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UNITARIES IN BANACH SPACES

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ABSTRACT. We study the abstract geometric notion of unitaries in a Banach space characterized in terms of the equivalence of the norm determined by the state space.

1. Introduction

Motivated by the recent work of Akemann and Weaver [2] we introduce and study an abstract geometric notion of a unitary in a Banach space defined as those unit vectors whose state space spans the dual. Because of the important role unitaries play in C^*-algebras, it is natural to study the properties of unitaries in general Banach spaces and to decide to what extent they determine the geometry and structure of such spaces.

We first compare unitaries with the well-studied notion of a vertex; the notions coincide for C^*-algebras. For Banach spaces we show that a vertex is a unitary if and only if the norm determined by the state space is an equivalent norm. As a consequence we conclude that in a complex Banach algebra the unit is a unitary and, just as in the case of C^*-algebras, unitaries remain unitaries in the bidual. We study the behavior of unitaries and the related notions of strongly extreme points and of weak*-unitaries in various settings including C^*-algebras, von Neumann algebras and L^1-preduals.

In Section 4 we consider unitaries in the space $C(X, E)$ of vector-valued continuous functions on a compact set X. If E is a function algebra then $f \in C(X, E)$ is a unitary if and only if $f(x)$ is a unitary for all $x \in X$. However, in general, a unitary-valued function need not be a unitary. This seems to be the first non-trivial example where a continuous function is pointwise in an extremal class but does not globally belong to that class.

In the last section of the paper we consider, for a Banach space E and $T \in \mathcal{L}(E)$, the relation between T^* being a unitary in $\mathcal{L}(E^*)$ and T being a unitary in $\mathcal{L}(E)$.

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2. Definitions and notations

Our notation and terminology is standard as found in [3] or [6]. For a Banach space E we denote by E_1 its closed unit ball and by $\partial_e E_1$ the set of extreme points of E_1. A point $e_0 \in E_1$ is called strongly extreme (or a point of local uniform convexity [3]) if for all sequences $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$ in E_1 such that $(x_n + y_n)/2 \to e_0$ we have $x_n - y_n \to 0$. With each norm one element $e_0 \in E_1$ we associate its state space $S_{e_0} \stackrel{\text{def}}{=} \{e^* \in E_1^* : e^*(e_0) = 1\}$; we call e_0 a *vertex* [3] if $\operatorname{span} S_{e_0}$ is weak*-dense in E^* and a *unitary* if $\operatorname{span} S_{e_0} = E^*$. We will denote by $\mathcal{U}(E)$ the set of unitaries of E.

DEFINITION 2.1. For a norm one element $e_0 \in E_1$ we define a seminorm p_{e_0} on E by

$$p_{e_0}(e) = \sup\{|e^*(e)| : e^* \in S_{e_0}\}.$$

It is clear that $p_{e_0} \leq \|\cdot\|$ and that p_{e_0} is a norm if and only if e_0 is a vertex. In some cases it may be useful to restrict the set of functionals to a norming subspace W of E^*. A closed subspace W of E^* is a norming subspace if $\|e\| = \sup\{e^*(e) : e^* \in W_1\}$ for all $e \in E$. We put $S_{e_0}^W \stackrel{\text{def}}{=} \{e^* \in W_1 : e^*(e_0) = 1\}$ and write $p_{e_0}^W$ in place of p_{e_0}, assuming that $p_{e_0}^W = 0$ if $S_{e_0}^W = \emptyset$. We call a norm one element $e_0 \in E_1$ a *W-unitary* if $S_{e_0}^W$ spans W. In particular, if $E = F^*$, we call an F-unitary a *weak*-unitary*.

We denote by $\mathbb{T}$ the unit circle in $\mathbb{C}$, by $\mathcal{L}(E,F)$ the spaces of all linear continuous maps between Banach spaces E and F and by $\mathcal{K}(E,F)$ the subspace consisting of compact maps; we write $\mathcal{L}(E)$ in place of $\mathcal{L}(E,E)$. $C(X,E)$ stands for the space of all continuous E-valued functions defined on a compact Hausdorff space X. The constant function $f(x) \equiv 1$ on X will be denoted by $\mathbf{1}$. By a uniform algebra A on X we mean a closed subalgebra of $C(X)$ which separates the points of X and contains the constant functions; ChA denotes the Choquet boundary of A [4]. A Banach space E such that E^* is isometric to $L^1(\mu)$ for some measure μ is called an L^1-predual space. We refer to [7, Chapter 7] for examples and properties of such spaces.

3. The geometry of unitaries

A norm one element e_0 of a Banach space E is a unitary if and only if any element e^* of E^* is a linear combination of finitely many elements of S_{e_0}:

$$e^* = \alpha_1 e_1^* + \cdots + \alpha_n e_n^*, \text{ for some } e_j^* \in S_{e_0}.$$

It is an obvious but useful observation that the number of elements of S_{e_0} taken in these linear combinations can always be limited to two in the real case and to four in the complex case. Indeed, since S_{e_0} is a convex set, we can group together all of the terms with the same sign. If e_0 is a unitary (and only in this case) we can define another norm on the dual space:

$$p^*_{e_0}(e^*) \stackrel{\text{def}}{=} \inf \left\{ \sum |\alpha_j| : e^* = \alpha_1 e^*_1 + \cdots + \alpha_n e^*_n, \text{ for some } e^*_j \in S_{e_0} \right\},$$

for $e^* \in E^*$.

A close look at the proof of Theorem 2 in [2] shows that in a C^*-algebra the notions of unitary and vertex coincide.

Theorem 3.1. *For a norm one element e_0 of a Banach space E the following conditions are equivalent:*

(1) *$e_0 \in E$ is a unitary.*
(2) *p_{e_0} is a complete norm on E.*
(3) *p_{e_0} is equivalent with the original norm of E.*
(4) *$p^*_{e_0}$ is a complete norm on E^*.*
(5) *$p^*_{e_0}$ is equivalent with the original norm of E^*,*

Proof. The equivalences (2) $\Longleftrightarrow$ (3) and (4) $\Longleftrightarrow$ (5) are obvious by the Open Mapping Theorem.

(1) $\Longrightarrow$ (5). Assume e_0 is a unitary and put $\mathcal{S} \stackrel{\text{def}}{=} \operatorname{conv}(S_{e_0} \cup -S_{e_0} \cup iS_{e_0} \cup -iS_{e_0})$. Since $\operatorname{span} S_{e_0} = E^*$ we have $E^* = \bigcup_{n=1}^{\infty} n\mathcal{S}$. As the set $\mathcal{S}$ is weak*-compact and hence norm closed, it follows from the Baire Category Theorem that there is a constant $K > 0$ such that $E^*_1 \subseteq K\mathcal{S}$. Hence

$$p^*_{e_0}(e^*) \leq K \text{ for any } e^* \in E^*_1.$$

(5) $\Longrightarrow$ (3). For any $e^* \in E^*$ and $e \in E$,

$$|e^*(e)| \leq p^*_{e_0}(e^*) p_{e_0}(e)$$

and hence, if $p^*_{e_0}(e^*) \leq K$ for any $e^* \in E^*_1$, then $p_{e_0}(e) \geq \frac{1}{K}\|e\|$.

(3) $\Longrightarrow$ (1). Assume that the norms p_{e_0} and $\|\cdot\|$ are equivalent and define $J : E \to C(S_{e_0})$ by $J(e)(e^*) = e^*(e)$. Notice that J is an isomorphism from E onto a Banach space $J(E)$ and $J(e_0) = \mathbf{1}$. Fix $e^* \in E^*$ and define a continuous linear functional Λ on $J(E)$ by $\Lambda = e^* \circ J^{-1}$. Let μ be a regular Borel measure on S_{e_0} representing a norm preserving extension of Λ to $C(S_{e_0})$ and let $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$, where the μ_j's are non-negative measures. Observe that the normalized measures $\mu_j/\|\mu_j\|$ are probability measures and hence are in the state space of $\mathbf{1} \in C(S_{e_0})$. Let Λ_j be the functional on $J(E)$ represented by $\mu_j/\|\mu_j\|$. Since S_{e_0} is convex and weak*-closed, it follows that $\Lambda_j \circ J \in S_{e_0}$, so $e^* \in \operatorname{span} S_{e_0}$. $\square$

Corollary 3.2. *Let W be a norming subspace of E^*. Then any W-unitary e_0 of E is a unitary. In particular, any weak*-unitary e^*_0 of E^* is a unitary.*

Proof. Exactly as in the proof of Theorem 3.1, putting $\mathcal{S} = \overline{\operatorname{conv}}(S^W_{e_0} \cup -S^W_{e_0} \cup iS^W_{e_0} \cup -iS^W_{e_0})$ and using the Category argument, it follows that if

$e_0 \in E$ is a W-unitary then there is a constant $K > 0$ such that $W_1 \subseteq K\mathcal{S}$. Since W is a norming subspace, $p^W_{e_0}$ is thus an equivalent norm on E. Since $p^W_{e_0} \leq p_{e_0}$, by Theorem 3.1, e_0 is a unitary. □

It should be noticed that in the case of a von Neumann algebra A weak*-unitaries and unitaries coincide [2, Theorem 3] and, as the following proposition shows, for any unitary u, $p_u^{A_*} = p_u$, where A_* is the predual of A. For this result we use the notations of [2] and write S^u for $S_u^{A_*}$.

Proposition 3.3. *Let A be a von Neumann algebra with e as the identity and let $u \in A$ be a unitary. Then S^u is weak*-dense in S_u.*

Proof. Since the isometry $a \to u^*a$ is also a weak*-homeomorphism it is enough to prove the statement for $u = e$. Furthermore, using the Gelfand-Naimark representation of A as a weak*-closed subalgebra of $\mathcal{L}(H)$, we may assume that $u = I \in \mathcal{L}(H)$.

S_I is the weak*-closed convex hull of the functionals of the form $x \otimes y$, where x, y are unit vectors in H satisfying $\langle x, y\rangle = 1$ (see, e.g., [8]). Since any such functional is in S^I, the latter space is weak*-dense in S_I. □

Notice that the concept of the unitary refers not only to a particular point but also to a specific space containing that point—if $e_0 \in E \subseteq F$ then e_0 may be a unitary in E and at the same time may not even be an extreme point in F. For example, if E is a proper M-summand in F, none of the points of E remain extreme in F. On the other hand, it is easy to see that if e_0 is a unitary in F it must be a unitary in E. The next corollary shows that in the case of $F = E^{**}$ we have both implications.

Corollary 3.4. *Let e_0 be a norm one element of a Banach space E. Then e_0 is a unitary in E if and only if it is a unitary in E^{**}.*

Proof. If $e_0 \in E$ is a unitary in E^{**} then by the above remark it is a unitary in E. Conversely, if e_0 is a unitary in E, then by the definition it is a weak*-unitary in E^{**}. By Corollary 3.2 it is a unitary in E^{**}. □

Corollary 3.5. *Let A be a complex Banach algebra with identity e_0. Then e_0, as well as any invertible element x such that $\|x\| = 1 = \|x^{-1}\|$, are unitaries.*

Proof. As p_{e_0} coincides with the numerical radius function V of [3], by [3, page 34, Theorem 1], we get that for a complex Banach algebra $ep_{e_0}(\cdot) \geq \|\cdot\|$. Thus e_0 is a unitary. Since $a \longmapsto ax^{-1}$ is a surjective isometry of A that maps x to e_0 the other conclusion follows. □

By Theorem 3.1 p_{e_0} and $\|\cdot\|$ are equivalent if and only if e_0 is a unitary. It is easy to notice that these norms are equal if and only if $E_1^* = \overline{\text{aco}}^{w^*}(S_{e_0})$, where aco denotes the absolutely convex hull. By Milman's Theorem this happens if and only if $|e^*(e_0)| = 1$ for any $e^* \in \partial_e E_1^*$. This is indeed the case for $E = C(X)$, or more generally if E is an L^1-predual space—see Theorem 3.12. Such a situation is however rather rare; for example, it happens for a C^*-algebra A if and only if A is commutative [9, page 277].

COROLLARY 3.6. *Any unitary is a strongly extreme point.*

Proof. Assume $x, y, e_0 \in E_1$ are such that $\|(x+y)/2 - e_0\| \le \varepsilon$ and let $e^* \in S_{e_0}$. We have

$$\frac{|(e^*(x)-1)+(e^*(y)-1)|}{2} = \left|e^*\left(\frac{x+y}{2} - e_0\right)\right| \le \varepsilon,$$

and

$$|e^*(x)| \le 1, \quad |e^*(y)| \le 1.$$

Hence

$$-2\varepsilon \le \operatorname{Re}(e^*(x)-1) \le 0, \quad -2\varepsilon \le \operatorname{Re}(e^*(y)-1) \le 0,$$

and

$$\max\{|\operatorname{Im}(e^*(x)-1)|, |\operatorname{Im}(e^*(y)-1)|\} \le \sqrt{2\,|\operatorname{Re}(e^*(x)-1)|} \le 2\sqrt{\varepsilon},$$

so

$$|e^*(x) - e^*(y)| \le 2\sqrt{\varepsilon},$$

and consequently

$$p_{e_0}(x-y) \le 2\sqrt{\varepsilon}.$$

Hence, if e_0 is a unitary and $(x_n + y_n)/2 \to e_0$ we get $x_n - y_n \to 0$. □

The next example shows that a vertex need not be strongly extreme and hence need not be a unitary.

EXAMPLE 3.7. Let E be the space of all convergent sequences with the norm defined by

$$\|(a_n)_{n=1}^\infty\| = \sup\left\{\frac{1}{n}\|(a_1,a_n)\|_1 + \left(1-\frac{1}{n}\right)\|(a_1,a_n)\|_\infty : n = 2,3,\ldots\right\},$$

where

$$\|(x,y)\|_1 = |x| + |y| \quad \text{and} \quad \|(x,y)\|_\infty = \max\{|x|,|y|\}.$$

Notice that

$$\frac{1}{2}\sup\{|a_n| : n = 2,3,\ldots\} \le \|(a_n)_{n=1}^\infty\| \le |a_1| + 2\sup\{|a_n| : n = 2,3,\ldots\},$$

so E is isomorphic with c_0 and E^* is isomorphic with ℓ^1. Let $\{e_n\}$ be the standard Schauder basis of E and $\{e_n^*\}$ the standard Schauder basis of E^*, that is, $e_n^*((a_n)_{n=1}^{\infty}) = a_n$. It is easy to check that the following functionals are in S_{e_1}:

$$e_1^*, e_1^* + \frac{1}{2}e_2^*, \ldots, e_1^* + \frac{1}{n}e_n^*, \ldots, \tag{3.1}$$

and the span of these functionals is weak*-dense in E^*, so e_1 is a vertex. The point e_1 is however not strongly extreme, since

$$\|e_1 \pm e_n\| = \frac{2}{n} + \left(1 - \frac{1}{n}\right) \to 1 \quad \text{while } \|e_n\| = 1.$$

The next easy example shows that the set of unitaries may not be closed even in a finite dimensional Banach space.

Example 3.8. Put

$$K = \left\{\left(\cos\frac{\pi}{n}, \eta \sin\frac{\pi}{n}\right) \in \mathbb{R}^2 : n = 2, 3, \ldots; \eta \in \{-1, 1\}\right\}$$

and let E be the two dimensional real Banach space whose unit ball is $B = \overline{\text{aco}}(K)$. Notice that B has an "edge" at each point of K, so all these points are unitaries. On the other hand, the limit point $(1, 0)$ has only a single supporting functional and consequently is not a unitary.

We now consider the notion of a strongly vertex point and examine its relation to a unitary. We say that $e_0 \in E_1$ is a strongly vertex point [1] if there exists $D \subseteq E_1$ such that

$$E_1 = \overline{\text{aco}}(D \cup \{e_0\}) \quad \text{and} \quad e_0 \notin \overline{\text{aco}}(D).$$

Lemma 3.9. *If $e_0 \in E_1$ is a strongly vertex point then e_0 is a unitary as well as a strongly exposed point of E_1.*

Proof. Assume $e^* \in \partial E_1^*$ separates e_0 from $\overline{\text{aco}}(D)$, that is,

$$\|e^*\| = 1 = e^*(e_0) > \sup\{|e^*(e)| : e \in D\} \stackrel{\text{def}}{=} \rho$$

Let $0 < r < (1 - \rho)/2$. If $\|x^* - e^*\| \le r$ then $|x^*(e_0)| = \|x^*\|$ [1] and consequently $x^*/\|x^*\| \in S_{e_0}$. That is, e_0 is a unitary.

Claim. *e^* strongly exposes e_0.*

Let $\varepsilon > 0$. We will show that there exists $\eta > 0$ such that if $x \in E_1$ and $\text{Re}(e^*)(x) > 1 - \eta$, then $\|x - e_0\| \le \varepsilon$. Since $E_1 = \overline{\text{aco}}(D \cup \{e_0\})$, $x = \lambda t e_0 + (1 - \lambda)z$ for some $\lambda \in [0, 1]$, $t \in \mathbb{T}$ and $z \in \overline{\text{aco}}(D)$. Then $\text{Re}(e^*)(x) = \lambda \text{Re}(t) + (1 - \lambda)\text{Re}(e^*)(z) > 1 - \eta$. If $0 < \eta < (1 - \rho)$, then $\text{Re}(e^*)(z) < 1 - \eta$, and hence, $\text{Re}(t) > 1 - \eta$. Since $|t| = 1$, $|1 - t|^2 = 2 - 2\text{Re}(t)^2 < 2\eta(2 - \eta)$. It also follows that

$$1 - \eta < \lambda \text{Re}(t) + (1 - \lambda)\text{Re}(e^*)(z) < \lambda + (1 - \lambda)\rho$$

and hence $(1-\lambda) < \eta/((1-\rho)$. Therefore

$$\begin{aligned}\|x-e_0\| &\leq \|x-te_0\| + |1-t| = \|(1-\lambda)(z-te_0)\| + |1-t| \\ &< \frac{2\eta}{1-\rho} + \sqrt{2\eta(2-\eta)} < \varepsilon\end{aligned}$$

for sufficiently small η. □

REMARK 3.10. Let E be a Banach space and $e \in E$ be a nonzero element. For any $0 < r < \|e\|$, if we define $B = \overline{\mathrm{aco}}(rE_1 \cup \{e_0\})$, then B is the unit ball of an equivalent norm on E in which e becomes a strongly vertex point.

This shows in particular that in any Banach space E, given any nonzero element e, E has an equivalent renorming in which e becomes a unitary.

Even though the notions of vertex and unitary coincide in the case of a C^*-algebra, we show that a C^*-algebra has no strongly vertex points.

THEOREM 3.11. *Let A be a unital complex C^*-algebra such that $A \neq \mathbb{C}$. Then A has no strongly vertex points.*

Proof. Let A be a C^*-algebra with identity e. Suppose $u \in A$ is a strongly vertex point. By Lemma 3.9, u is a unitary. From [2, Theorem 2] we get that $uu^* = u^*u = e$. By passing through the isometry $a \to au^*$ of A, if necessary, we may assume that $u = e$.

By spectral theory, there exists a nontrivial commutative C^*-subalgebra B of A containing e. By the Gelfand-Naimark Theorem [3], B is C^*-algebra isomorphic to $C(X)$ for a compact set X with at least two distinct points. Let us identify B with $C(X)$. Then e corresponds to the function $\mathbf{1}$. Let $p, q \in X$ be distinct and fix $f \in C(X)$ such that $0 \leq f \leq 1$, $f(p) = 1$ and $f(q) = 0$. Suppose $e = \mathbf{1}$ is a strongly vertex point. Let $D \subseteq A_1$ be a closed absolutely convex set such that $e \notin D$ and $A_1 = \overline{\mathrm{aco}}(D \cup \{e\})$. Since $e \notin D$, there exists $s \in \mathbb{T}$, $s \neq 1$, such that $f + s(e-f) \notin D$. But $f + s(e-f) \in A_1$, so

$$f + s(e-f) = \lambda te + (1-\lambda)z \tag{3.2}$$

for some $\lambda \in (0,1]$, $t \in \mathbb{T}$ and $z \in D$ as in Lemma 3.9. Note that this implies $z \in B$. Evaluating (3.2) at p, we get $1 = \lambda t + (1-\lambda)z(p)$, so that $t = 1$. Now, evaluating (3.2) at q, we get $s = \lambda + (1-\lambda)z(q)$. Since $s \neq 1$ and $|z(q)| \leq 1$, this is impossible. □

We thank the referee for this version of the above theorem.

We now show that if an L^1-predual space E satisfies the analogue of the Russo-Dye theorem [3], that is, if E_1 is the closed convex hull of unitaries, then E is a $C(X)$ space. Note that by Corollary 3.6 the hypothesis implies in particular that E_1 is the closed convex hull of its strongly extreme points.

THEOREM 3.12. *Let E be an L^1-predual space. If $E = \overline{\mathrm{span}}(\mathcal{U})$, then E is isometric to $C(X)$ for some compact Hausdorff space X.*

Proof. Let $e_0 \in \partial_e E_1$. Let S_{e_0} be equipped with the weak*-topology. It follows from [9, Theorem 1.5] that the map $\Psi : E \to A(S_{e_0})$ defined by $\Psi(e)(e^*) = e^*(e)$ is an onto isometry such that $\Psi(e_0) = \mathbf{1}$. Thus by Theorem 3.1 every extreme point of E_1 is a unitary. Moreover, since $\partial_e A(S_{e_0})_1^* = \mathbb{T}\partial_e S_{e_0}$ [7, Section 20] (S_{e_0} is embedded in $A(S_{e_0})_1^*$ by the canonical evaluation map), we get $|e^*(e_0)| = 1$ for any $e^* \in \partial_e E_1^*$.

We show that $\partial_e S_{e_0}$ is a weak*-closed set. Let $\{e_\alpha^*\} \subseteq \partial_e S_{e_0}$ be a net such that $e_\alpha^* \to e^*$ in the weak*-topology. Suppose $e^* = (e_1^* + e_2^*)/2$ for $e_i^* \in S_{e_0}$. For any $e \in \partial_e E_1$, by the previous paragraph, $|e^*(e)| = 1$. Thus $e^*(e) = e_1^*(e) = e_2^*(e)$. Since $E = \overline{\operatorname{span}}(\mathcal{U})$, we get $e^* = e_1^* = e_2^*$. Therefore $\partial_e S_{e_0}$ is a closed set.

Since E is a L^1-predual, when considered over the real scalar field, the space $A(S_{e_0})^*$ is a lattice. Thus S_{e_0} is a Choquet simplex [7]. Since $\partial_e S_{e_0}$ is closed, $a \to a|_{\partial_e S_{e_0}}$ is an onto isometry between $A(S_{e_0})$ and $C(\partial_e S_{e_0})$. Thus E is isometric to $C(\partial_e S_{e_0})$. □

We now give an example of a Banach algebra E with involution which is not a C^*-algebra in which an analogue of the Russo-Dye Theorem holds.

Example 3.13. Let $E = \ell^1(\mathbb{Z})$ with convolution as multiplication and $e_0 = \delta(0)$ as the identity. For $n \geq 1$, by taking $D \stackrel{\text{def}}{=} \{\delta(m) : m \neq n\}$, it is easy to see that $\delta(n)$ is a strongly vertex point. Since any point of $\partial_e E_1$ is of the form $t\delta(n)$ for some $n \in \mathbb{Z}$ and $t \in \mathbb{T}$, we get that any extreme point is a strongly vertex point, and hence a unitary. Clearly $E_1 = \overline{\operatorname{conv}}(\mathcal{U})$, and $E = \overline{\operatorname{span}}(\mathcal{U})$.

4. Unitaries in $C(X, E)$ spaces

Theorem 4.1. *Let E be a Banach space and X a compact Hausdorff space. If $f \in C(X, E)$ is a unitary, then for all $x \in X$, $f(x)$ is a unitary. Furthermore, if $p_f^*(F) \leq K\|F\|$ for all $F \in C(X, E)^*$, then $p_{f(x)}^*(e^*) \leq K\|e^*\|$ for all $x \in X$ and $e^* \in E^*$.*

Proof. Fix $x \in X$. Since the space $C(X, E)^*$ can be identified with the space of E^*-valued regular Borel measures equipped with the total variation norm, $P(F) = F|_{\{x\}} = \delta(x) \otimes F(\{x\})$ is a well-defined norm one projection. As in the case of scalar-valued measures, it is easy to check that

$$\|F\| = \|F|_{\{x\}}\| + \|F - F|_{\{x\}}\|, \quad F \in C(X, E)^*,$$
$$\|\Lambda\| = \max\{\|P^*(\Lambda)\|, \|\Lambda - P^*(\Lambda)\|\}, \quad \Lambda \in C(X, E)^{**}.$$

If $F \in S_f$ we get

$$\begin{aligned} 1 &= F(f) = F(\{x\})(f(x)) + (F - P(F))(f) \\ &\leq \|P(F)\|\|f(x)\| + \|F - F|_{\{x\}}\| \leq \|P(F)\| + \|F - F|_{\{x\}}\| = 1, \end{aligned}$$

so $\|F|_{\{x\}}\| = F(\{x\})(f(x)) \neq 0$ whenever $P(F) \neq 0$. Consequently

$$\frac{F(\{x\})}{\|F(\{x\})\|} \in S_{f(x)} \quad \text{if } F \in S_f \text{ and } F(\{x\}) \neq 0.$$

Let $0 \neq e^* \in E^*$ and let $\delta(x)\otimes e^* \in C(X,E)^*$ be defined by $(\delta(x)\otimes e^*)(g) = e^*(g(x))$. By our hypothesis, $\delta(x) \otimes e^* = \sum \alpha_i F_i$, where $F_i \in S_f$. Hence by evaluating at $\{x\}$ we have

$$e^* = P(\delta(x) \otimes e^*)(\{x\}) = \sum \alpha_i F_i(\{x\})$$

and all of $\alpha_i F_i(\{x\})$ are multiples of elements from $S_{f(x)}$. Thus at least one of the terms $\alpha_i F_i(\{x\})$ is nonzero and $f(x)$ is a unitary in E. Furthermore $\|e^*\| = \|\delta(x) \otimes e^*\|$ and $\sum \|\alpha_i F_i(\{x\})\| \leq \sum \|\alpha_i F_i\|$, so the second part of the theorem follows. □

The crucial part of the above proof was the ℓ^1-decomposition of an arbitrary functional F in $C(X,E)^*$ and the corresponding ℓ^∞ decomposition in the second dual. By Theorem 2.4.7 and Theorem 2.3.4 of [4], it can be seen that the same is true for the injective tensor product $A \otimes_\epsilon E \subseteq C(X) \otimes_\epsilon E \simeq C(X,E)$ of a uniform algebra $A \subseteq C(X)$ and any point x in the Choquet boundary ChA of that algebra. Thus we have a natural generalization of the last result.

Theorem 4.2. *Let E be a Banach space and A a uniform algebra. If $f \in (A \otimes_\epsilon E)$ is a unitary, then for any $x \in ChA$, $f(x)$ is a unitary.*

We now obtain a converse to the previous theorem.

Theorem 4.3. *Let E be a Banach space and A a uniform algebra on its Shilov boundary X. If $f \in A \otimes_\epsilon E$ is such that there exists $K > 0$ with $p_{f(x)}(e) \geq \frac{1}{K}\|e\|$ for all $e \in E$ and all x in X, then f is a unitary.*

In particular, in each of the following cases $f \in A \otimes_\epsilon E$ is a unitary if and only if $f(x)$ is a unitary for all $x \in X$: (i) *$f(X)$ is finite, in particular,* (a) *X is finite or* (b) *f is constant;* (ii) *E is a C^*-algebra;* (iii) *E is a function algebra;* (iv) *E is a L^1-predual; or* (v) *E is a $L^1(\mu)$ space.*

Proof. Since S_f is a w*-closed face of $(A \otimes_\epsilon E)_1^*$, $S_f = \overline{\mathrm{co}}^{w^*} \partial_e S_f$ and $\partial_e S_f \subseteq \partial_e C(X,E)_1^*$. It follows that

$$\partial_e S_f = \{e^* \otimes \delta(x) : x \in X,\, e^* \in S_{f(x)}\}l.$$

Therefore, for any $g \in A \otimes_\epsilon E$,

$$\begin{aligned} p_f(g) &= \sup\{|(e^* \otimes \delta(x))(g)| : x \in X,\, e^* \in S_{f(x)}\} \\ &= \sup\{|e^*(g(x))| : x \in X,\, e^* \in S_{f(x)}\} \\ &= \sup\{p_{f(x)}(g(x)) : x \in X\} \geq K\|g\|_\infty. \end{aligned}$$

Hence, f is a unitary.

It follows from our remarks in Section 3 and from Theorem 4.2 that if E is as in cases (ii)–(iv), then there exists $K > 0$ such that $p_u(e) \geq \frac{1}{K}\|e\|$ for all $e \in E$ and $u \in \mathcal{U}(E)$. Note that even in the case of $E = L^1(\mu)$, since extreme points of E_1 are given by atoms of μ and as extreme points of E_1^* are of absolute value 1 a.e, any extreme point is a unitary and $K = 1$ works. $\square$

We may notice that the above theorem would not be valid if we replaced *unitary* with *vertex*, even in the scalar case. For example, a norm one element f of the disc algebra such that $\{x \in \mathbb{T} : |f(x)| = 1\}$ has nonempty interior is a vertex, but $|f(x)|$ may not be equal to 1 on all of $\mathbb{T}$, which is the Choquet boundary of A.

The next example shows that in general a unitary-valued function need not be a unitary even when E is finite dimensional.

Example 4.4. Put

$$L = \left\{ \left(\cos \frac{\pi}{n}, \sin \frac{\pi}{n} \right) \in \mathbb{R}^2 : n = 2, 3, \dots \right\}$$

and let E be the two dimensional real Banach space whose unit ball is $W \stackrel{\text{def}}{=} \overline{\text{aco}}(L)$. Our Banach space is similar to that considered in Example 3.8—the unit ball W is identical with B in the first and the third quadrant, but in the second and the fourth one W coincides with the ℓ^1-ball. So this time $(1, 0)$ is a unitary; W still has an "edge" at each point of L and all these points are unitaries. However the angles of W at the points $(\cos \pi/n, \sin \pi/n)$, $n = 2, 3, \dots$ increase and tend to $\pi/2$; consequently these points are "less and less unitaries", that is, $p^*_{(\cos \pi/n, \sin \pi/n)}$ and $\|\cdot\|$ are "less and less equivalent". More precisely

$$\lim_n \sup \left\{ |e^*(0, 1)| : e^* \in S_{(\cos \pi/n, \sin \pi/n)} \right\} = 0,$$

and hence

$$\frac{p^*_{(\cos \pi/n, \sin \pi/n)}\left((0, 1)\right)}{\|(0, 1)\|} = p^*_{(\cos \pi/n, \sin \pi/n)}\left((0, 1)\right) \to \infty \text{ as } n \to \infty. \tag{4.1}$$

Let $X = \mathbb{N} \cup \{\infty\}$ be one point compactification of the discrete set $\mathbb{N}$, and put

$$f(n) = \begin{cases} (\cos \pi/n, \sin \pi/n) & \text{for } n \in \mathbb{N} \\ (1, 0) & \text{for } n = \infty \end{cases}.$$

For all n, including $n = \infty$, $f(n)$ is a unitary; however, by (4.1) and Theorem 4.1, $f \in C(X, E) = c(E)$ is not a unitary.

Arguments similar to Theorem 4.3 show that if $E_1, E_2, \dots, E_n$ are Banach spaces and $E = \oplus_\infty E_k$, then $e = (e_k) \in E$ with $\|e\|_\infty = 1$ is a unitary if and only if for all k, e_k is a unitary in E_k. While the necessity still holds for arbitrary ℓ^∞ sums, the above example shows that in an ℓ^∞ sum of even a single E there may be a vector that is not a unitary, but each of its coordinates is a unitary.

In the following proposition we show that any vertex-valued continuous function is a vertex. Thus the above is yet another example of a vertex that is not a unitary.

PROPOSITION 4.5. *Let $f \in C(X, E)$ be such that for all x in a dense subset X' of X, $f(x)$ is a vertex. Then f is a vertex.*

Proof. It suffices to show that S_f separates points of $C(X, E)$. Suppose $g \in C(X, E)$ and $g \neq 0$. Then, for some $x \in X'$, $g(x) \neq 0$. Since $f(x)$ is a vertex, there exists $e^* \in S_{f(x)}$ such that $e^*(g(x)) \neq 0$. Note that $\delta(x) \otimes e^* \in S_f$ and $(\delta(x) \otimes e^*)(g) = e^*(g(x)) \neq 0$. $\square$

We use this opportunity to present a short C^*-algebra proof of the following result of Grząślewicz [5, Theorem 1].

THEOREM 4.6. *Let H be a Hilbert space and let $f \in \partial_e C(X, \mathcal{L}(H))_1$. Then for all $x \in X$, $f(x) \in \partial_e \mathcal{L}(H)_1$.*

Proof. In a unital C^*-algebra A,

$$f \in \partial_e A_1 \Longleftrightarrow (1 - ff^*)A(1 - f^*f) = 0 \tag{4.2}$$

[10, Proposition 1.4.7]. Since, with the point-wise multiplication, $C(X, \mathcal{L}(H))$ is a C^*-algebra, we have $(1 - ff^*)C(X, \mathcal{L}(H))(1 - f^*f) = 0$. Evaluating at $1 \otimes \mathcal{L}(H)$, we have for any $x \in X$, $(1 - f(x)f(x)^*)\mathcal{L}(H)(1 - f(x)^*f(x)) = 0$; using (4.2) again, we get $f(x) \in \partial_e \mathcal{L}(H)_1$. $\square$

5. Unitaries in spaces of operators

For a Banach space E, the map $T \longmapsto T^*$ is an isometry of $\mathcal{L}(E)$ onto the space $\mathcal{L}_{w^*}(E^*)$ of weak*-continuous operators. Since surjective isometries preserve unitaries and, as noted before, a unitary in a Banach space is automatically a unitary in a subspace that contains it, we have

$$\begin{aligned}[][T^* \in \mathcal{L}(E^*) \text{ is a unitary}] &\Longrightarrow [T^* \in \mathcal{L}_{w^*}(E^*) \text{ is a unitary}] \\ &\Longleftrightarrow [T \in \mathcal{L}(E) \text{ is a unitary}]\,.\end{aligned}$$

It is natural to ask whether the converse of the first implication is also true. The following propositions provide a partial answer. We recall that a unitary is an extreme point.

PROPOSITION 5.1. *If $T \in \partial_e \mathcal{L}(\ell^1)_1$ then T^* is a unitary. If $T \in \partial_e \mathcal{L}(\ell^\infty)_1$ then T is a unitary.*

Proof. Assume $T \in \partial_e \mathcal{L}(\ell^1)_1$. Since the space $\mathcal{L}(\ell^1)$ may be identified with $\oplus_\infty \ell^1$, via $R \longmapsto (R(e_n))_{n=1}^\infty$, $T(e_n)$ is an extreme point of ℓ^1_1 for any $n \in \mathbb{N}$. Hence

$$T(e_n) = \alpha_n e_{j(n)}, \text{ for } n \in \mathbb{N}, \tag{5.1}$$

where $|\alpha_n| = 1$ and $j : \mathbb{N} \to \mathbb{N}$. Notice that $\mathcal{L}(\ell^\infty)$ can be linearly embedded into the space of bounded functions on $\partial_e \ell^\infty_1 \times \mathbb{N}$:

$$\Phi(S)(u, n) = S(u)(e_n), \text{ for } S \in \mathcal{L}(\ell^\infty), u \in \partial_e \ell^\infty_1, n \in \mathbb{N}.$$

Since $\ell^\infty_1 = \overline{\text{conv}} \partial_e \ell^\infty_1$, Φ is an isometry. For $S = T^*$, by (5.1) we get

$$|\Phi(T^*)(u, n)| = |T^*(u)(e_n)| = |u(T(e_n))| = 1, \text{ for } (u, n) \in \partial_e \ell^\infty_1 \times \mathbb{N}.$$

Thus $\Phi(T^*)$ is a function of absolute value one and hence a unitary; consequently T^* is a unitary.

If $T \in \partial_e \mathcal{L}(\ell^\infty)_1$, using the identification of $\mathcal{L}(\ell^\infty)$ as $\oplus_\infty (\ell^\infty)^*$, we again get that $T^*(e_n)$ is an extreme point of $(\ell^\infty)^*_1$, so $T^*(e_n) = t\delta(x)$ for some $x \in \beta(\mathbb{N})$ and $|t| = 1$. Thus $|\Phi(T)(u, n)| = |T^*(e_n)(u)| = 1$ as $|u| = 1$ on $\beta(\mathbb{N})$. Hence T is a unitary. □

Since $\mathcal{L}(c_0)$ can be isometrically embedded into $\mathcal{L}(\ell^1)$ via the adjoint map, we get that for every $T \in \partial_e \mathcal{L}(c_0)_1$, T^* is a unitary.

In general, it is not clear whether T^* is always a unitary, even if T is compact. The following proposition addresses a special case.

Proposition 5.2. *Let E be a Banach space such that the set $\mathcal{U}(E^*)$ of unitaries on E^* is closed. For any unitary $T \in \mathcal{K}(E, C(X))$, T^* is a vertex of $\mathcal{K}(C(X)^*, E^*)$.*

Proof. Since $\mathcal{K}(E, C(X))$ can be identified with $C(X, E^*)$ via the map $T \to T^*|_X$, we can assume that $T \simeq f \in C(X, E^*)$. Since T is a compact operator, T^* can also be identified with a $\widetilde{f} \in C(K, E^*) = C(K) \otimes_\epsilon E^* = C(X)^{**} \otimes_\epsilon E^* = \mathcal{K}(C(X)^*, E^*)$, where K is the Stone space of $C(X)^{**}$.

It is well-known that K is extremally disconnected and that X can be embedded as a discrete set in K [7, Section 11] and there exists a retract $\phi : K \to \beta(X)$ such that $\widetilde{f} = f \circ \phi$ (as $f(X)$ is a compact subset of E, f has a natural extension to $\beta(X)$ still denoted by f). For any $k \in K$, $\widetilde{f}(k) = f(\phi(k))$. Arguments similar to the ones given during the proof of Theorem 4.1 show that $f(X) \subseteq \mathcal{U}$. Since X is dense in $\beta(X)$ and $\mathcal{U}$ is closed, $\widetilde{f}(k)$ is a unitary and hence by Proposition 4.5, $\widetilde{f}$ is a vertex. □

Added in Proof. It has been pointed out to us that the results of [2] are essentially known and can be found in Chapter 9 of T. W. Palmer, "Banach algebras and the general theory of *-algebras, Volume II, *-algebras" (Encyclopedia of Mathematics and its Applications, vol. 79, Cambridge University Press, Cambridge, 2001). Theorem 9.5.9 and the remarks following 9.5.16 in that book show that the semi-norm we have considered is in the context of a *-algebra, an equivalent norm with the constant e replaced by 2.

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