\ell_1, X) = \Pi_1(\ell_1, X)$, which shows that $R_{vbv}(X)$ and $\Pi_2(\ell_1, X)$ can be identified as Banach spaces. We are now ready to prove the theorem.

Proof. (i)⇒(ii).

Step 1: Let $I : \ell_p(X) \to R_{vbv}(X)$ denote the inclusion map induced by (i). We show that I is continuous, i.e., there exists $c > 0$ such that

$$\|\bar{x}\|_{vbv} \le c\ \epsilon_p(\bar{x}),\ \forall\ \bar{x} \in \ell_p(X). \tag{1}$$

Invoking the denseness of $\varphi(X)$ in $\ell_p(X)$ and the completeness of $R_{vbv}(X)$, it suffices to verify (1) for $\bar{x} \in \varphi(X)$, the space of finitely non-zero sequences in X, defined by

$$\varphi(X) = \{\bar{x} = (x_n) :\ \exists\ N\ s.t.\ x_n = 0,\ \forall\ n > N\}.$$

Assume, on the contrary, that (1) does not hold for $\bar{x} \in \varphi(X)$. Then for each $n \ge 1$ there exists $H_n = \{x_i^{(n)}\}_{i=1}^{k(n)} \subset X$ with $\epsilon_p((x_i)_{i=1}^{k(n)}) \le 1$ such that for all vector measures $\mu : \Sigma \to X_0$ of bounded variation with $H_n \subset \mathrm{rg}(\mu)$, we have $\mathrm{tv}(\mu) \ge n^2$. Now the sequence having its terms in the set $H = \bigcup_{n=1}^\infty H_n/n$ is included inside $\ell_p(X)$, so that by (i) there exists $\mu_0 : \Sigma \to X_0$, a vector measure of bounded variation such that $H_n \subset n\,\mathrm{rg}(\mu_0) = \mathrm{rg}(n\mu_0)$ for all $n \ge 1$. But then $\mathrm{tv}(n\mu_0) \ge n^2$, or equivalently $\mathrm{tv}(\mu_0) \ge n$ for all $n \ge 1$, contradicting that μ_0 is of bounded variation. Thus I restricted to $\varphi(X)$ is continuous, and so is its (unique) extension to $\ell_p(X)$, which obviously coincides with the inclusion map I.

Step 2: $\Pi_2(X, \ell_1) \subset (R_{vbv}(X))^*$, i.e., each $S = \sum_{n=1}^{\infty} x_n^* \otimes e_n \in \Pi_2(X, \ell_1)$ defines a continuous linear functional ψ_S on $R_{vbv}(X)$ given by

$$\psi_S(\bar{x}) = \sum_{n=1}^{\infty} \langle x_n, x_n^* \rangle, \bar{x} = (x_n) \in R_{vbv}(X).$$

Thus, let $\bar{x} = (x_n) \in R_{vbv}(X)$. By the remarks preceding the proof, $T = T_{\bar{x}} \in \Pi_1(\ell_1, X)$ with $\pi_1(T) \leq \|\bar{x}\|_{vbv}$. An easy calculation shows that $ST = \sum_{n=1}^{\infty} \langle\, , f_n \rangle e_n$, where $f_n \equiv \{\langle x_i, x_n^* \rangle\}_{i=1}^{\infty} \in \ell_\infty, n \geq 1$. Since $\Pi_2^{(2)} \subset N$ (Proposition 2.2(c)), it follows that ST is nuclear and, in fact, $\nu(ST) = \sum_{n=1}^{\infty} \|f_n\|$ by [8, 1.15]. This gives

$$\begin{aligned} |\psi_S(\bar{x})| &\leq \sum_{n=1}^{\infty} |\langle x_n, x_n^* \rangle| \leq \sum_{n=1}^{\infty} \|f_n\| = \nu(ST) \\ &\leq \pi_2(S)\pi_2(T) \leq \pi_2(S)\|\bar{x}\|_{vbv}, \end{aligned}$$

which yields the continuity of ψ_S on $R_{vbv}(X)$. Combining the above conclusions, it follows that ψ_S is continuous on $\ell_p(X)$. But it is well known that $(\ell_p(X))^* \simeq N(\ell_p, X^*)$ (see [3, Proposition 3] and [2, Proposition 1]). Further, under this correspondence, ψ_S gives rise to a nuclear map $\eta_S : \ell_p \to X^*$, where $\eta_S(\bar{\alpha}) = \sum_{n=1}^{\infty} \alpha_n x_n^*, \bar{\alpha} = (\alpha_n) \in \ell_p$, such that for each $x \in X$ we have

$$\begin{aligned} \langle \eta_S^*(x), \alpha \rangle &= \langle x, \eta_S(\alpha) \rangle \\ &= \left\langle x, \sum_{n=1}^{\infty} \alpha_n x_n^* \right\rangle \\ &= \sum_{n=1}^{\infty} \alpha_n \langle x, x_n^* \rangle \\ &= \left\langle \sum_{n=1}^{\infty} \langle x, x_n^* \rangle e_n, \alpha \right\rangle \\ &= \langle S(x), \alpha \rangle. \end{aligned}$$

This yields that $S = \eta_S^* \mid_X$ is nuclear.

(ii)⇒(iii): By (ii), the map

$$T \in \Pi_2(X, \ell_1) \to i_{1q} \circ T \in N(X, \ell_q)$$

is well-defined and continuous, by the closed graph theorem. Thus, there exists $c > 0$ such that

$$\nu(i_{1q} \circ T) \leq c\, \pi_2(T),\ \forall\, T \in \Pi_2(X, \ell_1).$$

To show that (iii) holds, fix $(x_i)_{i=1}^n \subset X, (x_i^*)_{i=1}^n \subset X^*$, and define

$$u : X \to \ell_q^n,\ v : \ell_q^n \to X,$$

where

$$u(x) = \sum_{i=1}^{n} \langle x,\ x_i^* \rangle e_i,\ v(\bar{\alpha}) = \sum_{i=1}^{n} \alpha_i x_i, x \in X,\ \bar{\alpha} \in \ell_q^n.$$

Denoting by $i_{q1}^n : \ell_q^n \to \ell_1^n$ the identity map, by $i_n : \ell_1^n \to \ell_1$ the canonical inclusion and by $\hat{v}$ the 'canonical' extension of v to ℓ_q, we note that $\|\nu\| = \epsilon_p((x_i)_{i=1}^n)$ and that $\hat{v} i_{1q} i_n i_{q1}^n u = v \circ u$. This gives

$$\begin{aligned}
\left| \sum_{i=1}^{n} \langle x_i, x_i^* \rangle \right| = \operatorname{trace}(v \circ u) &\leq \nu(\hat{v} i_{1q} i_n i_{q1}^n u) \\
&\leq \|\hat{v}\| \nu(i_{1q} i_n i_{q1}^n u) \\
&\leq c \|v\| \pi_2(i_n i_{q1}^n u) \\
&= c\ \pi_2 \left(\sum_{i=1}^{n} x_i^* \otimes e_i : X \to \ell_1^n \right) \epsilon_p((x_i)_{i=1}^n).
\end{aligned}$$

(iii)$\Rightarrow$(i): Define a map $\psi : (\varphi(X), \epsilon_p) \to \Pi_2(\ell_1, X)$ by

$$\psi(\bar{x}) = T_{\bar{x}}.$$

ψ is clearly a well-defined linear map. We show that ψ is continuous. To this end, for $n \geq 1$ let $\varphi_n(X)$ denote the subspace of $\varphi(X)$ consisting of all sequences which are zero after the nth term. Letting $\psi_n : \varphi_n(X) \to \Pi_2(\ell_1^n, X)$ denote the map given by $\psi_n(\bar{x}) = T_{\bar{x}}$ and using trace duality, we see that for each $\bar{x} \in \varphi(X)$ we can write $\bar{x} = (x_1, x_2, \ldots, x_n, 0, 0)$, such that using (iii) we get

$$\begin{aligned}
\pi_2(\psi(\bar{x})) &= \pi_2(\psi_n(\bar{x})) \\
&= \pi_2(T_{\bar{x}}) \\
&= \sup\{|\operatorname{trace}(T_{\bar{x}} \circ S)| : S \in \pi_2(X, \ell_1^n), \pi_2(S) \leq 1\} \\
&= \sup \left\{ \left| \sum_{i=1}^{n} \langle x_i, S^* e_i^* \rangle \right| : S = \sum_{i=1}^{n} S^* e_i^* \otimes e_i \in \pi_2(X, \ell_1^n), \pi_2(S) \leq 1 \right\} \\
&\leq c\ \sup \left\{ \pi_2 \left(\sum_{i=1}^{n} S^* e_i^* \otimes e_i : X \to \ell_1^n \right) \epsilon_p((x_i)_{i=1}^n), \pi_2(S) \leq 1 \right\} \\
&\leq c\ \epsilon_p((x_i)_{i=1}^n).
\end{aligned}$$

This shows that ψ is continuous and, therefore, has a (unique) continuous extension—denoted again by ψ—to $\ell_p(X)$, which contains $\varphi(X)$ as a dense subspace. It follows from our construction that the (extended) map $\psi : \ell_p(X) \to \Pi_2(\ell_1, X) = \Pi_1(\ell_1, X)$ is again given by

$$\psi(\bar{x}) = T_{\bar{x}},$$

which in light of the above Fact means that $\ell_p(X) \subset R_{vbv}(X)$. This completes the proof of the theorem. $\square$

Corollary 3.2. *For* $1 \leq p \leq 2, \ell_p(X) \subset R_{vbv}(X)$.

Proof. By the above theorem, it suffices to show that $\Pi_2(X, \ell_1) \subset N(X, \ell_q)$, where $q = p^*$. But that is an easy consequence of Grothendieck's theorem, which says that the inclusion map $i_{12} : \ell_1 \to \ell_2$ is absolutely summing. Now for $T \in \Pi_2(X, \ell_1), i_q \circ T$ can be factorised as $i_q \circ T = i_{2q} \circ i_{12} \circ T$ as long as $1 \leq p \leq 2$. But then as a composite of 2-summing maps, the indicated map is nuclear by Proposition 2.2(c). □

Remark 3.3. The above corollary is a strengthening of a well-known theorem of Diestel and Anantharaman [1], which says that weakly 2-summable sequences in a Banach space X are included inside the range of an X-valued measure. This follows from the above Fact that if $\bar{x} \in R_{vbv}(X)$, then $T_{\bar{x}} : \ell_1 \to X$ is absolutely summing and, therefore, factors over a Hilbert space. But it is well-known (see [1]) that the unit ball of a Hilbert space H lies inside the range of an H-valued vector measure.

Remark 3.4. The case when p-unconditionally Cauchy sequences in a Banach space X are contained inside the range of a vector measure has been treated by C. Pineiro in [15]. Our main theorem provides a refinement of this work for the case of vector measures of bounded measures taking values in a superspace of X.

Remark 3.5. In the limiting case when $p = 1$, the above theorem translates into the statement that $c_0(X) \subset R_{vbv}(X)$ if and only if $\Pi_2(X, \ell_1) = N(X, \ell_1)$. The fact that each of these conditions is equivalent to X being Hilbertian follows from Theorem 4.1 to be proved in the next section. C. Pineiro [14] had arrived at the same conclusion by employing methods which are different from our approach (see also [12]).

4. Relationship with nuclear operators

We start with an alternative approach to a proof of the following theorem of C. Pineiro (see [12] and [14]).

Theorem 4.1. *For a Banach space X, the following assertions are equivalent:*

(i) X *is Hilbertian.*
(ii) $B_X \subset R_{vbv}(X)$.
(iii) $\ell_\infty(X) \subset R_{vbv}(X)$.
(iv) $c_0(X) \subset R_{vbv}(X)$.
(v) $\Pi_2(X, \ell_1) = N(X, \ell_1)$.

Proof. We shall prove this theorem, invoking Theorem 1.1 by showing that under each of these conditions, $N(X) = \Pi_2^{(2)}(X)$, so that as a consequence of

Proposition 2.2 (e), each nuclear operator on X shall have absolutely summable eigenvalues.

It is clear that (ii)$\Rightarrow$(iii)$\Rightarrow$(iv). We begin by showing (i)$\Rightarrow$(ii). Applying Grothendieck's theorem to X, which is assumed to be a Hilbert space, we have

$$L(\ell_1(\wedge), X) = \Pi_1(\ell_1(\wedge), X),$$

where the index set $\wedge$ is chosen such that it has the same cardinality as B_X. Combining this equality with the above Fact gives that B_X is included inside the range of a vector bv-measure, i.e., $B_X \subset R_{vbv}(X)$.

(iv)$\Rightarrow$(v): Let $T \in \Pi_2(X, \ell_1)$. Thus, as proved in Step 2 of Theorem 3.1, the map

$$\psi_T(S) = \sum_{n=1}^{\infty} \langle x_n, x_n^* \rangle$$

defines a continuous linear functional on $\Pi_2(\ell_1, X) = \Pi_1(\ell_1, X)$. Recall that T and S can be written as

$$T(x) = \sum_{n=1}^{\infty} \langle x, x_n^* \rangle e_n, \;\; S(\alpha) = \sum_{n=1}^{\infty} \alpha_n x_n,$$

where $x \in X, \alpha \in \ell_1, (x_n)_{n=1}^{\infty} \subset X$ and $(x_n^*)_{n=1}^{\infty} \subset X^*$. Further, it is easy to see that the inclusion in (iv) is already continuous, so that ψ_T is continuous when restricted to $c_0(X)$. In other words, there exists $(y^*)_{n=1}^{\infty} \in (c_0(X))^* = \ell_1\{X^*\}$—the space of absolutely convergent sequences in X^*—such that for each $\bar{x} = (x_n)_{n=1}^{\infty} \in c_0(X)$,

$$\psi_T(\bar{x}) = \sum_{n=1}^{\infty} \langle x_n, x_n^* \rangle = \sum_{n=1}^{\infty} \langle x_n, y_n^* \rangle.$$

This gives $(x_n^*)_{n=1}^{\infty} = (y_n^*)_{n=1}^{\infty}$, so that $\sum_{n=1}^{\infty} \|x_n^*\| = \sum_{n=1}^{\infty} \|y_n^*\| < \infty$, which yields that $T \in N(X, \ell_1)$.

(v)$\Rightarrow$(i): By the closed graph theorem, there exists $c > 0$ such that $\nu(S) \leq c \; \pi_2(S)$ for all $S \in \Pi_2(X, \ell_1)$. We show that $L(\ell_1, X) = \Pi_2(\ell_1 X)$. To this end, let $T \in L(\ell_1, X)$. Then for $T_n = T \mid_{\ell_1^n}$, we get, using trace duality,

$$\begin{aligned}
\pi_2(T_n) &= \sup\{\text{trace}(T_n S_n) : S_n \in L(X, \ell_1^n), \pi_2(S_n) \leq 1\} \\
&\leq c \;\; \sup\{\text{trace}(T_n S_n) : S_n \in L(X, \ell_1^n), \nu(S_n) \leq 1\} \\
&= c \; \|T_n\| \leq c \; \|T\|, \; \forall \; n \geq 1.
\end{aligned}$$

In other words, we get

$$\pi_2(T) = \sup_{n \geq 1} \pi_2(T_n) \leq c \; \|T\|.$$

This gives

$$L(\ell_1, X) = \Pi_2(\ell_1, X). \tag{2}$$

Finally, let $T \in N(X)$. Then $T = T_2 D T_1$, where $D : \ell_\infty \to \ell_1$ is a diagonal (nuclear) operator and $T_1 : X \to \ell_\infty$, $T_2 : \ell_1 \to X$ are bounded linear operators. By (2), $T_2 \in \Pi_2(\ell_1, X)$, so that $T = T_2 D T_1 \in \Pi_2 \circ N(X) \subset \Pi_2^{(2)}(X)$ (by Proposition 2.2(a) and (c)). In other words, $N(X) = \Pi_2^{(2)}(X)$, so that by the eigenvalue theorem (Theorem 1.1), X is a Hilbert space. This completes the proof of the theorem. $\square$

REMARK 4.2. It is natural to ask for a description of Banach spaces for which Theorem 4.1(v) holds with ℓ_1 replaced by ℓ_2. However, it is not difficult to see that there are no infinite dimensional Banach spaces X for which $\Pi_2(X, \ell_2) = N(X, \ell_2)$. Indeed, given $T \in N(X)$, so that $T = T_2 D T_1$ as in the proof of (v)$\Rightarrow$(i) above, we can further factorise D as $D_2 D_1$, where $D_1 : \ell_\infty \to \ell_2$ and $D_2 : \ell_2 \to \ell_1$ are diagonal operators induced by sequences in ℓ_2. But then D_1 is 2-summing by [13, 1.6.2], so that $DT_1 \in \Pi_2(X, \ell_2) = N(X, \ell_2)$, which means that DT_1 factorises over a nuclear operator into ℓ_1. Invoking Grothendieck's theorem yet again, it follows that $DT_1 \in N \circ \Pi_1(X, \ell_2)$, which yields that $T = T_2 D T_1 \in N \circ \Pi_1(X) \subset \Pi_2^{(2)}(X)$, so that X is a Hilbert space by Theorem 1.1. Finally, recalling that on an infinite dimensional Hilbert space there always exist non-nuclear Hilbert-Schmidt (= 2-summing) operators, we conclude that X is finite dimensional. We isolate the above conclusion in the form of

PROPOSITION 4.3. *For a Banach space* $X, \Pi_2(X, \ell_2) = N(X, \ell_2)$ *if and only if* $\dim X < \infty$.

In light of the above result, it is still conceivable that there may exist infinite dimensional Banach spaces X on which ℓ_2-valued absolutely summing maps are nuclear. In the theorem below, we give a complete description of these spaces.

THEOREM 4.4. *For a Banach space X, the following statements are equivalent:*

(i) $\Pi_1(X, \ell_2) = N(X, \ell_2)$.
(ii) X^* *has (GT) and (GL).*

Before we proceed to prove the theorem, we recall that a Banach space X verifies (GT) if it satisfies *Grothendieck's theorem*, i.e., if all X-valued bounded linear maps on an L_1-space are absolutely summing. X is said to have (GL) (*Gordon-Lewis property*) if all absolutely summing maps on X factorise over an L_1-space. For further details see [17, Chapter 6] and [4, Chapter 17].

Proof. (i)$\Rightarrow$(ii): Using the fact that X has (GL) if and only if X^* has it (Proposition 17.9 in [4]) and noting that under the given hypothesis absolutely summing maps on X factorise over a nuclear map into ℓ_1, we see that X^* has (GL).

To show that X^* has (GT), the given condition yields $c > 0$ such that $\nu(T) \leq c\ \pi_1(T)$ for all $T \in \Pi_1(X, \ell_2)$. Equivalently,

$$\nu(T) \leq c\ \pi_1(T),\ \forall\ T \in \Pi_1(X, \ell_2^m),\ \forall\ m \geq 1. \tag{3}$$

It suffices to show that $\pi_1(T) \leq c\ \|T\|$ for all $T \in L(X^*, \ell_2^m), m \geq 1$.

Fix $m \geq 1, n \geq 1, T \in L(X^*, \ell_2^m)$ and $(x_j^*)_{j=1}^n \subset X^*$ arbitrarily. We can write $T = \sum_{i=1}^m \langle x_i, \cdot\rangle e_i$ for some $x_i \in X, 1 \leq i \leq m$. For $1 \leq i \leq m, 1 \leq j \leq n$, put $a_{ij} = \overline{\langle x_i, x_j^*\rangle}/B_i$, where $B_j = (\sum_{i=1}^m |\langle x_i, x_j^*\rangle|^2)^{1/2}$, and let $M = (a_{ij})$ denote the corresponding operator $M : \ell_1^n \to \ell_2^m$. Let $S : X \to \ell_1^n$ be the map given by $S(x) = \sum_{i=1}^n \langle x, x_i^*\rangle e_i, x \in X$. Then $\|S\| = \epsilon_1((x_i^*)_{i=1}^n)$. By Grothendieck's theorem we have

$$\pi_1(M) \leq K_G\|M\|, \tag{4}$$

where K_G is the Grothendieck constant. Let us estimate the nuclear norm of the composite map $\ell_2^m \xrightarrow{T^*} X \xrightarrow{S} \ell_1^n \xrightarrow{M} \ell_2^m$. Indeed, combining (3) and (4) and noting that $\|M\| \leq 1$, we get

$$\begin{aligned} \nu(MST^*) \leq \nu(MS)\|T\| &\leq c\ \|T\|\pi_1(MS) \\ &\leq c\ \pi_1(M)\|T\|\ \|S\| \\ &\leq c\ K_G\|T\|\epsilon_1((x_i^*)_{i=1}^n). \end{aligned} \tag{5}$$

Further, we note that, for each $x \in X$,

$$T^*MS(x) = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij}\langle x, x_j^*\rangle\right) x_i,$$

which yields

$$\begin{aligned} \nu(MST^*) \geq \operatorname{trace}(MST^*) &= \operatorname{trace}(T^*MS) \\ &= \sum_{i=1}^m \sum_{j=1}^n a_{ij}\langle x_i, x_j^*\rangle \\ &= \sum_{j=1}^n \left(\sum_{i=1}^m |\langle x_i, x_j^*\rangle|^2\right)^{1/2}. \end{aligned} \tag{6}$$

After combining (5) and (6) we get

$$\begin{aligned} \sum_{i=1}^n \|Tx_i^*\| = \sum_{i=1}^n \left(\sum_{j=1}^m |\langle x_j, x_i^*\rangle|^2\right)^{1/2} &\leq \nu(MST^*) \\ &\leq C\ K_G\|T\|\epsilon_1((x_i^*)_{i=1}^n). \end{aligned}$$

In other words,

$$\pi_1(T) \leq C\ K_G\|T\|,\ \forall\ T \in L(X^*, \ell_2^m),$$

which establishes (ii).

(ii)⇒(i): By the (GL)-property of X^* or equivalently of X, every $T \in \Pi_1(X, \ell_2)$ factors over an L_1-space: $T = T_2T_1$, where $T_1 \in L(X, L_1), T_2 \in L(L_1, \ell_2)$. Also, the (GT)-property of X^* yields $L(X, L_1) = \Pi_2(X, L_1)$ by Proposition 6.2 of [17]. This gives $T_1 \in \Pi_2(X, L_1)$, and by Grothendieck's theorem $T_2 \in \Pi_2(L_1, \ell_2)$, so that finally $T = T_2T_1 \in \Pi_2^{(2)}(X, \ell_2) \subset N(X, \ell_2)$, which completes the proof. □

Corollary 4.5. *Under the assumptions of Theorem 4.4, X^* has cotype* 2.

Proof. The (GT) property of X^* gives $L(X^*, \ell_2) = \Pi_1(X^*, \ell_2)$, so that, in particular, $\Pi_2(X^*, \ell_2) = \Pi_1(X^*, \ell_2)$. Now, in the presence of (GL), it follows from Theorem 17.11 of [4] that X^* has cotype 2. □

Remark 4.6. Regarding the converse of Corollary 4.5, we see that ℓ_2 is a cotype 2 space having (GL) but which is not (GT). The fact that there are cotype 2 spaces having (GT) but not (GL) follows from a highly non-trivial theorem of J. Bourgain to the effect that the dual $(A(D))^*$ of the disc algebra $A(D)$ has (GT) and cotype 2, but lacks the (GL)-property! The latter statement is a famous theorem of A. Pelczynski, which can be 'deduced' from the (GT) property of $(A(D))^*$ combined with the supposed hypothesis that $(A(D))^*$ has (GL). Indeed, these two conditions put together yield, by virtue of Theorem 4.4, the equality $\Pi_1(A(D), \ell_2) = N(A(D), \ell_2)$, which, in conjunction with the existence of a surjection in $\Pi_1(A(D), \ell_2)$ (see [17, 5.f]), leads to a contradiction, in view of the compactness of a nuclear operator.

We conclude with an application of Theorem 4.4 to the theory of vector measures. As already stated in the introduction, a Banach space X has the property that every X-valued measure is of bounded variation (if and) only if X is finite dimensional. The locally convex analogue of this statement is an old result of M. Duchon [6], which says that the stated condition holds in a Fréchet space exactly when it is nuclear. This together with Remark 3.3 motivates the more general question of characterising Banach spaces X which enjoy the following property:

Property $P(X, Y)$. A (bounded) set $A \subset X$ included inside the range of an X-valued vector measure already lies inside the range of a Y-valued measure of bounded variation via an isometric embedding of X into the Banach space Y.

We list below a set of equivalent conditions for $P(X, X^{**})$.

THEOREM 4.7. *For a Banach space X, the following assertions are equivalent.*

(i) *X has $P(X, X^{**})$.*
(ii) *$L(X, \ell_1) = \Pi_1(X, \ell_1)$.*
(iii) *Both X and X^* have (GT).*
(iv) *$A \subset R_c(X) \Rightarrow A \subset R_{bv}(X)$. (Here $A \subset R_c(X)$ means that A lies inside the range of an X-valued measure having relatively compact range.)*

Proof. It is proved in [7] that (ii), (iii) and (iv) are all equivalent. To see that (i)$\Leftrightarrow$(ii), let $A \subset R(X)$. By the definition of $P(X, X^{**})$ we get $A \subset R^*_{bv}(X)$. In particular, $A \subset R_{vbv}(X)$. By [11, Theorem 6] it follows that $\Pi_2(X, \ell_1) = \Pi_1(X, \ell_1)$. Further, if $A \subset R_{vbv}(X)$, then, as seen above, $A \subset R(X)$. But then, by $P(X, X^{**})$, $A \subset R^*_{bv}(X)$. Invoking Theorem 6 of [14], this yields that X^* has (GT). Equivalently, $L(X, \ell_1) = \Pi_2(X, \ell_1)$. Combining this with the above equality, we get (ii). Retracing the above argument yields the reverse implication. □

Combining the above result with Theorem 4.4 yields the following

COROLLARY 4.8. *A (GL)-space X with $P(X, X^{**})$ is finite dimensional.*

Proof. Theorem 4.8 yields that both X and X^* are (GT)-spaces. Now under the assumption of the (GL)-property, Theorem 4.4 gives $\Pi_1(X, \ell_2) = N(X, \ell_2)$. Combining this with the (GT)-property of X, namely that $L(X, \ell_2) = \Pi_1(X, \ell_2)$, leads to the equality $L(X, \ell_2) = N(X, \ell_2)$, which, by virtue of Proposition 4.3, amounts to the finite dimensionality of X. □

The above corollary may be compared with Chapter 10 of [17], where in Section (d) the author lists certain conditions which do not hold in an infinite dimensional Banach space violating Grothendieck's conjecture. One of these properties is the 'Local unconditional structure', which is stronger than the (GL)-property, so that Corollary 4.9 can be viewed as a strengthening of the indicated result of [17].

REMARK 4.9. The above corollary does not hold in the absence of (GL)-property. In fact, the existence of infinite dimensional Banach spaces with these properties has been a remarkable discovery of G. Pisier in his ground-breaking work pertaining to 'Grothendieck's Conjecture'! (See [17, Chapter 10].) Incidentally, the above corollary shows that 'Pisier's space' does not have the Gordon-Lewis property. Finally, a useful result that can be gleaned from the above considerations is given in

PROPOSITION 4.10. *Let X be a Banach space such that X^* has (GT). Then X has (GT) if and only if $\Pi_2(X, \ell_2) = \Pi_1(X, \ell_2)$.*

We conclude with the following conjecture, which is motivated by the above discussion.

Conjecture. *For a Banach space X, $P(X,X)$ holds (if and) only if $\dim X < \infty$.*

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