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ON THE DERIVATIVE OF INFINITE BLASCHKE PRODUCTS

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ABSTRACT. A well known result of Privalov shows that if f is a function that is analytic in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, then the condition $f' \in H^1$ implies that f has a continuous extension to the closed unit disc. Consequently, if B is an infinite Blaschke product, then $B' \notin H^1$. This has been proved to be sharp in a very strong sense. Indeed, for any given positive and continuous function ϕ defined on $[0, 1)$ with $\phi(r) \to \infty$ as $r \to 1$, one can construct an infinite Blaschke product B having the property that

$$(*) \qquad M_1(r, B') \overset{\text{def}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} |B'(re^{it})|\, dt = \mathrm{O}\left(\phi(r)\right), \quad \text{as } r \to 1.$$

All examples of Blaschke products constructed so far to prove this result have their zeros located on a ray. Thus it is natural to ask whether an infinite Blaschke product B such that the integral means $M_1(r, B')$ grow very slowly must satisfy a condition "close" to that of having its zeros located on a ray. More generally, we may formulate the following question: Let B be an infinite Blaschke product and let $\{a_n\}_{n=1}^{\infty}$ be the sequence of its zeros. Do restrictions on the growth of the integral means $M_1(r, B')$ imply some restrictions on the sequence $\{\mathrm{Arg}(a_n)\}_{n=1}^{\infty}$?

In this paper we prove that the answer to these questions is negative in a very strong sense. Indeed, for any function ϕ as above we shall construct two new and quite different classes of examples of infinite Blaschke products B satisfying $(*)$ with the property that every point of $\partial\Delta$ is an accumulation point of the sequence of zeros of B.

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1. Introduction and main results

Let Δ denote the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. For $0 < r < 1$ and g analytic in Δ we set

$$M_p(r,g) = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| g(re^{i\theta}) \right|^p \, d\theta \right)^{1/p}, \quad 0 < p < \infty,$$
$$M_\infty(r,g) = \max_{|z|=r} |g(z)|.$$

For $0 < p \leq \infty$ the Hardy space H^p consists of those functions g that are analytic in Δ and satisfy

$$||g||_{H^p} = \sup_{0<r<1} M_p(r,g) < \infty.$$

We refer to [2] for the theory of Hardy spaces. We recall that if a sequence $\{a_n\} \subset \Delta \setminus \{0\}$ satisfies the "Blaschke condition"

$$\sum (1 - |a_n|) < \infty,$$

then the product

$$B(z) = \prod_n \frac{\overline{a}_n}{|a_n|} \frac{a_n - z}{1 - \overline{a}_n z}$$

defines an H^∞ function, called the Blaschke product with zeros $\{a_n\}$.

A classical result of Privalov [2, Th. 3.11] asserts that a function f that is analytic in Δ has a continuous extension to the closed unit disc $\overline{\Delta}$, whose boundary values are absolutely continuous on $\partial\Delta$ if and only if $f' \in H^1$. In particular,

$$f' \in H^1 \quad \Longrightarrow \quad f \in \mathcal{A},$$

where, as usual, $\mathcal{A}$ denotes the disc algebra, that is, the space of all functions f that are analytic in Δ and have a continuous extension to the closed unit disc $\overline{\Delta}$.

Since the boundary values of a Blaschke product have modulus 1 almost everywhere [2], it is clear that if B is an infinite Blaschke product, then $B \notin \mathcal{A}$ and, hence, $B' \notin H^1$. This is best-possible, as the following theorem shows.

Theorem A. *Let ϕ be a positive and continuous function defined on* $[0,1)$ *with $\phi(r) \to \infty$ as $r \to 1$. Then there exists an infinite Blaschke product B with positive zeros having the property that*

$$M_1(r, B') = \mathrm{O}(\phi(r)), \quad \text{as } r \to 1. \tag{1}$$

Different proofs of this result have been given in [3], [4] and [5]. It is natural to ask whether an infinite Blaschke product B such that the integral means $M_1(r, B')$ grow very slowly must satisfy a condition "close" to that of having its zeros located on a ray. More generally, we may formulate the following question:

Let B be an infinite Blaschke product and let $\{a_n\}_{n=1}^{\infty}$ be the sequence of its zeros. Do restrictions on the growth of the integral means $M_1(r, B')$ imply some restrictions on the sequence $\{\operatorname{Arg}(a_n)\}_{n=1}^{\infty}$?

We shall prove that the answer to these questions is negative in a very strong sense. Indeed, for any function ϕ as in Theorem A we shall construct two new and quite different classes of examples of infinite Blaschke products B satisfying (1) with the property that every point of $\partial\Delta$ is an accumulation point of the sequence of zeros of B. Our first construction is given in Theorem 1.

THEOREM 1. *Let ϕ be a positive and continuous function defined on $[0, 1)$ with $\phi(r) \to \infty$ as $r \to 1$. Then there exists an increasing sequence $\{r_k\}_{k=1}^{\infty} \subset (0, 1)$ with $\sum_{k=1}^{\infty}(1 - r_k) < \infty$ such that if, for every k, a_k is a complex number with $|a_k| = r_k$ and B is the Blaschke product whose sequence of zeros is $\{a_k\}_{k=1}^{\infty}$, then B satisfies* (1).

Notice that if $\{r_k\}_{k=1}^{\infty}$ is the sequence constructed in Theorem 1, $\{\theta_k\}_{k=1}^{\infty}$ is any sequence of real numbers that is dense in $\mathbb{R}$ and we set $a_k = r_k e^{i\theta_k}$ $(k \geq 1)$, then every point of $\partial\Delta$ is an accumulation point of the sequence $\{a_k\}$ and the Blaschke product with zeros $\{a_k\}$ satisfies (1).

Our second class of examples is given in Theorem 2. The Blaschke products B constructed in Theorem 1 have the property that for any $r \in (0, 1)$ at most one zero of B lies on the circle $\{|z| = r\}$. The Blaschke products that we construct in Theorem 2 are quite different: If B is any of these products, then there exist a sequence $\{r_k\} \uparrow 1$ and a sequence of natural numbers $\{n_k\} \uparrow \infty$ such that, for all k, n_k of the zeros of B lie on the circle $\{|z| = r_k\}$.

THEOREM 2. *Let ϕ be a positive and continuous function defined on $[0, 1)$ with $\phi(r) \to \infty$ as $r \to 1$. Then there exist an increasing sequence $\{r_k\}_{k=1}^{\infty} \subset (0, 1)$ and a sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} n_k = \infty$ satisfying*

$$\sum_{k=1}^{\infty} n_k(1 - r_k) < \infty,$$

such that if B is the Blaschke product whose zeros are

$$\left\{ r_k e^{2\pi i j/n_k} : j = 0, 1, \dots, n_k - 1,\ k = 1, 2, \dots \right\},$$

that is,

$$B(z) = \prod_{k=1}^{\infty} \frac{r_k^{n_k} - z^{n_k}}{1 - r_k^{n_k} z^{n_k}}, \quad z \in \Delta, \tag{2}$$

then $M_1(r, B') = \mathrm{O}(\phi(r))$ as $r \to 1$.

We mention that Blaschke products like those constructed in Theorem 2 were used by Lohwater and Piranian [6] (see also Theorem 2.22 on p. 43 of [1]) to show that Fatou's theorem is best possible and by Piranian [11] to construct a Blaschke product B with $\iint_\Delta |B'(z)|dxdy = \infty$.

2. Proof of Theorem 1

If f is an analytic function in Δ, we let $n(r, f)$ $(0 < r < 1)$ denote the number of zeros of f in the disc $\{z : |z| \leq r\}$. Our proof of Theorem 1 will be based on the following result, which is an extension of Theorem 1 on p. 3 of [5].

THEOREM 3. *Given $\alpha \in (0, 1)$ there exist two positive constants $C_1(\alpha)$ and $C_2(\alpha)$ such that if $\{a_n\}_{n=1}^{\infty}$ is any sequence in $\Delta \setminus \{0\}$ satisfying*

$$(1 - |a_{n+1}|) \leq \alpha(1 - |a_n|), \quad n \geq 1, \tag{3}$$

and B is the Blaschke product whose sequence of zeros is $\{a_n\}_{n=1}^{\infty}$, then, for all r sufficiently close to 1,

$$C_1(\alpha)n(r, B) \leq M_1(r, B') \leq C_2(\alpha)n(r, B). \tag{4}$$

Proof. Take $\alpha \in (0, 1)$ and let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\Delta \setminus \{0\}$ satisfying (3). Let B be the Blaschke product whose sequence of zeros is $\{a_n\}_{n=1}^{\infty}$. Define

$$r_{2k-1} = |a_k|, \quad k = 1, 2, 3, \dots \tag{5}$$

and

$$r_{2k} = \frac{r_{2k-1} + r_{2k+1}}{2} = \frac{|a_k| + |a_{k+1}|}{2}, \quad k = 1, 2, 3, \dots. \tag{6}$$

Set $\beta = \frac{1}{2}(1 + \alpha)$. Then $0 < \beta < 1$ and it is easy to see that we have

$$1 - r_{k+1} \leq \beta(1 - r_k), \quad \text{for all } k.$$

Using Theorem 9.2 of [2], we see that the sequence $\{r_k\}$ is uniformly separated, that is, there exists a constant $\delta > 0$ such that

$$\prod_{\substack{j=1 \\ j \neq k}}^{\infty} \left| \frac{r_j - r_k}{1 - r_j r_k} \right| \geq \delta, \quad \text{for all } k. \tag{7}$$

Actually, an examination of the proof of Theorem 9.2 on pp. 155–156 of [2] shows that the constant δ depends only on β (or, equivalently, on α). Using the lemma on p. 154 of [2], we see that

$$\min_{|z|=r} \left| \frac{a_j - z}{1 - \overline{a_j} z} \right| \geq \left| \frac{|a_j| - r}{1 - |a_j| r} \right|, \quad 0 < r < 1, \quad j = 1, 2, \dots$$

and, hence,

$$\min_{|z|=r} |B(z)| \geq \prod_{j=1}^{\infty} \left| \frac{|a_j| - r}{1 - |a_j| r} \right| = \prod_{j=1}^{\infty} \left| \frac{r_{2j-1} - r}{1 - r_{2j-1} r} \right|, \quad 0 < r < 1.$$

Taking $r = r_{2k}$ and using (7), we obtain

$$\text{(8)} \quad \min_{|z|=r_{2k}} |B(z)| \geq \prod_{j=1}^{\infty} \left| \frac{r_{2j-1} - r_{2k}}{1 - r_{2j-1} r_{2k}} \right| \geq \prod_{\substack{j=1 \\ j \neq 2k}}^{\infty} \left| \frac{r_j - r_{2k}}{1 - r_j r_{2k}} \right| \geq \delta, \quad k = 1, 2, \dots$$

Once (8) has been established, the argument used on pp. 5–6 of [5] gives that there exists $\varrho_1 \in (0,1)$ such that

$$M_1(r, B') \geq \frac{\delta}{2} n(r, B), \quad \rho_1 < r < 1.$$

This gives the first inequality of (4) for all $r \in (\rho_1, 1)$ with $C_1(\alpha) = \delta/2$.

The second inequality with $C_2(\alpha) = 5$ follows from the argument on pp. 6–7 of [5]. □

Proof of Theorem 1. With Theorem 3 established, the proof of Theorem 1 follows the lines of the proof of Theorem A in [5]. Let ϕ be as in Theorem 1. We may assume without loss of generality that $\phi(0) < 1$. Define

$$\text{(9)} \qquad b_n = \max\{r \in (0,1) : \phi(r) = n\}, \quad n = 1, 2, 3, \dots.$$

It is clear that the sequence $\{b_n\}_{n=1}^{\infty}$ is well defined, increasing, and that $b_n \to 1$ as $n \to \infty$.

Given $r \in (0,1)$, let $N(r)$ denote the number of elements of the sequence which are smaller than or equal to r. It is clear that

$$n > \phi(r) \quad \Longrightarrow \quad b_n > r,$$

and thus

$$\text{(10)} \qquad N(r) \leq \phi(r).$$

Since $b_n \uparrow 1$, we can extract a subsequence $\{b_{n_k}\}$ of $\{b_n\}$ such that

$$\text{(11)} \qquad (1 - b_{n_{k+1}}) \leq \frac{1}{2}(1 - b_{n_k}), \quad k \geq 1.$$

Set $r_k = b_{n_k}$ ($k \geq 1$) and let $\{a_k\}_{k=1}^{\infty}$ be a sequence of complex numbers with $|a_k| = r_k$ for all k. Notice that (11) implies that $\{a_k\}$ satisfies the Blaschke condition. Let B be the Blaschke product whose sequence of zeros is $\{a_k\}_{k=1}^{\infty}$. Since $\{|a_k|\}$ is a subsequence of $\{b_n\}$, it is clear that

$$n(r, B) \leq N(r), \quad \text{for all } r \in (0,1).$$

Then (10) shows that

$$n(r, B) \leq \phi(r), \quad 0 < r < 1,$$

which, using Theorem 3 with $\alpha = 1/2$, gives

$$M_1(r, B') = \mathrm{O}(\phi(r)), \quad \text{as } r \to 1.$$

This finishes the proof. □

3. Proof of Theorem 2

The proofs of Theorem A in [3] and [4] make essential use of certain sequences introduced by K. I. Oskolkov in several contexts (see [7], [8], [9] and [10]). The proof given in [5] is simpler and independent of the Oskolkov's sequences. However, for the proof of Theorem 2 we shall again need to make use of Oskolkov's sequences.

Definition 1. Let $\omega : [0, 1] \to [0, \infty)$ be a continuous function with $\omega(0) = 0$ and

$$\frac{\omega(\delta)}{\delta} \to \infty, \quad \text{as } \delta \to 0. \tag{12}$$

Take a fixed number λ with $0 < \lambda < 1$ and consider the sequence of numbers $\{\delta_j\}_{j=0}^{\infty}$, defined inductively by

$$\begin{cases} \delta_0 = 1, \\ \delta_{j+1} = \min \left\{ \delta \in [0, 1) : \max \left[\frac{\omega(\delta)}{\omega(\delta_j)}, \frac{\omega(\delta_j)\delta}{\delta_j \omega(\delta)} \right] = \lambda \right\}, \quad j \geq 0. \end{cases} \tag{13}$$

Then $\{\delta_j\}_{j=0}^{\infty}$ is called the "λ-Oskolkov sequence associated with ω".

It is clear that the definition of $\{\delta_j\}$ makes sense. The main properties of the sequence $\{\delta_j\}$ that will be used in the sequel are stated and proved in Lemma 2 of [4]. We state them here for the sake of completeness.

Lemma 1. *Let $\omega : [0, 1] \to [0, \infty)$ be a continuous function with $\omega(0) = 0$ satisfying* (12). *Let $0 < \lambda < 1$ and let $\{\delta_j\}_{j=0}^{\infty}$ be the "λ-Oskolkov sequence associated with ω". Then $\{\delta_j\}$ is a decreasing sequence of positive numbers with $\delta_j \to 0$ as $j \to \infty$. Moreover, for all $j \geq 0$, we have*

$$\omega(\delta_{j+1}) \leq \lambda \omega(\delta_j), \tag{14}$$

$$\delta_{j+1} \leq \lambda^2 \delta_j, \tag{15}$$

$$\omega(\delta_{j+1})\delta_{j+1} \leq \lambda^3 \omega(\delta_j)\delta_j, \tag{16}$$

$$\frac{\omega(\delta_j)}{\delta_j} \leq \lambda^{k-j} \frac{\omega(\delta_k)}{\delta_k}, \quad 0 \leq j \leq k, \tag{17}$$

$$\omega(\delta_j) \leq \lambda^{j-k} \omega(\delta_k), \quad j \geq k. \tag{18}$$

In the following lemma we obtain an upper bound for the integral means $M_1(r, B')$ of Blaschke products B of the type considered in Theorem 2. It is similar to an inequality proved by D. Protas on p. 394 of [12].

LEMMA 2. *Let $\{r_k\}_{k=1}^{\infty}$ be an increasing sequence of numbers in $(0,1)$ and let $\{n_k\}_{k=1}^{\infty}$ be a sequence of natural numbers with $\lim_{k\to\infty} n_k = \infty$ satisfying*

$$\sum_{k=1}^{\infty} n_k(1-r_k) < \infty. \tag{19}$$

Let B be the Blaschke product whose zeros are

$$\left\{ r_k e^{2\pi i j/n_k} : j = 0, 1, \dots, n_k - 1,\ k = 1, 2, \dots \right\},$$

that is,

$$B(z) = \prod_{k=1}^{\infty} \frac{r_k^{n_k} - z^{n_k}}{1 - r_k^{n_k} z^{n_k}}, \quad z \in \Delta. \tag{20}$$

Then

$$M_1(r, B') \le 4 \sum_{j=1}^{\infty} \frac{n_j(1 - r_j^{n_j})}{(1-r) + (1 - r_j^{n_j})}, \quad 0 < r < 1. \tag{21}$$

Proof. We have

$$|B'(z)| = \left| \sum_{j=1}^{\infty} \frac{-n_j z^{n_j - 1}(1 - r_j^{2n_j})}{(1 - r_j^{n_j} z^{n_j})^2} \prod_{\substack{k=1\\k\ne j}}^{\infty} \frac{r_k^{n_k} - z^{n_k}}{1 - r_k^{n_k} z^{n_k}} \right| \tag{22}$$

$$\le \sum_{j=1}^{\infty} \frac{n_j(1 - r_j^{2n_j})}{|1 - r_j^{n_j} z^{n_j}|^2} \le 2 \sum_{j=1}^{\infty} \frac{n_j(1 - r_j^{n_j})}{|1 - r_j^{n_j} z^{n_j}|^2}, \quad z \in \Delta.$$

Now, a simple calculation shows that

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{|1 - r_j^{n_j} r^{n_j} e^{i n_j t}|^2} &= \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{|1 - r_j^{n_j} r^{n_j} e^{it}|^2} \\ &= \frac{1}{1 - r_j^{2n_j} r^{2n_j}} \le \frac{2}{(1 - r^{n_j}) + (1 - r_j^{n_j})} \\ &\le \frac{2}{(1-r) + (1 - r_j^{n_j})}, \quad 0 < r < 1, \end{aligned}$$

which together with (22) gives (21). This finishes the proof. □

Proof of Theorem 2. We may assume without loss of generality that $\phi(r) \ge 1$, $0 \le r < 1$. Define

$$\phi_1(r) = \min\left(\phi(r), \frac{2}{(1-r)^{1/2}} \right), \quad 0 < r < 1,$$

and let ϕ_2 denote the highest increasing minorant of ϕ_1, that is,

$$\phi_2(r) = \inf_{r \le s < 1} \phi_1(s), \quad 0 \le r < 1.$$

Then it is clear that ϕ_2 is a positive, continuous and increasing function on $[0,1)$ with $\phi_2(r) \ge 1$ for all $r \in [0,1)$. Also,

$$\phi_2(r) \to \infty \quad \text{and} \quad (1-r)\phi_2(r) \to 0, \quad \text{as } r \to 1.$$

Let $\omega : [0,1] \to \mathbb{R}$ be defined by

$$\begin{cases} \omega(0) = 0, \\ \omega(\delta) = \delta\phi_2(1-\delta), \quad 0 < \delta \le 1. \end{cases} \tag{23}$$

Hence,

$$\phi_2(r) = \frac{\omega(1-r)}{1-r}, \quad 0 < r < 1. \tag{24}$$

Clearly, ω is positive and continuous on $[0,1]$ and satisfies

$$\omega(\delta) \ge \delta \text{ for all } \delta \in [0,1] \quad \text{and} \quad \frac{\omega(\delta)}{\delta} \to \infty \text{ as } \delta \to 0.$$

Take and fix a real number λ with $0 < \lambda < 1$ and let $\{\delta_j\}_{j=0}^{\infty}$ be the "λ-Oskolkov sequence associated with ω". Set

$$n_j = E\left[\min\left(\frac{\omega(\delta_j)}{\delta_j}, \frac{1}{\lambda^{2j}}\right)\right], \quad j \ge 1, \tag{25}$$

where, for $x \ge 0$, $E[x]$ denotes the greatest integer which is $\le x$. It is clear that $n_j \to \infty$, as $j \to \infty$, and that there exists a positive integer N such that $\omega(\delta_j) < 1$ for all $j \ge N$. Define

$$r_j = (1 - \delta_j\omega(\delta_j))^{1/n_j}, \quad j \ge N. \tag{26}$$

Using (25) and (18), we easily obtain that

$$\sum_{j=N}^{\infty} n_j(1-r_j) < \infty.$$

Consequently, the infinite product

$$B(z) = \prod_{j=N}^{\infty} \frac{r_j^{n_j} - z^{n_j}}{1 - r_j^{n_j} z^{n_j}}$$

is in fact a Blaschke product of the type considered in Lemma 2.

Using Lemma 2, we have

$$M_1(r, B') \le 4 \sum_{j=N}^{\infty} \frac{n_j(1-r_j^{n_j})}{(1-r) + (1-r_j^{n_j})}. \tag{27}$$

Define now

$$\varrho_j = 1 - \delta_j, \quad j \geq N. \tag{28}$$

Then $\varrho_j \uparrow 1$ as $j \uparrow \infty$. From now on we shall use the convention that C will denote a constant which may be different at distinct occurrences. From (28), (27) and (26) we obtain

$$M_1(\varrho_{k+1}, B') \leq C \sum_{j=N}^{\infty} \frac{n_j \delta_j \omega(\delta_j)}{\delta_{k+1} + \delta_j \omega(\delta_j)}, \quad k \geq N. \tag{29}$$

Using (17) and (25) we deduce that, for $k \geq N$,

$$\begin{aligned} \sum_{j=N}^{k} \frac{n_j \delta_j \omega(\delta_j)}{\delta_{k+1} + \delta_j \omega(\delta_j)} &\leq \frac{\omega(\delta_k)}{\delta_k} \sum_{j=N}^{k} \lambda^{k-j} \frac{n_j \delta_j}{\omega(\delta_j)} \\ &\leq \frac{\omega(\delta_k)}{\delta_k} \sum_{j=N}^{k} \lambda^{k-j} \leq \frac{\omega(\delta_k)}{\delta_k} \sum_{j=0}^{\infty} \lambda^j \leq C \frac{\omega(\delta_k)}{\delta_k}. \end{aligned} \tag{30}$$

Using (18), (15) and (25), we obtain

$$\begin{aligned} \sum_{j=k+1}^{\infty} \frac{n_j \delta_j \omega(\delta_j)}{\delta_{k+1} + \delta_j \omega(\delta_j)} &\leq \sum_{j=k+1}^{\infty} \frac{n_j \delta_j \omega(\delta_j)}{\delta_{k+1}} \\ &\leq \frac{\omega(\delta_k)}{\delta_k} \sum_{j=k+1}^{\infty} \lambda^{-2j} \lambda^{2(j-k-1)} \lambda^{j-k} \delta_k \\ &= \frac{\omega(\delta_k)}{\delta_k} \sum_{j=k+1}^{\infty} \lambda^{j-k} \lambda^{-2(k+1)} \delta_k \\ &\leq \lambda^{-2} \frac{\omega(\delta_k)}{\delta_k} \sum_{j=k+1}^{\infty} \lambda^{j-k} \leq \lambda^{-2} \frac{\omega(\delta_k)}{\delta_k} \sum_{j=0}^{\infty} \lambda^j \\ &\leq C \frac{\omega(\delta_k)}{\delta_k}, \qquad k \geq N, \end{aligned} \tag{31}$$

which, together with (28), (30), (29) and (24), gives

$$M_1(\varrho_{k+1}, B') \leq C \phi_2(\varrho_k), \quad k \geq N. \tag{32}$$

Since $M_1(r, B')$ and $\phi_2(r)$ are increasing functions of r and $\phi_2(r) \leq \phi(r)$ for all r, (32) yields $M_1(r, B') \leq C\phi(r)$ if $r \geq \varrho_N$. This finishes the proof. □

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